Algebra Universalis, 12 (1981) 70–75 0002-5240/81/001070-06$01.50+0.20/0

Subdirectly irreducible pseudocomplemented de Morgan algebras

ANNA ROMANOWSKA

1. Introduction

The purpose of this note is to give a characterisation of finite subdirectly irreducible pseudocomplemented de Morgan algebras. Such an algebra L has trivial or four-element center and the intersection of the set of dense elements in L and the set of dual dense elements in L is one-element or is a simple de Morgan algebra.

The class of all pseudocomplemented de Morgan algebras is equational and is generated by its finite members. Nevertheless, the above description does not hold in the infinite case, which show quoted example.

Interesting examples of pseudocomplemented de Morgan algebras are Lukasiewicz and Post algebras. They are Stone algebras (see [7]) and de Morgan algebras satisfying the condition $x \wedge x' \leq y \vee y'$ for every x, y in such an algebra (see [1]). Moreover, it is possible to characterise Post algebras by some Stone and de Morgan algebras. It is easy to check with help of the Theorem 3.11 in [7], that the following conditions are equivalent:

(i) L is a Post algebra of order n;
(ii) L is a Stone algebra of order n and a de Morgan algebra satisfying $x \wedge x' \leq y \vee y'$ for every x, y in L;
(iii) L is a dual Stone algebra of order n and a de Morgan algebra satisfying $x \wedge x' \leq y \vee y'$ for every x, y in L.

2. Preliminaries

An algebra $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ is called a p-algebra if $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded lattice such that for every $a \in L$, the element $a^* \in L$ is the pseudocomplement of a; i.e. $x \leq a^*$ iff $a \wedge x = 0$. An algebra $\langle L; \vee, \wedge, {}^*, {}^+, 0, 1\rangle$ is

Presented by G. Grätzer. Received September 20, 1976. Accepted for publication in final form July 30, 1979.

called a double p-algebra (briefly dp-algebra) if $\langle L; \vee, \wedge, {}^*, 0, 1\rangle$ is a p-algebra and $\langle L; \vee, \wedge, {}^+, 0, 1\rangle$ is a dual p-algebra; i.e. $x \geq a^+$ iff $a \vee x = 1$. A dp-algebra is called distributive if it is distributive as a lattice; a distributive dp-algebra will be referred to as a ddp-algebra. For the rules of computations and other properties of p-algebras and distributive lattices we refer to [1].

Let L be a ddp-algebra. We let $D^*(L) = \{x \in L\colon x^* = 0\}$; such elements are called dense. Also, $D^+(L) = \{x \in L\colon x^+ = 1\}$. The sets $\{x \in L\colon x^* = x^+\}$, $\{x \in L\colon X^{+*} = x\}$ both coincide with the center $C(L)$ of L.

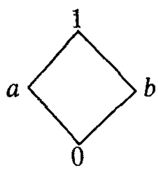
A de Morgan algebra is an algebra of the form $\langle L; \vee, \wedge, ', 0, 1\rangle$, where $\langle L; \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice and $'$ is a unary operation on L satisfying

(M_1) $x'' = x$

and

(M_2) $(x \wedge y)' = x' \vee y'$.

The mapping $x \mapsto x'$ is an antiautomorphism and an involution of L[1]. We will call de Morgan algebras M-algebras. Let M_0 denote the two-element M-algebra, M_1 the three-element M-algebra and M_2 the four-element M-algebra



with $a = a'$ and $b = b'$.

LEMMA 1. ([2], [6]) *A M-algebra L is subdirectly irreducible iff L is isomorphic to M_0, M_1 or M_2.*

Note that M_0, M_1 and M_2 are simple. A pseudocomplemented de Morgan algebra will be called a pM-algebra. Let L be a pM-algebra. The lattice of all congruences on L is denoted by $K(L)$. It is known that the congruence lattice of a distributive lattice is distributive, hence the congruence lattice of any pM-algebra is distributive. If $a, b \in L$, then $\theta(a, b)$, $\theta_{lat}(a, b)$, $\theta^{*+}(a, b)$, $\theta'(a, b)$ denotes the smallest congruences that identify the elements a and b respectively on the following algebras: pM-algebra L, the lattice L, dp-algebra L, M-algebra L. ω

---

[1] Conversely, if $\varphi: L \to L$ is an involution of the bounded lattice L which is at the same time an antiautomorphism of L, then L can be made into a de Morgan algebra by defining $x' = \varphi(x)$ for $x \in L$.

denotes the least and i the greatest element of $K(L)$. a/b is defined to be $\{x \in L: b \leq x \leq a\}$; $a \succ b$ means a covers b.

