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Modal operators on Heyting algebras

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§0. Background

DEFINITION 0.1. Let L be a $\wedge$-semi lattice. A modal operator on L is a function $f: L \to L$ such that, $\forall x, y \in L$,

(1) $x \leq f(x)$ (f is inflationary)
(2) $ff(x) = f(x)$ (f is idempotent)
(3) $f(x \wedge y) = f(x) \wedge f(y)$ (f is $\wedge$-preserving).

The purpose of this paper is to investigate aspects of the general algebraic theory of modal operators on Heyting algebras where the theory is particularly interesting. It is based on part of the author's Ph.D. thesis [8].

Historically, the notion of a modal operator as such has its main source in the theory of topoi and sheafification due to Grothendieck, Lawvere and Tierney; see [4], [6], [7], [11] for details. Modal operators are also referred to as j-operators because j was the symbol commonly used to denote them. The term modal is due to Lawvere.

From a quite different starting point, Dowker and Papert [2] developed the concept of a frame and of a frame map. These were intended to mirror algebraically the notion of a topology and of maps which preserved the open set operations – finite intersections and arbitrary unions. An examination of [2], however, shows that a frame map on a frame is simply a modal operator on a complete Heyting algebra.

Modal operators have also occasionally appeared in general lattice theory under the name of multiplicative closure operators.

The paper is organised as follows. §1 contains the elementary properties of modal operators including an alternative characterisation in terms of a single identity. §2 introduces the algebra $M(H)$ of all modal operators on a Heyting algebra H, and the concepts of admissible filter and modal subalgebra. In §3 an

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important subclass of modal operators is investigated in detail. §4 analyses admissible filters and modal subalgebras in terms of a Galois connexion. The final two sections relate the structure of $M(H)$ to separation properties on H.

An elementary knowledge of lattice theory and in particular Heyting algebras is assumed. We use $\Rightarrow$ to denote the implication or relative pseudocomplement on a Heyting algebra.

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§1. Introduction

We recall that a Heyting algebra $(H, \wedge, \vee, 0, 1, \Rightarrow)$ is a lattice with least element 0 and greatest element 1 such that, for all $a, b, x \in H$, $x \wedge a \leq b$ if and only if $x \leq a \Rightarrow b$. Equivalently, $a \Rightarrow b$ is the greatest element of $\{x \in H : x \wedge a \leq b\}$. A Heyting algebra is necessarily distributive. For an arbitrary 0,1-distributive lattice L the pseudocomplement of $b \in L$ is defined to be the greatest element of $\{x \in L : x \wedge b = 0\}$, if it exists. If every element of L has a pseudocomplement, L is called a pseudocomplemented lattice. The pseudocomplement of b is denoted by b^*. Every Heyting algebra is a pseudocomplemented lattice in which $b^* = b \Rightarrow 0$. Lemma 1.1 lists the standard properties of Heyting algebras which we require.

LEMMA 1.1. *Let H be any Heyting algebra. Then the following hold, for all elements a, b, c of H.*

(i) $b \leq a \Rightarrow b$,

(ii) $a \wedge (a \Rightarrow b) = a \wedge b$,

(iii) $a \leq b \Rightarrow c$ *iff* $b \leq a \Rightarrow c$,

(iv) $b \leq c$ *implies* $a \Rightarrow b \leq a \Rightarrow c$,

(v) $a \leq (a \Rightarrow b) \Rightarrow b$,

(vi) $a \leq b$ *implies* $b \Rightarrow c \leq a \Rightarrow c$,

(vii) $(a \Rightarrow b) \Rightarrow (a \Rightarrow c) = a \Rightarrow (b \Rightarrow c) = b \Rightarrow (a \Rightarrow c) = (a \wedge b) \Rightarrow c$,

(viii) $(a \Rightarrow b) \wedge (b \Rightarrow c) \leq a \Rightarrow c$,

(ix) $a \leq b$ *iff* $a \Rightarrow b = 1$,

(x) $a \Rightarrow (b \wedge c) = (a \Rightarrow b) \wedge (a \Rightarrow c)$,

(xi) $(a \vee b) \Rightarrow c = (a \Rightarrow c) \wedge (b \Rightarrow c)$.

COROLLARY 1

(i) $a \Rightarrow b^* = (a \wedge b)^*$, (ii) $a \leq b$ *implies* $b^* \leq a^*$,

(iii) $a \leq b$ *implies* $a^{**} \leq b^{**}$, (iv) $a \leq a^{**}$,

(v) $a^* = a^{***}$, (vi) $a \Rightarrow a^* = a^*$,
(vii) $a^* \Rightarrow a = a^{**}$, (viii) $(a \vee b)^* = a^* \wedge b^*$.

(*Properties* (ii)–(v), (viii) *of the Corollary hold in any pseudocomplemented lattice.*)

$\{x \in H : x = x^{**}\}$ is called the set of regular elements of H. Note that an element $b \in H$ is called complemented iff $b \vee b^* = 1$, in which case b^* is its complement. Complemented elements are sometimes called boolean. If $a \in H$ is such that $a^* = 0$, then a is called a dense element of H. For further details, see [5] or [9].

Modal operators were defined in §0. Theorem 1.2 gives their basic algebraic properties.

THEOREM 1.2. *Let f be a modal operator on a Heyting algebra H. Then the following hold, where x, y are any elements of H.*

(i) $f(x \vee y) = f(x \vee f(y)) = f(f(x) \vee f(y))$,
(ii) $f(x \Rightarrow y) \leq f(x) \Rightarrow f(y) = f(f(x) \Rightarrow f(y)) = x \Rightarrow f(y) = f(x \Rightarrow f(y))$,
(iii) $x \vee f(0) \leq f(x) \leq (x \Rightarrow f(0)) \Rightarrow f(0)$,
(iv) $(x \vee f(0))^{**} = f(x)^{**}$.

Proof outline

(i) $f(x \vee y) \leq f(x \vee f(y)) \leq f(f(x) \vee f(y)) \leq ff(x \vee y) = f(x \vee y)$
(ii) $f(x) \wedge f(x \Rightarrow y) = f(x \wedge (x \Rightarrow y)) = f(x \wedge y) \leq f(y)$.

Hence $f(f(x) \Rightarrow f(y)) \leq ff(x) \Rightarrow ff(y) = f(x) \Rightarrow f(y)$

$\leq x \Rightarrow f(y) \leq f(x \Rightarrow f(y)) \leq f(x) \Rightarrow f(y)$.

Hence $f(x \Rightarrow f(y)) = f(f(x) \Rightarrow f(y)) = f(x) \Rightarrow f(y) = x \Rightarrow f(y)$.

(iii) Clearly, $x \vee f(0) \leq f(x)$.

Also, $f(x) \leq (f(x) \Rightarrow f(0)) \Rightarrow f(0) = (x \Rightarrow f(0)) \Rightarrow f(0)$, by (ii).

(iv) $x^* \leq x \Rightarrow f(0) = f(x) \Rightarrow f(0)$.

Hence $f(x) \wedge x^* \wedge f(0)^* = 0$.
Hence $f(x) \wedge (x \vee f(0))^* = 0$.
Hence $f(x) \leq (x \vee f(0))^{**}$.
We thus have $(x \vee f(0)) \leq f(x) \leq (x \vee f(0))^{**}$ and hence $(x \vee f(0))^{**} = f(x)^{**}$.

For any $b \in H$, the operator u_b defined by $u_b(x) = b \vee x$ is easily shown to be modal. The following Corollary to Theorem 1.2 shows that for boolean algebras these are the only modal operators.

