Algebra Universalis, **12** (1981) 242–246 0002-5240/81/002242-05$01.50+0.20/0

Embedding modular lattices into relation algebras

ROGER MADDUX

In [4], Jónsson asked whether every modular lattice is isomorphic to a lattice of commuting equivalence elements of some relation algebra. This note provides an affirmative answer.

A *relation algebra* is an algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, \breve{}, 1' \rangle$ where $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra, ; is a binary associative operation on A, 1' is an identity element for ;, and for all $x, y, z \in A$ the following conditions are equivalent; $x \cdot (y; z) = 0$, $y \cdot (x; z^{\smile}) = 0$, and $z \cdot (y^{\smile}; x) = 0$. An important example of a relation algebra is the *algebra of subrelations* $\mathcal{S}\ell E$ of an equivalence relation E, where $\mathcal{S}\ell E = \langle SbE, \cup, \cap, \sim, \emptyset, E, |, ^{-1}, Id \rangle$, $\sim$ is complementation with respect to E, $|$ is relative product, $^{-1}$ is conversion, and Id is the identity relation on the field of E. (E must be an equivalence relation to insure that SbE is closed under $|$ and $^{-1}$.) A relation algebra is *representable* if it is isomorphic to a subalgebra of some $\mathcal{S}\ell E$.

Let $\mathfrak{A}$ be a relation algebra. An element $x \in A$ is an *equivalence element* of $\mathfrak{A}$ if $x ; x \leq x = x^{\smile}$. Eq $\mathfrak{A}$ is the set of equivalence elements of $\mathfrak{A}$. Notice that $R \in$ Eq $\mathcal{S}\ell E$ *iff* $R \subseteq E$ and R is an equivalence relation. We will be concerned only with equivalence elements which contain 1', so we define $\mathrm{Eq}^+ \mathfrak{A} = \{x \in A : x ; x \leq x = x^{\smile} \geq 1'\}$. It was noted in [1], p. 383, and [4], p. 463, that if $B \subseteq \mathrm{Eq}^+ \mathfrak{A}$, B is closed under ; and $\cdot$, and $x ; y = y ; x$ for all $x, y \in B$, then $\langle B, ;, \cdot \rangle$ is a modular lattice with ; as join and $\cdot$ as meet. Jónsson called $\langle B, ;, \cdot \rangle$ a *lattice of commuting equivalence elements*, and asked ([4], p. 463) whether every modular lattice is isomorphic to such a lattice. It is easy to see that if $\mathfrak{A}$ is representable, then $\langle B, ;, \cdot \rangle$ is isomorphic to a lattice of commuting equivalence relations, i.e. $\langle B, ;, \cdot \rangle$ has a representation of type 1 (see [3], p. 97). Jónsson proved in [4] that a lattice has a representation of type 1 iff it satisfies a certain infinite set of implications, and some modular lattices (such as the lattice of subspaces of a non-Desarguesian projective plane) have no such representation. On the other hand, there are non-representable relation algebras (first proved in [5]), so it was

Presented by B. Jónsson. Received August 7, 1979. Accepted for publication in final form November 9, 1979.

reasonable to ask whether every modular lattice is isomorphic to a lattice of commuting equivalence elements.

A relation algebra $\mathfrak{A}$ is *symmetric* if $x^{\smile} = x$ for all $x \in A$. It is easy to show (see [1], Theorem 2.2) that an algebra $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, ;, ^{\smile}, 1' \rangle$ is a symmetric relation algebra iff the following conditions hold:

(a) $\langle A, +, \cdot, -, 0, 1 \rangle$ is a Boolean algebra,
(b) ; is commutative and associative,
(c) $x; 1' = x = 1'; x$ for all $x \in A$,
(d) for all $x, y, z \in A$, if $x \cdot (y; z) = 0$ then $y \cdot (x; z) = 0$,
(e) $x^{\smile} = x$ for all $x \in A$.

If $\mathfrak{A}$ is a symmetric relation algebra, then $\mathrm{Eq}^+\mathfrak{A} = \{x \in A : x ; x \leq x \geq 1'\}$, and $\langle B, ;, \cdot \rangle$ is a modular lattice whenever $B \subseteq \mathrm{Eq}^+ \mathfrak{A}$, and B is closed under ; and $\cdot$.

Let $\mathscr{L} = \langle L, \vee, \wedge \rangle$ be a lattice with minimum element $e \in L$. Then $\mathfrak{A}(L)$ is the algebra

$$\langle Sb\, L, \cup, \cap, \sim, \emptyset, L, ;, ^{\smile}, \{e\} \rangle$$

where $Sb\, L$ is the set of subsets of L, $\sim$ is complementation with respect to L, $X^{\smile} = X$ for all $X \subseteq L$, and

$$X ; Y = \{z \in L : z \vee x = z \vee y = x \vee y \text{ for some } x \in X,\ y \in Y\}$$

for all $X, Y \subseteq L$. Furthermore, let I be the function defined by

$$I(x) = \{y \in L : y \leq x\}$$

for every $x \in L$.