3. Finite subdirectly irreducibles

LEMMA 2. *Let L be a pM-algebra. If $a' = a^*$, then L is isomorphic to $a/0 \times a'/0$.*

Proof. If $a' = a^*$, then $a' = a^* \in C(L)$. It is easy to see that $a/0$ and $a^*/0$ form pM-algebras, where for $x \leqq a$, $x^* = x^* \wedge a$ and $x' = x' \wedge a$ holds and for $x \leqq a^*$, similarly. Hence $L = a/0 \times a'/0$.

LEMMA 3. *Every pM-algebra is a dp-algebra.*

Proof. By self-duality of L, it is easy to check that $x'^{*\prime}$ is a dual pseudocomplement of x, i.e. $x^+ = x'^{*\prime}$.

Note, that each congruence on L is a $^{*+}$-congruence on L. By Lemma 3, $C(L) = \{x \in L: x^{*\prime} = x'^{*}\}$ for any pM-algebra L. If $a \in C(L)$, then $a^* = a''^{*} = a'^{*\prime}$; hence $a' \in C(L)$.

COROLLARY 4. ([5]) *Let L be a pM-algebra and $a, b \in D^*(L) \cap D^+(L)$. Then $\theta_{lat}(a, b) = \theta^{*+}(a, b)$.*

LEMMA 5. *Let L be a subdirectly irreducible pM-algebra and let $D^*(L) \cap D^+(L)$ have the greatest element p and the least element q. Then $a = a'$ for each element $a \in D^*(L) \cap D^+(L)$ such that $a \neq p, q$.*

Proof. Let $a \in D^*(L) \cap D^+(L)$. Then $a' \in D^*(L) \cap D^+(L)$ too. By Corollary 4, if a is not comparable with a', then $\theta(a, a \vee a') \wedge \theta(a', a \vee a') = \theta'(a, a \vee a') \wedge \theta'(a', a \vee a') = \omega$ (see Lemma 5, p. 213 in [1]). If $a < a'$, then $\theta(q, a) \wedge \theta(a, a') = \theta'(q, a) \wedge \theta'(a, a') = (\theta_{lat}(q, a) \vee \theta_{lat}(a', p)) \wedge \theta_{lat}(a, a') = (\theta_{lat}(q, a) \wedge \theta_{lat}(a, a')) \vee (\theta_{lat}(a', p) \wedge \theta_{lat}(a, a')) = \omega$, because of Corollary 4 and Lemma 4, p. 213 in [1], which shows that $\theta'(x, y) = \theta_{lat}(x, y) \vee \theta_{lat}(y', x')$ for $x, y \in L$ and $x \leq y$ (the join is taken in the congruence lattice of the lattice L). The case $a' < a$ is similar to the previous. Hence $a = a'$.

LEMMA 6. *Let L be a finite subdirectly irreducible pM-algebra. Let q be the join of the atoms of L and p the meet of the dual atoms. Then $p \geqslant q$.*

Proof. Let $x = \bigvee \{a \in L: a \succ 0 \text{ and } a \not\leq p\}$, $y = \bigwedge\{b \in L: 1 \succ b \text{ and } b \not\geq q\}$ and $x_1 = \bigvee\{a \in L: a \succ 0 \text{ and } a \leq p\}$. Then $x \vee y = x \vee x_1 \vee y = 1$ and $x \wedge y = 0$. By selfduality of L, $y' = y^* = x$, which is impossible by Lemma 2. Hence $p \geq q$.

THEOREM 7. ([5]) *Let L be a finite ddp-algebra. Then L is subdirectly irreducible iff $C(L) = \{0, 1\}$ and $1 \leq |D^*(L) \cap D^+(L)| \leq 2$.*

THEOREM 8. *Let L be a finite pM-algebra with $|L| > 4$. Then the following are equivalent:*

(1) *L is subdirectly irreducible;*
(2) *Either*

(i) *$C(L) = \{0, 1\}$ and $|D^*(L) \cap D^+(L)| = 1$ or $D^*(L) \cap D^+(L)$ is a simple M-algebra,*

or

(ii) *$C(L) = \{0, 1, a = a', a^*\}$ and $|D^*(L) \cap D^+(L)| = 1$ or $D^*(L) \cap D^+(L)$ is a four-element simple M-algebra.*

L is simple iff $|D^*(L) \cap D^+(L)| = 1$, and $C(L)$ is as described in (2).