COROLLARY 1. *If f is a modal operator on a boolean algebra B, then $f = u_{f(0)}$.*

Proof. Immediate from (iv) of the Theorem.

We will denote by 0 and 1 the operators u_0 and u_1.

For Heyting algebras there is a useful alternative characterisation of modal operators, (which improves a result of Freyd that a closure operator f on a Heyting algebra is modal iff it satisfies $x \Rightarrow f(y) = f(x) \Rightarrow f(y)$).

THEOREM 1.3. *The following are equivalent for any function f on a Heyting algebra H.*

(1) *f is a modal operator on H.*

(2) *For all x, y in H, $x \Rightarrow f(y) = f(x) \Rightarrow f(y)$.*

Proof. (1)→(2). This is contained in (ii) of Theorem 1.2.

(2)→(1). On putting (i) $y = x$, (ii) $x \leq y$, (iii) $x = f(y)$, it is easily seen that f is (i) inflationary, (ii) monotone, (iii) idempotent. To show that f preserves $\wedge$, observe first that, since f is inflationary, $x \wedge y \leq f(x \wedge y)$. Hence

$$y \leq x \Rightarrow f(x \wedge y) = f(x) \Rightarrow f(x \wedge y),$$

so that

$$y \wedge f(x) \leq f(x \wedge y).$$

Similarly,

$$f(y) \wedge f(x) \leq f(x \wedge y).$$

Since f is monotone, it follows that f preserves $\wedge$ and is thus a modal operator on H.

Using Theorem 1.3 we can introduce two further classes of modal operators on an arbitrary Heyting algebra H.

LEMMA 1.4. *For any $a \in H$, let $v_a, w_a : H \to H$ be defined by $v_a(x) = a \Rightarrow x$, $w_a(x) = (x \Rightarrow a) \Rightarrow a$. Then v_a, w_a are modal operators on H.*

Proof. It is straightforward to verify that v_a, w_a satisfy the criterion of Theorem 1.3. (A direct proof for w_a may be found in [10].)

§2. The algebra of operators $M(H)$

We denote by $M(H)$ the set of all modal operators on a Heyting algebra H. For general H, $M(H)$ is an $\wedge$-semi lattice (with $\wedge$ defined pointwise), but need neither be pseudocomplemented nor a lattice. However, if H is complete, then, for any set I, $\bigwedge\{f_i \in M(H): i \in I\}$, (again defined pointwise), exists and is modal, so that $M(H)$ is a complete lattice. Because the meet in $M(H)$ is pointwise so also is the order so that $f \leq g$ if and only if $f(x) \leq g(x)$ for all x in H. We will denote the join in $M(H)$ by $\sqcup$. It is almost never obtained pointwise. Since $f_1f_2, f_2f_1 \leq f_1 \sqcup f_2$, it follows that if f_1f_2 is modal, then $f_1f_2 = f_1 \sqcup f_2$ and similarly for f_2f_1. Also, if $f_1f_2 \leq f_2f_1$, then f_2f_1 is modal. We thus obtain the next Lemma.

LEMMA 2.1. *Let H be a Heyting algebra and let $f \in M(H)$. Then*
(i) $u_a \sqcup f = fu_a$,
(ii) $v_a \sqcup f = v_af$,
(iii) *if $g \in M(H)$ and g preserves $\Rightarrow$, then $g \sqcup f = gf$.*

Proof. (i) holds since $u_af \leq fu_a$.
(ii) follows from (iii) since v_a preserves $\Rightarrow$, by Lemma 1.1(vii).
(iii) may be verified by showing that gf satisfies the criterion of Theorem 1.3.

We leave until §3 the behaviour of the w_a operators with respect of joins.

Using Lemma 2.1, we can obtain more information about the structure of $M(H)$ when H is complete. For $f_1, f_2 \in M(H)$, let $f = \bigwedge\{v_{f_1(a)} \sqcup u_a \sqcup f_2 : a \in H\}$; i.e., by Lemma 2.1,

$$f(x) = \bigwedge\{f_1(a) \Rightarrow f_2(x \vee a) : a \in H\}.$$

LEMMA 2.2. *For $g \in M(H)$, $g \wedge f_1 \leq f_2$ if and only if $g \leq f$.*

Proof. For any $x \in H$,

$$\begin{aligned}(f \wedge f_1)(x) &= f(x) \wedge f_1(x)\\ &\leq f_1(x) \wedge (v_{f_1(x)} \sqcup u_x \sqcup f_2)(x)\\ &= f_1(x) \wedge (f_1(x) \Rightarrow f_2(x))\\ &= f_1(x) \wedge f_2(x) \leq f_2(x).\end{aligned}$$

Hence $f \wedge f_1 \leq f_2$; i.e. $g \leq f$ implies $g \wedge f_1 \leq f_2$. Now suppose that $g \wedge f_1 \leq f_2$. We show that, $\forall a \in H$, $g \leq v_{f_1(a)} \sqcup u_a \sqcup f_2$ which is equivalent to showing that, $\forall x \in H$, $g(x) \leq f_1(a) \Rightarrow f_2(a \vee x)$. Thus we have to show that, $\forall a, x \in H$, $g(x) \wedge f_1(a) \leq f_2(a \vee x)$. Now, for $x \leq a$,

$$\begin{aligned} g(x) \wedge f_1(a) &\leq g(a) \wedge f_1(a) \\ &\leq f_2(a), \quad \text{since} \quad g \wedge f_1 \leq f_2 \\ &= f_2(a \vee x). \end{aligned}$$

Hence, for any $x, a \in H$,

$$\begin{aligned} g(x) \wedge f_1(a) &\leq g(x) \wedge f_1(a \vee x) \\ &\leq f_2(a \vee x \vee x) \\ &= f_2(a \vee x). \end{aligned}$$

(Lemma 2.2 is essentially the same as Lemma 2 in [2], but the proof is different.)

THEOREM 2.3. *If H is a complete Heyting algebra, then $M(H)$ is also a complete Heyting algebra.*

If we denote the implication in $M(H)$ also by $\Rightarrow$, then $f_1 \Rightarrow f_2$ is given by the formula of Lemma 2.2. Rarely is $f_1 \Rightarrow f_2$ obtained pointwise. A description of how the implication in $M(H)$ can be obtained from the pointwise function

$$(f_1 \dot{\Rightarrow} f_2)(x) = f_1(x) \Rightarrow f_2(x)$$

is given in [8] and may also be found in [1]. Briefly,

$$f_1 \Rightarrow f_2 = \bigwedge \{w_a : f_1(a) \Rightarrow f_2(a) = a\}.$$

By no means any complete Heyting algebra can occur as $M(H)$ for suitable complete H as the next Theorem and Corollary show.

THEOREM 2.4. *If H is a complete Heyting algebra and $f \in M(H)$, then $f = \bigsqcup \{u_{f(a)} \wedge v_a : a \in H\}$.*

Proof. Let $g = \bigsqcup \{u_{f(a)} \wedge v_a : a \in H\}$.

Then, for each $x \in H$, $g \geq u_{f(x)} \wedge v_x$ so that $g(x) \geq f(x)$. Hence $g \geq f$.
Also, for each $a \in H$,

$$\begin{aligned} v_a f(x) = a \Rightarrow f(x) &= f(a) \Rightarrow f(x) \\ &= v_{f(a)} f(x), \end{aligned}$$

so that

$$v_a f = v_{f(a)} f.$$

Hence $v_a \leq v_{f(a)} f = v_{f(a)} \sqcup f$, by Lemma 2.1.
Hence $u_{f(a)} \wedge v_a \leq f$.
It follows that $g \leq f$ and hence $g = f$.