LEMMA. *Let $\mathscr{L}$ be a lattice with minimum element e.* Then $\mathrm{Eq}^+ \mathfrak{A}(\mathscr{L})$ *is the set of non-empty ideals of $\mathscr{L}$.*

Proof. Let X be a non-empty ideal of $\mathscr{L}$. Then $e \in X$, so $\{e\} \subseteq X$. Let $y \in X ; X$. Then $y \vee x = y \vee x' = x \vee x'$ for some $x, x' \in X$. But then $y \leq x \vee x' \in X$, so $X ; X \subseteq X$. Thus $X \in \mathrm{Eq}^+ \mathfrak{A}(\mathscr{L})$.

Now suppose $X \in \mathrm{Eq}^+ \mathfrak{A}(\mathscr{L})$. Then $\{e\} \subseteq X$, so $X \neq \emptyset$. If $y \leq x \in X$, then $y \vee x = y \vee x = x \vee x$, so $y \in X ; X \subseteq X$. If $x, y \in X$, then $(x \vee y) \vee x = (x \vee y) \vee y = x \vee y$, so $x \vee y \in X ; X \subseteq X$. Thus X is an ideal.

THEOREM. *Let $\mathscr{L}$ be a modular lattice with minimum element. Then*
(1) *$\mathfrak{A}(\mathscr{L})$ is a symmetric relation algebra,*

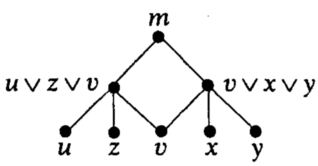
(2) *I is an embedding of $\mathscr{L}$ into* $\langle \mathrm{Eq}^+ \mathfrak{A}(L), ;, \cap \rangle$,
(3) $\langle \mathrm{Eq}^+ \mathfrak{A}(\mathscr{L}), ;, \cap \rangle$ *is the lattice of non-empty ideals of $\mathscr{L}$.*

Proof. (1). It obviously suffices to verify that $\mathfrak{A}(\mathscr{L})$ satisfies (b), (c), and (d). But ; is clearly commutative, and (c) and (d) are easy to check, so we only prove ; is associative.

Let X, Y, $Z \subseteq L$ and $u \in (X; Y); Z$. Then there are $v \in X; Y$ and $z \in Z$ such that

(i) $u \vee v = u \vee z = v \vee z$.
Since $v \in X; Y$, there are $x \in X$ and $y \in Y$ such that
(ii) $v \vee x = v \vee y = x \vee y$.
Let $m = u \vee z \vee v \vee x \vee y$. Then by (i) and (ii),

$$u \vee x \vee y = u \vee v \vee x \vee y = u \vee z \vee v \vee x \vee y = m,$$



and we see, similarly, that the join of any three elements in $\{u, z, v, x, y\}$, other than $u \vee z \vee v$ and $v \vee x \vee y$, is m. Let $w = (u \vee x) \wedge (y \vee z)$. By modularity, $w \vee y = (u \vee x \vee y) \wedge (y \vee z) = m \wedge (y \vee z) = y \vee z$, and $w \vee z = (u \vee x \vee z) \wedge (y \vee z) = m \wedge (y \vee z) = y \vee z$, so $w \in Y; Z$. Similarly, $u \vee x = u \vee w = x \vee w$, so $u \in X; (Y; Z)$. Thus $(X; Y); Z \subseteq X; (Y; Z)$ for all X, Y, $Z \subseteq L$. Since ; is commutative, it follows that ; is associative.

(2) For every $x \in L$, $I(x)$ is a non-empty ideal, so $I(x) \in \mathrm{Eq}^+ \mathfrak{A}(\mathscr{L})$ by the Lemma. It is easy to prove that I is one-to-one and $I(x \wedge y) = I(x) \cap I(y)$, so we only show $I(x \vee y) = I(x); I(y)$ for all x, $y \in L$.

Let $u \in I(x \vee y)$. Set $v = x \wedge (u \vee y) \in I(x)$ and $w = y \wedge (u \vee x) \in I(y)$. By modularity, $u \vee v = (u \vee x) \wedge (u \vee y) = u \vee w$. Using this, modularity, and $u \leq x \vee y$, we get $u \vee v = (x \vee y) \wedge (u \vee v) = (x \vee y) \wedge (u \vee x) \wedge (u \vee y) = (x \vee [y \wedge (u \vee x)]) \wedge (u \vee y) = [x \wedge (u \vee y)] \vee [y \wedge (u \vee x)] = v \vee w$. Hence $u \in I(x); I(y)$.

On the other hand, if $u \in I(x); I(y)$, then there are $v \in I(x)$ and $w \in I(y)$, such that $u \leq u \vee v = u \vee w = v \vee w \leq x \vee y$, so $u \in I(x \vee y)$.