Proof. (1)$\Rightarrow$(2). Let L be subdirectly irreducible pM-algebra with $|L| > 4$. Let q be the join of the atoms and p the meet of the dual atoms of L. By Lemma 6, $p \geq q$. This implies that $D^*(L) \cap D^+(L) = p/q \neq \emptyset$. Suppose $|D^*(L) \cap D^+(L)| > 4$. Then there exist elements a, b, c in $D^*(L) \cap D^+(L)$, different from p and q. By Lemma 5, $a = a'$, $b = b'$, $c = c'$. Again by Lemma 5 and distributivity of L at least two elements among a, b, c must be equal. Hence $|D^*(L) \cap D^+(L)| \leq 4$. If $|D^*(L) \cap D^+(L)| = 4$, then, by Lemma 5, $D^*(L) \cap D^+(L)$ is isomorphic to M_2.

Suppose $a \in C(L)$ and $a \neq 0, 1$. Then $a^{*\prime} = a^{\prime *}$ and by Lemma 2, $a^* \neq a'$. Hence $(a^* \vee a')^* = a^{**} \wedge a'^* = a \wedge a^{*\prime} = (a' \vee a^*)'$ and $a^* \vee a' \in C(L)$. It follows from Lemma 2 and the hypothesis of the theorem that $a^* \vee a' = 1$. Substituting a' for a, we get $a'^* \vee a = 1$. Therefore, since $a \in C(L)$, $a^* \wedge a' = 0$. This implies $a = a' = a^{**}$. Then since $C(L)$ is a Boolean algebra and $'$ restricted to $C(L)$ is a dual isomorphism, one has $C(L) = \{0, 1\}$ or $C(L) = \{0, 1, a = a', a^*\}$.

If $C(L) = \{0, 1, a = a', a^*\}$, then the ddp-algebra L is isomorphic to $a^*/0 \times a/0$. Since $1 \leq |D^*(L) \cap D^+(L)| \leq 4$ and $D^*(L) = D^*(a^*/0) \times D^*(a/0)$, $D^+(L) = D^+(a^*/0) \times D^+(a/0)$, then $|D^*(L) \cap D^+(L)| = 1$ or 4.

For to prove the converse, we need some definition and lemmas. Let L_1, L_2 be algebras of the same type, φ_1 a congruence on L_1 and φ_2 a congruence on L_2. Then the congruence $\varphi_1 \times \varphi_2$ on $L_1 \times L_2$ is defined by $(a, b) \equiv (c, d)(\varphi_1 \times \varphi_2)$ iff $a \equiv c(\varphi_1)$ and $b \equiv d(\varphi_2)$. Congruences of this form we call p-congruences.

LEMMA 9. ([4]) *If K is the congruence distributive equational class, then each congruence on a finite product of algebras from K is a p-congruence.*

COROLLARY 10. *Each congruence on a finite product of ddp-algebras is a p-congruence.*

LEMMA 11. *Let L be subdirectly irreducible ddp-algebra and L^d its dual. Then $L \times L^d$ can be made into a de Morgan algebra, which as a pM-algebra is subdirectly irreducible. Moreover, if L is simple pM-algebra, then $L \times L^d$ is simple pM-algebra, if no, then $L \times L^d$ is subdirectly irreducible, not simple pM-algebra.*

Proof. Let L be subdirectly irreducible *ddp*-algebra and L^d its dual. The de Morgan operation on $L \times L^d$ is clearly $(x, y) \mapsto (y, x)$. Every *pM*-congruence on $L \times L^d$ is a *ddp*-algebra congruence on $L \times L^d$ and so by Corollary 10, it is of the form $\theta_1 \times \theta_2$, where θ_1 and θ_2 are congruences on L. The de Morgan operation shows that $\theta_1 > \omega$ iff $\theta_2 > \omega$. Hence, if θ is the least nontrivial *ddp*-congruence on L, $\theta \times \theta$ is the least nontrivial *pM*-congruence on $L \times L^d$ and therefore, if L is subdirectly irreducible (simple) *ddp*-algebra, $L \times L^d$ is subdirectly irreducible (resp. simple) *pM*-algebra.