A zero-dimensional lattice is one in which every element is a supremum of complemented elements; i.e. the complemented elements form a base for the lattice.

A *regular* lattice is a lattice L with 0 and 1 such that given any $a, b \in L$ with $a < b$, there exist $x, y \in L$ satisfying

$$x \not\leq a, \qquad b \vee y = 1, \qquad x \wedge y = 0.$$

COROLLARY 1. *For any complete Heyting algebra H,*
(i) *$M(H)$ is a zero-dimensional lattice,*
(ii) *$M(H)$ is a regular lattice.*

Proof. (i) By Lemma 2.1, u_a and v_a are complements. Hence every $f \in M(H)$ is a supremum of complemented elements; i.e., $M(H)$ is zero-dimensional.

(ii) If L is a zero-dimensional lattice and $a < b$ then there exists a complemented element x such that $x \leq b$, $x \not\leq a$. Now put $y = x^*$. Then $b \vee y = 1$ and $x \wedge y = 0$. Hence L is a regular lattice.

COROLLARY 2. *If H is finite, then $M(H)$ is a boolean algebra.* (There is a direct proof of this Corollary, giving additional information, in [8].)

The characterisation of modal operators in Theorem 2.4 requires the algebra to be complete. It also requires the use of the awkward join operation $\sqcup$. There is, however, a more useful characterisation in terms of w_a operators which holds for any Heyting algebra and which requires only the simpler meet operation.

THEOREM 2.5. *Let H be any Heyting algebra and let $f \in M(H)$. Then*
(i) *for all $a \in H$, $f \leq w_a$ iff $f(a) = a$,*
(ii) *$\bigwedge \{w_a : f \leq w_a\}$ exists and equals f.*

Proof. (i) That $f \leq w_a$ implies $f(a) = a$ is straightforward. Conversely, for any

$a \in H$,

$$\begin{aligned} f(x) &\leq (f(x) \Rightarrow f(a)) \Rightarrow f(a) \\ &= (x \Rightarrow f(a)) \Rightarrow f(a) \\ &= (x \Rightarrow a) \Rightarrow a \\ &= w_a(x), \end{aligned}$$

so that $f(a) = a$ implies $f \leq w_a$.

(ii) This now follows on noting that, for any $x \in H$, $w_{f(x)}(x) = f(x)$, so that the infimum exists at each point x and equals $f(x)$ there.

For $f \in M(H)$, let $H_f = \{x \in H : f(x) = x\}$. Since f is idempotent, H_f is simply the image of H under f. Then Theorem 2.5 states that $f = \bigwedge \{w_a : a \in H_f\}$. Clearly H_f uniquely determines f. It is called the *fixed algebra* of f. We now have the following Corollary to Theorem 2.5.

COROLLARY 1. *Let X by any (non-empty) subset of a complete Heyting algebra H, and set $f = \bigwedge \{w_a : a \in X\}$. Then H_f is the least extension of X to the fixed algebra of a modal operator on H.*

Proof. It is clear from the proof of the Theorem that $X \subseteq H_f$. Suppose now that g is a modal operator on H such that $X \subseteq H_g$. Then, by the Theorem,

$$g = \bigwedge \{w_a : a \in H_g\} \leq \bigwedge \{w_a : a \in X\} = f.$$

Hence $gf = f$ giving $H_f \subseteq H_g$.

(Note that for any modal operators f, g we have $g \leq f$ iff $gf = f$ iff $H_f \subseteq H_g$.)

Theorem 1.2 shows that H_f is a Heyting algebra whose meet and implication are those induced from H, $f(0)$ being the zero of H_f, but whose join $\sqcup$ is given by $f(a) \sqcup f(b) = f(f(a) \vee f(b))$.

If X is a subset of H such that there exists $f \in M(H)$ with $X = H_f$, then we will call X a *modal subalgebra* of H. The following Lemma is proved in [8] and may also be found in [4], and in [2] for complete H.

LEMMA 2.6. *If $X \subseteq H$, then X is a modal subalgebra of H if and only if*
(i) *for each $a \in H$, $\bigwedge \{x \in X : a \leq x\}$ exists and lies in X,*
(ii) *for all $a \in H$ and $b \in X$, $a \Rightarrow b \in X$.*

For any $f \in M(H)$, the set $J_f = f^{-1}(1)$ is a filter on H. An arbitrary filter J on H will be called *admissible* provided there exists $f \in M(H)$ with $J = f^{-1}(1)$. For boolean algebras, it follows from Theorem 1.2, Corollary 1, that the only admissible filters are principal filters. Since $[a, 1] = v_a^{-1}(1)$, principal filters are admissible for any Heyting algebra. From Theorem 2.5 it follows that every admissible filter is an infimum of filters of form $W_a = w_a^{-1}(1)$.

If $f \leq g$ it is clear that $J_f \subseteq J_g$. The converse of this is, however, false. We do have the following result.

LEMMA 2.7. *Let J_1, J_2 be two admissible filters such that J_1 has a minimal associated modal operator j_1. Let j_2 be any modal operator whose filter is J_2. Then $J_1 \subseteq J_2$ implies $j_1 \leq j_2$.*

Proof. The modal operator $j_1 \wedge j_2$ has filter $J_1 \cap J_2 = J_1$. Hence $j_1 \wedge j_2 = j_1$ since j_1 is minimal. Hence $j_1 \leq j_2$.

COROLLARY 1. *If an admissible filter in a Heyting algebra has a minimal associated modal operator, then it has a least associated modal operator.*

Proof. Put $J_1 = J_2$ in the Lemma.

An admissible filter J may have many associated modal operators. The set $M_j = \{f \in M(H) : f^{-1}(1) = J\}$ will be called the *block* associated with J. If $f_1, f_2 \in M_J$, they are called *companions*. If M_J is a singleton, the corresponding operator is said to be *alone*. For a complete algebra H, each block in $M(H)$ has a least element. In general a block does not have a greatest element but for finite algebras we have the following result.

LEMMA 2.8. *Let J be an (admissible) filter in a finite algebra H. Let $\mathscr{C} = \{c : c \notin J$, and if $c < x$, then $x \in J\}$. Then $\bigwedge \{w_a : a \in \mathscr{C}\}$ is the greatest element in M_J.*

Lemma 2.8 can be extended to those Heyting algebras for which the ascending chain condition holds.

LEMMA 2.9. *Let H be an arbitrary Heyting algebra and let f be a modal operator on H which preserves $\Rightarrow$. Then f is the least operator in its block.*

Proof. Let g be any companion of f. Then, for all $x \in H$, $f(f(x) \Rightarrow x) = 1$ so that $g(f(x) \Rightarrow x) = 1$. Thus by Theorem 1.2(ii), $1 = g(f(x) \Rightarrow x) \leq gf(x) \Rightarrow g(x)$ so that $gf(x) \leq g(x)$. Hence $gf = g$ and so $f \leq g$.

COROLLARY 1. *v_a is the least operator associated with $[a, 1]$.*

The converse of Lemma 2.9 is false for general algebras but does hold in chains. We note that if H is complete then the least operator in M_J can be represented as $\bigsqcup\{v_a : a \in J\}$. An alternative description will be given in §4.

§3. The w_a operators

We now look at the behaviour of these operators in detail.