(3). Since, by the Lemma, $\mathrm{Eq}^+ \mathfrak{A}(\mathscr{L})$ is the set of non-empty ideals of $\mathscr{L}$, and the meet of any two ideals is merely their intersection, it suffices to show that $X; Y$ is the join of any two ideals X and Y, that is, $X; Y = \{z \in L : z \leq x \vee y$ for $x \in X$ and $y \in Y\}$.

One inclusion is easy, for if $z \in X; Y$, then $z \leq z \vee x = z \vee y = x \vee y$ for some $x \in X$ and $y \in Y$. For the other inclusion, suppose $z \leq x \vee y$, $x \in X$, and $y \in Y$. Then $z \in I(x \vee y) = I(x); I(y)$ by (2), and $I(x) \subseteq X$ and $I(y) \subseteq Y$ since X and Y are ideals, so $z \in I(x); I(y) \subseteq X; Y$.

This theorem solves Jónsson's problem, for if $\mathscr{L}$ is a modular lattice with no minimum element, then the theorem may be applied instead to the lattice $\mathscr{L}'$ obtained by adding a minimum element to $\mathscr{L}$.

If $\mathfrak{A}(\mathscr{L})$ is representable, then there is an embedding F of $\mathfrak{A}(\mathscr{L})$ into some $\mathscr{S}\mathscr{E} E$, and $F \circ I$ is a representation of type 1 for $\mathscr{L}$. However, $F \circ I$ has some strong additional properties. For example, $F \circ I$ fails to preserve joins of chains. To see this, let $x_0 \leq x_1 \leq \cdots < y = \sum_{k<\omega} x_k$ where $x_k \neq y$ for all $k < \omega$. Then $FI(x_0) \subseteq FI(x_1) \subseteq \cdots \subseteq \bigcup_{k<\omega} FI(x_k) \subseteq FI(y)$. But $y \in I(y)$, so $F(\{y\}) \subseteq FI(y)$, and $F(\{y\}) \cap FI(x_k) = \emptyset$ for every $k < \omega$ since $y \notin I(x_k)$. Consequently $\bigcup_{k<\omega} FI(x_k) \neq FI(y)$. Further differences appear even for simple finite lattices. The two-element chain **2** has a representation of type 1 over a two-element set, but if $\mathfrak{A}(\mathbf{2})$ can be embedded in $\mathscr{S}\mathscr{E} E$, then the field of E must have at least three elements.

PROBLEM. Is $\mathfrak{A}(\mathscr{L})$ representable whenever $\mathscr{L}$ has a representation of type 1?

If $\mathscr{L}$ is the lattice of subspaces of a non-Desarguesian projective plane, then $\mathfrak{A}(\mathscr{L})$ is not representable. This is a somewhat roundabout way of constructing non-representable relation algebras. For more direct constructions see [2], [7], [8], [9], and [6]. The latter paper is of particular interest here. In it, Lyndon constructs a relation algebra using an arbitrary projective geometry, and proves that the resulting algebra is representable iff the geometry can be embedded as a hyperplane in a geometry with one more dimension. He consequently obtains non-representable relation algebras from lines which cannot be embedded in a projective plane. Lyndon uses the points of the geometry as atoms, and adds one extraneous element to serve as the atom 1'. The construction presented here uses all the subspaces of the geometry as atoms, with $\{\emptyset\}$ as the atom 1'. In particular, if $\mathscr{L}$ is the lattice of subspaces of a line, then $\mathfrak{A}(\mathscr{L})$ has one more atom than Lyndon's algebra for that line, namely an atom for the line itself. It turns out that $\mathfrak{A}(\mathscr{L})$ is representable, although Lyndon's algebra may not be.

REFERENCES

[1] L. H. CHIN and A. TARSKI, *Distributive and modular laws in the arithmetic of relation algebras.* University of California Publications in Mathematics, new series, vol. 1, no. 9 (1951), pp. 341–384.

[2] S. D. COMER, *Integral relation algebras via pseudogroups.* Notices Amer. Math. Soc., vol 23 (1976), p. A-659.

[3] P. CRAWLEY and R. P. DILWORTH, *Algebraic theory of lattices.* Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973.

[4] B. JÓNSSON, *Representation of modular lattices and relation algebras.* Trans. Amer. Math. Soc., vol. 92 (1959), pp. 449–464.

[5] R. C. LYNDON, *The representation of relational algebras.* Ann. of Math., ser. 2, vol. 51 (1950), pp. 707–729.

[6] R. C. LYNDON, *Relation algebras and projective geometries.* Michigan Math. J., vol. 8 (1961), pp. 21–28.

[7] R. D. MADDUX, *Some non-representable relation algebras.* Notices Amer. Math. Soc., vol. 23 (1976), pp. A-431, A-557.

[8] R. MCKENZIE, *Representations of integral relation algebras.* Michigan Math. J., vol. 17 (1970), pp. 279–287.

[9] U. WOSTNER, *Finite relation algebras.* Notices Amer. Math. Soc., vol. 23 (1976), p. A-482.

Iowa State University
Ames, Iowa
U.S.A.