Note that, if L is finite, then $\theta \times \theta$ is the unique nontrivial congruence on *pM*-algebra $L \times L^d$.

Proof of Theorem 8. (2)$\Rightarrow$(1). Let the condition (2i) be satisfied. Assuming $C(L) = \{0, 1\}$, take $c > d$ in L and let φ be *pM*-congruence generated by (c, d). By considering $FD(\mathbf{2}+\mathbf{2})$, if $(q \vee c) \wedge p > q \vee (d \wedge p)$, then as p/q is a simple *M*-algebra, $\theta(p, q) = \theta((q \vee c) \wedge p,\ q \vee (d \wedge p)) \leq \varphi$.

If $(q \vee c) \wedge p = q \vee (d \wedge p)$ (in particular if $p = q$) then either $q \wedge d < q \wedge c$ or $p \vee d < p \vee c$. Since $q/0$ and $1/p$ are Boolean lattices we must have in either case (use $'$ in the second case), an atom $a > 0$ with $\theta(0, a) \leq \varphi$.

Since a is an atom, $a \leq a^{*+}$ and we obtain an increasing sequence $a_0 \leq a_1 \leq \cdots a_n \leq a_{n+1} \cdots \leq 1$ with $a_0 = a$, $a_{n+1} = a_n^{*+}$. This gives $(0, a_n) \in \theta(0, a) \leq \varphi$ for all $n < \omega$ and, since L is finite, $a_n = a_{n+1}$ for some n. Since $C(L) = \{0, 1\}$, $a_n = a_{n+1} < 1$ is impossible. Therefore we have $i = \theta(0, a) \leq \varphi$ and L is simple.

Now let the condition (2ii) be satisfied. In this case the lattices $a/0$ and $a^*/0$ are antiisomorphic (the antiisomorphism is given by $x \mapsto x' \wedge a^*$). $a/0$ does not contain different from a and 0 element of L complemented in $a/0$, since $|C(L)| = 4$. Similarly, $C(a^*/0) = \{0, a^*\}$. Moreover $1 \leq |D^*(a/0) \cap D^+(a/0)| = |D^*(a^*/0) \cap D^+(a^*/0)| \leq 2$. Hence (see Theorem 7), $a/0$ and $a^*/0$ are subdirectly irreducible *ddp*-algebras. By Lemma 11, L is subdirectly irreducible *pM*-algebra.

Theorem 8 does not hold for infinite *pM*-algebras, which shows the following.

EXAMPLE. Let L be infinite, subdirectly irreducible ddp-algebra, that is not simple and has $D^*(L) \cap D^+(L) = \emptyset$. (Examples of such algebras are given in [3].) By Lemma 11, the pM-algebra $L \times L^d$ with the operation $'$ defined as in the proof of the Lemma 11, is subdirectly irreducible, not simple and $D^*(L \times L^d) \cap D^+(L \times L^d) = \emptyset$. If $|D^*(L) \cap D^+(L)| = 1$, then $L \times L^d$ is subdirectly irreducible, but not simple.

REFERENCES

[1] R. BALBES and P. DWINGER, *Distributive lattices*, Univ. of Missouri Press (1974).
[2] A. BIALYNICKI-BIRULA, *Remarks on Quasi-Boolean algebras*, Bull. de l'acad. Polon. des Sc., Ser. Sci. Math. Astron. Phys. *6* (1957), 615–619.
[3] B. A. DAVEY, *Subdirectly irreducible distributive double p-algebras*, Algebra Univ. *8* (1978), 73–88.
[4] G. FRASER and A. HORN, *Congruence relations in direct products*, Proc. Amer. Math. Soc. *26* (1970), 390–394.
[5] R. FREESE and A. ROMANOWSKA, *Subdirectly irreducible modular double p-algebras*, Houston J. Math. *3* (1977), 109–112.
[6] J. KALMAN, *Lattices with involution*, Trans. Amer. Math. Soc. *87* (1958), 485–491.
[7] T. KATRIŇÁK and A. MITSCHKE, *Stonesche Verbände der Ordnung n und Postalgebren*, Math. Ann. *199* (1972), 13–30.
[8] T. KATRIŇÁK, *The structure of distributive double p-algebras. Regularity and congruences*, Algebra Univ. *3* (1973), 238–246.

Warsaw Technical University
Warsaw
Poland