LEMMA 3.1. *Let H be any Heyting algebra, let $f \in M(H)$ and let b be any element of H.*

Then $w_b f = w_b u_{f(0)}$ if and only if $f(b) \wedge (f(0) \Rightarrow b) = b$.

Proof. $w_b f = w_b u_{f(0)}$
iff $w_b u_{f(0)} \geq w_b f$
iff $w_b u_{f(0)} \geq f$
iff $w_b \geq f \wedge v_{f(0)}$, using Lemma 2.1,
iff $b = f(b) \wedge (f(0) \Rightarrow b)$, by Theorem 2.5(i)
We now have the first main result of this section.

THEOREM 3.2. *Each w_a operator is the greatest in its block.*

Proof. Suppose first that f, w_a are companions with $u_a \leq f$. Then $a \leq f(0)$, so that $a \leq f(0) \leq f(a)$, and hence $f(a) \wedge (f(0) \Rightarrow a) = a$. Hence, by Lemma 3.1, $w_a u_{f(0)} = w_a f$. Hence

$$w_a u_{f(0)}(x) = 1 \quad \text{iff} \quad w_a f(x) = 1 \quad \text{iff} \quad w_a(x) = 1,$$

since w_a and f are companions. Now

$$\begin{aligned} w_a u_{f(0)}(f(0) \Rightarrow a) &= ((f(0) \Rightarrow a) \wedge ((f(0) \Rightarrow a) \Rightarrow a)) \Rightarrow a \\ &= a \Rightarrow a = 1. \end{aligned}$$

Hence $w_a(f(0) \Rightarrow a) = 1$ so that $f(0) \leq a$ and hence $f(0) = a$. Then $f(a) = ff(0) = f(0) = a$, giving $f \leq w_a$, by Theorem 2.5(i).

Now let f be any companion of w_a. Then so also is $f \sqcup u_a$ $(= fu_a)$ since if $f(a \vee x) = 1$, we have $w_a(x) = w_a(a \vee x) = 1$ and conversely.

Hence $f \leq f \sqcup u_a \leq w_a$.

COROLLARY 1. *w_0 (double-negation) is alone.*

Proof. w_0 preserves $\Rightarrow$ and hence by Lemma 2.9 is the least in its block.

While in general w_a does not preserve $\Rightarrow$, it does possess a useful weaker property which we now derive with the aid of a preliminary Lemma.

LEMMA 3.3. *Let H be any Heyting algebra. Suppose $b, c \in H$ with $b \geq c$. Then, for all $a \in H$, $(a \Rightarrow b) \Rightarrow c = ((a \Rightarrow c) \Rightarrow c) \wedge (b \Rightarrow c)$.*

Proof. Since $b \leq a \Rightarrow b$, we have $(a \Rightarrow b) \Rightarrow c \leq b \Rightarrow c$.
Since $b \geq c$, $a \Rightarrow c \leq a \Rightarrow b$ and hence $(a \Rightarrow b) \Rightarrow c \leq (a \Rightarrow c) \Rightarrow c$.
Hence $(a \Rightarrow b) \Rightarrow c \leq ((a \Rightarrow c) \Rightarrow c) \wedge (b \Rightarrow c)$.
Also,

$$\begin{aligned}(b \Rightarrow c) \Rightarrow ((a \Rightarrow b) \Rightarrow c) &= ((a \Rightarrow b) \wedge (b \Rightarrow c)) \Rightarrow c, \text{ by 1.1(vii)}\\ &\geq (a \Rightarrow c) \Rightarrow c, \text{ by 1.1(viii)}\end{aligned}$$

Hence $(b \Rightarrow c) \wedge ((a \Rightarrow c) \Rightarrow c) \leq (a \Rightarrow b) \Rightarrow c$.

THEOREM 3.4. *Let H be any Heyting algebra. Let $b \in H$. Then, for all $x, y \in H$ with $b \leq y$, $w_b(x \Rightarrow y) = w_b(x) \Rightarrow w_b(y)$.*

Proof. By Lemma 3.3,

$$\begin{aligned}w_b(x \Rightarrow y) &= (((x \Rightarrow b) \Rightarrow b) \wedge (y \Rightarrow b)) \Rightarrow b\\ &= ((x \Rightarrow b) \Rightarrow b) \Rightarrow ((y \Rightarrow b) \Rightarrow b)\\ &= w_b(x) \Rightarrow w_b(y).\end{aligned}$$

We can now obtain a formula for $w_a \sqcup f$.

LEMMA 3.5. *Let H be any Heyting algebra. Then, for all $a \in H$ and for all $f \in M(H)$, $w_a \sqcup f = w_a f u_a = w_a u_{f(a)}$.*

Proof. We use Theorem 1.3 to show that $w_a f u_a$ is modal. For all $x, y \in H$,

$$\begin{aligned}w_a f u_a(x) \Rightarrow w_a f u_a(y) &= w_a(f u_a(x) \Rightarrow f u_a(y)), \text{ by Theorem 3.4}\\ &= w_a(u_a(x) \Rightarrow f u_a(y))\\ &= w_a(x \Rightarrow f u_a(y))\\ &= w_a(x) \Rightarrow w_a f u_a(y)\\ &= x \Rightarrow w_a f u_a(y).\end{aligned}$$

The equality $w_a f u_a = w_a u_{f(a)}$ follows from Lemma 3.1 since if $g = f u_a = f \sqcup u_a$, then $g \in M(H)$, $g(0) = f(a)$, and $g(a) \wedge (g(0) \Rightarrow a) = a$.
Finally, $w_a \sqcup f = w_a f u_a$, since $w_a, f \leq w_a f u_a \leq w_a \sqcup f \sqcup u_a = w_a \sqcup f$.

LEMMA 3.6. *Let H be any Heyting algebra. Then, for all $a, b \in H$, $w_a u_b = w_{w_a(b)}$.*

Proof. For all $x \in H$,

$$\begin{aligned} w_{w_a(b)}(x) &= (x \Rightarrow ((b \Rightarrow a) \Rightarrow a)) \Rightarrow ((b \Rightarrow a) \Rightarrow a) \\ &= ((b \Rightarrow a) \wedge ((b \Rightarrow a) \Rightarrow (x \Rightarrow a))) \Rightarrow a \\ &= ((b \Rightarrow a) \wedge (x \Rightarrow a)) \Rightarrow a \\ &= ((b \vee x) \Rightarrow a) \Rightarrow a \\ &= w_a u_b(x). \end{aligned}$$

COROLLARY 1. $w_a \sqcup f = w_{w_a f(a)} = u_{f(a)} \sqcup w_a = (u_{f(u)} \wedge v_a) \sqcup w_a$.

COROLLARY 2. *The w_a operators form a final section in $M(H)$. Hence $w_a \leq f$ implies $f = w_{f(0)}$*

COROLLARY 3. $f \Rightarrow w_a = v_{f(a)} \sqcup w_a = w_{f(a) \Rightarrow a}$.

Proof. $f \Rightarrow w_a = (f \sqcup w_a) \Rightarrow w_a = (u_{f(a)} \sqcup w_a) \Rightarrow w_a = u_{f(a)} \Rightarrow w_a = v_{f(a)} \sqcup w_a$.

COROLLARY 4. $w_a^* \sqcup w_a^{**} = 1$; *i.e., the w_a operators are Stone elements of $M(H)$.*

Proof. By Lemma 1.1, Corollary (viii), $w_a^{**} = w_a^* \Rightarrow w_a = v_k \sqcup w_a$, where $k = w_a^*(a)$.

Then $w_a^{**} \sqcup w_a^* = v_k \sqcup w_a \sqcup w_a^* = v_k \sqcup w_a \sqcup u_k = 1$.

COROLLARY 5. $(f \Rightarrow w_a) \Rightarrow w_a = (v_{f(a)} \sqcup w_a) \Rightarrow w_a = v_{f(a)} \Rightarrow w_a = f \sqcup w_a$.

Theorem 3.4 can also be used in another direction.

LEMMA 3.7. *If f is a companion of w_a, then $f \sqcup u_a = w_a$.*

Proof. By Theorem 3.2, $f \sqcup u_a \leq w_a$.

By Theorem 3.4, for all $x \in H$, $w_a(x) \Rightarrow u_a(x) \in W_a = w_a^{-1}(1)$. Hence, for a given x, there exists $b \in W_a$ such that $w_a(x) \Rightarrow u_a(x) = b$.

Hence $w_a(x) \leq b \Rightarrow u_a(x) = (v_b \sqcup u_a)(x) \leq (f \sqcup u_a)(x)$, since $f(b) = 1$ so that $f \sqcup u_b = 1$ and hence $v_b \leq f$, since u_b and v_b are complements.

Hence, for all $x \in H$, $w_a(x) \leq (f \sqcup u_a)(x)$.

Hence $w_a \leq f \sqcup u_a$ and so $w_a = f \sqcup u_a$.

COROLLARY 1. *If* W_a *is the principal filter* $[b, 1]$, *then* $w_a = v_b \sqcup u_a$.

Finally, Lemma 3.5 can be used to give a characterisation of those complete algebras H for which $M(H)$ is a boolean algebra.

LEMMA 3.8. *If* w_a *is complemented, then* $w_a^*(a)$ *is the least element in* $W_a \cap [a, 1]$; *i.e.*, $W_a \cap [a, 1] = [w_a^*(a), 1]$.

Proof. Since $w_a \sqcup w_a^* = 1$, we have $w_a w_a^* u_a = 1$ so that $w_a w_a^*(a) = 1$ and hence $w_a^*(a) \in W_a \cap [a, 1]$.

Also $w_a \wedge w_a^* = 0$, so that if $w_a(x) = 1$, then $w_a^*(x) = x$.

Hence $w_a^*(a) \leq x$ when $a \leq x$ and $x \in W_a$.

THEOREM 3.9. *Let* H *be a complete Heyting algebra. Then* $M(H)$ *is a boolean algebra if and only if for each* $a \in H$, $W_a \cap [a, 1]$ *has a least element, namely* $w_a^*(a)$.

Proof. If $M(H)$ is a boolean algebra, then the result follows from Lemma 3.8.

Suppose then that $W_a \cap [a, 1]$ has a least element d_a.

Then $w_a \cap [a, 1] = v_{d_a}$ by Theorem 3.4 and Lemma 2.9.

Hence $w_a = w_a u_a = v_{d_a} u_a$ and hence w_a is complemented.

Further $w_a^*(a) = (u_{d_a} \wedge v_a)(a) = d_a va = d_a$.

Since $f = \bigwedge \{w_a : f \leq w_a\}$ for any $f \in M(H)$, it follows that $M(H)$ is a boolean algebra.

(There is a different proof of the first half of Theorem 3.9 in [1].)

The properties of w_a operators given in Theorem 3.4 and Lemma 3.6, Corollary 1 can be generalised as follows.

Let H be any Heyting algebra. (i) A modal operator f on H will be called *nice* if $f|[f(0), 1]$ preserves $\Rightarrow$.

(ii) An element g of a Heyting algebra H will be called a *gem* iff, for each $b \in H$, there exists a complemented element c_b such that $b \vee g = c_b \vee g$ and $c_b \leq b$.

We then have the following Theorems.

THEOREM 3.10. *The following hold for any nice modal operator* f *on a Heyting algebra* H.

(i) $\forall x, y \in H$ *with* $f(0) \leq y$, $f(x \Rightarrow y) = f(x) \Rightarrow f(y)$.

(ii) $\forall x \geq f(0)$, $f(f(x) \Rightarrow x) = 1$.

(iii) *If* k *is a companion of* f *and* $k \leq f$, *then* $f = k \sqcup u_{f(0)}$.

(iv) *For any* $g \in M(H)$, $f \sqcup g = fgu_{f(0)}$.

(v) *If* f *is maximal in its block, then* f *is the greatest in its block.*

(vi) f *is complemented iff* $\{x \geq f(0) : f(x) = 1\}$ *has a least element.*

[The concept of a nice operator and Theorem 3.10 were suggested to me by H. Simmons.]

THEOREM 3.11. *The following hold for any gem in a Heyting algebra H.*

(i) $b \Rightarrow g = c_b^* \vee g$,
(ii) $g^* \vee g^{**} = 1$,
(iii) $(b \Rightarrow g) \Rightarrow g = b \vee g$,
(iv) $b \vee (b \Rightarrow g) = 1$,
(v) $(g \Rightarrow b) \vee ((g \Rightarrow b) \Rightarrow b) = 1$,
(vi) $(b_1 \wedge b_2) \Rightarrow g = (b_1 \Rightarrow g) \vee (b_2 \Rightarrow g)$
(vii) $(g \Rightarrow b) \Rightarrow b = c_{g \Rightarrow b}^* \vee g \vee b$

It follows from Lemma 3.6, Corollary 1, and Theorem 2.5(ii) that $M(H)$ is infimum-generated by a final section of gems. This implies zero-dimensionality.

§4. Admissible filters and modal subalgebras

The condition given in §2 for a filter to be admissible is not in practice easy to use. In this section we obtain, for complete algebras, a different criterion which is of a more intrinsic nature. This approach also reveals an unexpected connection between admissible filters and modal subalgebras.

The key is the following relation R on a Heyting algebra H.

DEFINITION 4.1. For all $x, y \in H$, xRy if and only if $w_x(y) = 1$; i.e. $y \Rightarrow x = x$.
Given R we can construct the following operations on subsets of H.
For $X \subseteq H$, set

$$X^* = \{y \in H : \forall x \in X, xRy\}$$
$$= \{y \in H : \forall x \in X, w_x(y) = 1\}.$$

For $Y \subseteq H$, set

$$Y^0 = \{x \in H : \forall y \in Y, xRy\}$$
$$= \{x \in H : \forall y \in Y, w_x(y) = 1\}.$$

Then (R, X^*, Y^0) is a Galois connection on H.
Let $\bar{X} = X^{0*} = \{x \in H : w_b(x) = 1, \forall b$ such that for all $y \in H$, $w_b(y) = 1\}$.

LEMMA 4.1. *For any* $X, Y \subseteq H$,
(i) $X \subseteq \bar{X}$, (ii) $\bar{\bar{X}} = \bar{X}$, (iii) $X \subseteq Y$ *implies* $\bar{X} \subseteq \bar{Y}$.

LEMMA 4.2. *For any* $X \subseteq H$, $\bar{X}$ *is a filter on* H.

Proof. $\bar{X} = \bigcap\{W_a : a \in X^0\}$ and hence is a filter.

The next Theorem gives the relationship between the Galois connection and admissible filters.

THEOREM 4.3. *If* X *is a subset of a Heyting algebra* H *and if* J *is any admissible filter on* H *such that* $X \subseteq J$, *then* $\bar{X} \subseteq J$.

Proof. Let f be any modal operator in M_J and let $b \in H_f$, so that $b = f(b)$. Then, for all $x \in X$,

$$\begin{aligned} x \Rightarrow b &= x \Rightarrow f(b) \\ &= f(x) \Rightarrow f(b) \\ &= 1 \Rightarrow f(b), \quad \text{since} \quad X \subseteq J \\ &= b. \end{aligned}$$

Hence $b = f(b)$ implies that, $\forall x \in X$, $w_b(x) = 1$.
Hence $H_f \subseteq X^0$.
By Theorem 2.5, $J = \{x \in H : w_b(x) = 1, \ \forall b \in H_f\}$.
Since $\bar{X} = \{x \in H : w_b(x) = 1, \ \forall b \in X^0\}$, it follows that $\bar{X} \subseteq J$.

COROLLARY 1. *If* K *is an admissible filter on a Heyting algebra* H, *then* $K = \bar{K}$.

Proof. Set $J = X = K$ in the Theorem.

For complete algebras the converse of the above Corollary is also true.

THEOREM 4.4. *If* J *is a subset of a complete Heyting algebra such that* $J = \bar{J}$, *then* J *is an admissible filter on* H.

Proof. Let $f = \bigwedge \{w_b : b \in J^0\} = \bigwedge \{w_b : \forall y \in J, \ w_b(y) = 1\}$.
Then f is modal, and $x \in J$ implies that $f(x) = 1$.
Also, if $f(x) = 1$, then $w_b(x) = 1$, $\forall b \in J^0$. Hence $x \in \bar{J}$ and so $x \in J$.
Hence $J = f^{-1}(1)$ and is thus admissible.

COROLLARY 1. *If* X *is any subset of a complete Heyting algebra* H, *then* $\bar{X}$ *is the least admissible filter containing* X.

Proof. $\bar{X}$ is admissible by the Theorem and Lemma 4.1.
It is the least admissible filter containing X by Theorem 4.3.

COROLLARY 2. *The operator f defined in the Theorem is the least operator in* M_J.

Proof. Let $g \in M_J$. Then, as in the proof of Theorem 4.3, $H_g \subseteq J^0$. Now $f = \bigwedge \{w_b : b \in J^0\}$ and $g = \bigwedge \{w_b : b \in H_g\}$. Hence $f \leq g$.

If H is not complete, then $J = \bar{J}$ is not sufficient to imply that J is an admissible filter. This may be seen by considering the boolean algebra of all finite or cofinite sets of integers, and taking J to be the filter of all cofinite sets containing the even integers. Then $J = \bar{J}$, but J is not principal and thus is not admissible.

We now use the Galois connection to relate admissible filters to modal subalgebras.

THEOREM 4.5. *Let X by any non-empty subset of a complete Heyting algebra. Then,*

(i) X^* *is the filter of the modal operator associated with the least extension of X to a modal subalgebra of H,*

(ii) X^0 *is the fixed algebra of the least modal operator associated with the filter which is the least extension of X to an admissible filter.*

Proof. (i) X^* is the associated filter of $\bigwedge \{w_a : a \in X\}$ which, by Theorem 2.5, Corollary 1, is the modal operator of the least extension of X to a modal subalgebra of H.

(ii) The proofs of Theorems 4.3 and 4.4, Corollary 2, show that $\bigwedge \{w_a : a \in X^0\}$ is the least modal operator associated with the least extension of X to an admissible filter. We thus have only to show that X^0 is a modal subalgebra of H. We verify the conditions of Lemma 2.6.

(1) X^0 is $\wedge$-complete since if I is any set, and $\mathscr{C} = \{c_i \in H : j \in I\}$, then $y \Rightarrow \bigwedge_I \mathscr{C} = \bigwedge_I \{y \Rightarrow c_i : i \in I\}$, for any $y \in H$.

(2) Suppose that $x \in H$ and $a \in X^0$. Then, $\forall y \in X$, $y \Rightarrow (x \Rightarrow a) = x \Rightarrow (y \Rightarrow a) = x \Rightarrow a$, so that $x \Rightarrow a \in X^0$.

COROLLARY 1. *If X is a subset of a complete Heyting algebra H, then* $X = X^{*0}$ *iff X is the fixed algebra of a modal operator which is the least such operator associated with its filter.*

Proof. (a) Suppose $X = X^{*0}$. Then the result follows from (ii) of the Theorem.

(b) Suppose X is the fixed algebra of a least such operator. Then $X = J^0$ where J is the associated filter. But $J = X^*$ by (i) of the Theorem. Hence $X = X^{*0}$.

The Theorem and Corollary show that for those algebras in which each block is a singleton there is a precise duality between admissible filters and modal subalgebras. Such algebras will be identified in the next section.

§5. Separation properties of H and the block structure of $M(H)$

It is well-known that certain separation properties of topological spaces – for example, regularity or normality – are equivalent to algebraic properties of their open set lattices. For regularity the property is that given before the proof of the Corollary to Theorem 2.4. For an arbitrary Heyting algebra H three properties of this type are closely related to the structure of the blocks in $M(H)$. They are as follows.

S_{wu}: Given $a, b \in H$ with $a < b$, there exists $x \in H$ such that

$$w_a(x) \neq 1, \qquad u_b(x) = 1.$$

S_{uu}: Given $a, b \in H$ with $a < b$, there exists $x \in H$ such that

$$u_a(x) \neq 1, \qquad u_b(x) = 1.$$

S_{ww}: Given $a, b \in H$ with $a < b$, there exists $x \in H$ such that

$$w_a(x) \neq 1, \qquad w_b(x) = 1.$$

It is easy to see that the property S_{wu} implies each of the others and examples can be found to show that no other implications hold.

S_{wu} has the following equivalent formulations.

LEMMA 5.1. *The following are equivalent for any Heyting algebra H.*

(1) *H has property* S_{wu}.

(2) *For any* $a, b \in H$ *with* $a < b$, *there exists* $x \in H$ *such that*

$$x \leq a \quad \text{and} \quad b \vee (x \Rightarrow a) = 1.$$

(3) *For any* $a, b \in H$ *with* $a < b$, *there exists* $x, y \in H$ *such that*

$$x \nleq a, \qquad b \vee y = 1, \qquad x \wedge y \leq a.$$

Version (3) shows that every regular lattice is an S_{wu} lattice.

The property S_{uu} is the obvious lattice analogue of the T_1 separation axiom and appears as such (as the axiom T_1') in (3). However, it is strictly weaker than T_1. S_{uu} lattices are also called conjunctive lattices.

As noted at the end of §4 if every block of operators is a singleton then the structure of $M(H)$ has a simplicity not present in the general case. It turns out that the algebras with this property are exactly those with property S_{wu}.

THEOREM 5.2. *The following are equivalent for any Heyting algebra H.*

(1) *H has property* S_{wu}.

(2) *Every modal operator on H is alone.*

(3) *Every operator of the form* u_a *is the least in its block.*

Proof. (1)→(2). We show first that if f, g are modal operators such that there exists $x \in H$ with $f(x) < g(x)$, then f and g belong to different blocks, i.e. have different filters.

Suppose then that f, g are as described above. Then, since H has property S_{wu}, there exists $y \in H$ such that $y \Rightarrow f(x) \neq 1$, $g(x) \vee (y \Rightarrow f(x)) = 1$, using Lemma 5.1.

Let $z = y \Rightarrow f(x)$. Then $f(z) = z$, by Theorem 1.2, and hence $f(z) \neq 1$.

Also,

$$\begin{aligned} 1 &= g(g(x) \vee z) \\ &= g(x \vee z), \text{ by Theorem 1.2,} \\ &= g(z) \text{ since } x \leq z. \end{aligned}$$

Hence f, g are in different blocks.

Now suppose that J is any admissible filter and that $j, k \in M_J$. Then $j \wedge k \in M_J$ and hence, by the above, $j \wedge k = j$ and $j \wedge k = k$. Hence $j = k$. Hence every block is a singleton.

(2)→(3). Trivial.

(3)→(1). Suppose $a, b \in H$ with $a < b$. Then $w_a(0) < u_b(0)$. But u_b is the least in its block. Hence w_a, u_b belong to different blocks J_a and J_b. By Lemma 2.7, $J_b \not\subseteq J_a$.

Hence there exists $x \in H$ such that $w_a(x) \neq 1$, $u_b(x) = 1$.

Hence H has property S_{wu}.

Thus if H is the open set lattice of a T_3 topological space, then all modal operators on H are alone.

We now turn to the S_{uu} or conjunctive algebras.

THEOREM 5.3. *The following are equivalent for any Heyting algebra H.*

(1) *H is conjunctive.*

(2) *Every* u_a *operator is maximal in its block.*

(3) *The identity operator is alone.*

(4) *If a modal operator f has a pseudocomplement f^*, then f^* is the greatest in its block.*

(5) *Every u_a operator is the greatest in its block.*

(6) *Every v_a operator is alone.*

(7) *If f, g are companions and f^*, g^* exist, then $f^* = g^*$.*

Proof. We establish the equivalences as follows.

(1)→(2)→(3)→(4)→(5)→(1). (4)→(6)→(3). (3)↔(7).

(1)→(2). Let $a \in H$ and let f be any modal operator on H such that there exists $x \in H$ with $u_a(x) < f(x)$. Then, since H is conjunctive, there exists $y \in H$ such that $u_a(x) \vee y \neq 1$, $f(x) \vee y = 1$. Set $z = x \vee y$. Then $u_a(z) \neq 1$, but $f(z) = f(f(x) \vee y) = 1$. Hence f, u_a belong to different blocks. Hence u_a is maximal in its block.

(2)→(3). Since u_0 preserves $\Rightarrow$, it is the least in its block and hence is alone.

(3)→(4). Let f have filter J and let J_1 be any filter such that $J \cap J_1 = \{1\}$. Then if $f_1 \in M_{J_1}$ we have $f \wedge f_1 = 0$ and hence $f_1 \leq f^*$. Since $J \cap J^* = \{1\}$, where J^* is the filter of f^*, it follows that f^* is the greatest in its block.

(4)→(5). For any $a \in H$, $u_a = v_a^*$.

(5)→(1). Let $a, b \in H$ with $a < b$. Let u_a, u_b have filters J_a, J_b. Then $J_a \subset J_b$ but $J_a \neq J_b$. Hence $\exists x \in H$ such that $u_a(x) \neq 1$, $u_b(x) = 1$. Hence H is a conjunctive algebra.

(4)→(6). For any $a \in H$, $v_a = u_a^*$ and hence each v_a is the greatest in its block. Hence by Lemma 2.9, Corollary 1, each v_a is alone.

(6)→(3). Since v_1 is the identity operator, this is trivial.

(3)→(7). Suppose $f, g \in M_J$ and that f^*, g^* exist. Let J_1 be any admissible filter such that $J \cap J_1 = \{1\}$ and let $j \in M_{J_1}$.

Then $j \wedge f = j \wedge g = 0$ so that $j \leq f^*$, $j \leq g^*$. By taking J_1 to be either the filter of f^* or the filter of g^* we obtain $f^* \leq g^*$ and $g^* \leq f^*$. Hence $f^* = g^*$.

(7)→(3). Let f be a companion of 0. Then $f^* = 0^* = 1$.

Hence $f^{**} = 0$ and thus $f = 0$. Hence the identity operator is alone.

We note that in a complete conjunctive algebra, if $f \in M_J$, then $f^* = \bigwedge \{u_a : a \in J\}$.

From Theorems 5.2, 5.3 we see that the S_{wu} and S_{uu} algebras may be characterised as those for which (i) all u_a operators, (ii) all v_a operators are alone. Our third class of algebras is that for which all w_a operators are alone.

THEOREM 5.4. *The following are equivalent for a Heyting algebra H.*

(1) *H has property S_{ww}.*

(2) *Every w_a operator is alone.*

Proof. (1)→(2). By Theorem 3.2, every w_a operator is the greatest in its block. We have thus to show that the w_a operators are the least in their blocks.

Let a be any element of H and let f be any modal operator on H such that $f < w_a$. Then there exists $x \in H$ such that $f(x) < w_a(x)$.

Hence there exists $y \in H$ such that $w_{f(x)}(y) \neq 1$, $w_{w_a(x)}(y) = 1$.

By Lemma 3.6, $w_{w_a(x)}(y) = w_a u_x(y) = w_a(x \vee y)$.

Set $z = x \vee y$. Then $w_a(z) = 1$.

Also,

$$f(z) = f(x \vee y) = fu_x(y) \leq w_{fu_x(0)}(y)$$
$$= w_{f(x)}(b) \neq 1.$$

Hence f and w_a are in different blocks.

Hence w_a is minimal in its block. Hence by Lemma 2.7, Corollary 1, w_a is the least in its block.

(2)→(1). Let a, b be any elements of H with $a < b$. Then $w_a(0) < w_b(0)$. If J_a, J_b are the filters of w_a, w_b, then by Lemma 2.7, $J_b \not\subseteq J_a$.

Hence there exists $x \in H$ such that $w_a(x) \neq 1$, $w_b(x) = 1$. Hence H has property S_{ww}.

The following table summarises some of the above information.

	Max.	Min.	Alone
u_a	S_{uu}	S_{wu}	S_{wu}
v_a	S_{uu}	all	S_{uu}
w_a	all	S_{ww}	S_{ww}

§6. Modal operators on open set lattices

When H is the open set lattice $O(S)$ of a topological space S (and is thus a complete Heyting algebra in which $U \Rightarrow V = (U' \cup V)^0$, where ^{0} is the interior operation), we can define a fourth class of modal operators as follows. For any $A \subseteq S$, $X \in O(S)$, set $I_A(X) = (A \cup X)^0$. Then I_A is a modal operator on $O(S)$. For A open, $I_A = u_A$, and for A closed, $I_A = v_{A'}$.

Our first result shows the importance of the I_A operators.

LEMMA 6.1. *Every regular modal operator in the open set lattice $O(S)$ of a topological space S is of the form I_K for some subset K of S.*

Proof. Let f be any modal operator on $O(S)$.

By Lemma 2.2, if $X \in O(S)$, then

$$f^*(X) = \bigwedge \{f(A) \Rightarrow (A \cup X) : A \in O(S)\}.$$

Now, for any $A, X \in O(S)$, $f(A) \Rightarrow (A \cup X) = (B \cup X)^0$, where $B = f(A)' \cup A$. Let $\mathscr{C} = \{B : B = f(A)' \cup A$ for some $A \in O(S)\}$. Then, for all $X \in O(S)$,

$$\begin{aligned} f^*(X) &= \bigwedge \{(B \cup X)^0 : B \in \mathscr{C}\}. \\ &= (\bigcap \{(B \cup X)^0 : B \in \mathscr{C}\})^0 \\ &= (\bigcap \{(B \cup X) : B \in \mathscr{C}\})^0 \\ &= (X \cup \bigcap \{B : B \in \mathscr{C}\})^0 \\ &= (K \cup X)^0, \text{ where } K = \bigcap \{B : B \in \mathscr{C}\}. \end{aligned}$$

Hence $f^* = I_K$.

It is not in general the case that every I_A operator on an open set lattice is regular, nor is it in general the case that for an operator I_A, we have $I_A^* = I_{A'}$.

To investigate the I_A operators further, we introduce an auxiliary topology for a given topological space.

DEFINITION. For a given topological space S, let $\beta = \{U \cap V : U \in O(S), V \in C(S)\}$, where $C(S)$ is the set of closed sets of S. Then we define S^* to be the space with the same points as S and with β as a basis for the open sets.

The connection between this * topology and the I_A operators is described in the next Lemma.

LEMMA 6.2. *For a given topological space S and subset A of S, let $A^* = \bigcap \{(A \cup V)^{0\prime} \cup V : V \in O(S)\}$, so that $I_A^* = I_{A^*}$. Let $\XBox$ denote the interior operation in S^*. Then $A^* = A^{\XBox\prime}$.*

Proof. For any point $p \in S$,

$p \notin A^*$

iff $\exists V \in O(S)$ such that $p \notin (A \cup V)^{0\prime} \cup V$
iff $\exists V \in O(S)$ such that $p \in (A \cup V)^0$ and $p \in V'$
iff $\exists U, V \in O(S)$ such that $p \in U \subseteq A \cup V$ and $p \in V'$
iff $\exists U \in O(S)$, $C \in C(S)$ such that $p \in U \subseteq A \cup C'$ and $p \in C$
iff $\exists U \in O(S)$, $C \in C(S)$ such that $p \in U \cap C \subseteq a$
iff $p \in A^{\XBox}$.
Hence $A^{*\prime} = A^{\XBox}$ and hence $A^* = A^{\XBox\prime}$.

LEMMA 6.3. *If S^* is a discrete space, then, for any subset A of S, $I_A^* = I_{A'}$ and I_A is a regular operator.*

Proof. S^* is discrete if and only if $A^{\texttt{x}} \simeq A$ for all subsets A of S. Clearly, $I_A^{**} = I_{A'}^* = I_A$.

It is easy to see that S^* is a discrete space if and only if every point of S is an intersection of an open set and a closed set, which is one of the definitions of a T_D space. Other definitions are that $\{p\}^- - \{p\}$ is closed for any point p, or that $\{p\} \cup \{p\}'^0$ is open for any point p.

LEMMA 6.4. *The following are equivalent for any topological space S.*

(1) *S is a T_D space.*

(2) *S^* is a discrete space.*

We now have the main result on I_A operators.

THEOREM 6.5. *The following are equivalent for a topological space S.*

(1) *S is a T_D space.*

(2) *For any subset A of S, $I_A^* = I_{A'}$.*

(3) *If f is the map $P(S) \to M(O(S))$ defined by $f(A) = I_A$, $A \in P(S)$, then f preserves pseudocomplements.*

(4) *f is one-to-one.*

(5) *f is a boolean isomorphism between the boolean power set algebra $P(S)$ and the boolean algebra of regular elements of $M(O(S))$.*

Proof. (1)$\to$(2). This follows from Lemmas 6.3 and 6.4. (3) is a restatement of (2). We show that (2)$\to$(4) and (4)$\to$(1).

(2)$\to$(4). Suppose $I_A = I_B$. Then $I_A^* = I_B^*$ and hence $I_{A'} = I_{B'}$.

Now $I_C \wedge I_D = I_{C \cap D}$ for any sets C, D.

Hence $I_{A \cap B'} = I_A \wedge I_{B'} = I_A \wedge I_{A'} = 0$.

Hence $I_{A' \cup B} = 1$. It follows that $A' \cup B = S$ and hence $A \subseteq B$. Similarly, $B \subseteq A$. Hence $A = B$.

Hence f is one-to-one.

(4)$\to$(1). Let p be any point of S. Since f is one-to-one, $\exists X \in O(S)$ such that $(\{p\} \cup X)^0 \neq (\phi \cup X)^0 = X$. Hence $\exists X \in O(S)$ such that $p \notin X$ and $X \cup \{p\}$ is open.

Hence $X \subseteq (S - \{p\})^0$ and $X \cup \{p\}$ is open.

Hence $(S - \{p\})^0 \cup \{p\}$ is open.

Hence S is a T_D space.

(2)$\to$(5). By Lemma 6.1, f is onto the regular algebra of $M(O(S))$. Since f

preserves pseudocomplements and intersections, it preserves unions. Hence f is the required boolean isomorphism.

(5)→(4). Trivial.

We collect without proof some additional facts in the next Lemma.

LEMMA 6.6. *Let S be a topological space and let A, B be any subsets of S. Then,*

(i) *$A^{\text{¤}}$ is the least member of $\{X \in P(S) : I_X = I_A\}$,*

(ii) *$I_A = I_B$ iff $A^{\text{¤}} = B^{\text{¤}}$,*

(iii) *If $p \in S$ and $I_{\{p\}} \neq 0$, then $I^*_{\{p\}} = I_{S-\{p\}}$,*

(iv) *$I^*_{S-\{p\}} = I_{\bar{p}}$, where $\bar{p} = \{p\}^- \cap \langle p\rangle$, and $\langle p\rangle = \bigcap\{X \in O(S) : p \in X\}$,*

(v) *if S is a T_0 space, then, $\forall p \in S$, $I_{\{p\}}$ is a regular operator,*

(vi) *S is a conjunctive space if and only if for any $f \in M(O(S))$ with filter J, $f^* = I_B$, where $B = \bigcap\{A : A \in J\}$.*

While all I_A operators are regular in T_D spaces, the converse of this is false – consider, for example, any space with a finite topology. We now determine necessary and sufficient conditions on a space S in order that all its I_A operators are regular. We first look at the associated space S^*.

LEMMA 6.7. *The following are equivalent for a topological space S.*

(1) *Every I_A operator on S is regular.*

(2) *$O(S^*)$ is a boolean algebra.*

Proof. (1)→(2). By Lemma 6.6(ii), $I_A^{**} = I_A$ iff $A^{\text{¤}\prime\text{¤}\prime\text{¤}} = A^{\text{¤}}$.

Hence if A is open in S^*, then $A = A^{c\text{¤}}$, where ^{c} denotes closure in S^*, so that A is regular. Hence all open sets in S^* are regular and hence $O(S^*)$ is a boolean algebra.

(2)→(1). If $O(S^*)$ is a boolean algebra then every closed set is open in S^* and hence $A^{\text{¤}\prime\text{¤}\prime\text{¤}} = A^{\text{¤}\prime\prime\text{¤}} = A^{\text{¤}}$ for any set A, so that $I_A^{**} = I_A$.

We are thus left with the task of finding an elementary condition on S equivalent to the booleanness of $O(S^*)$.

LEMMA 6.8. *For any subset of a topological space S,*

$$I_A = \bigwedge \{I_{S-\{p\}} : p \notin A\}.$$

Proof

$$\begin{aligned}\forall X \in O(S),\ I_A(X) &= \left(\bigcap\{S-\{p\} : p \notin A\} \cup X\right)^0 \\ &= \left(\bigcap\{(S-\{p\}) \cup X : p \notin A\}\right)^0 \\ &= \left(\bigcap\{((S-\{p\}) \cup X)^0 : p \notin A\}\right)^0 \\ &= \bigwedge \{I_{S-\{p\}}(X) : p \notin A\}.\end{aligned}$$

COROLLARY 1. *In a topological space S, all I_A operators are regular if and only if all operators of the form $I_{S-\{p\}}$ are regular.*

THEOREM 6.9. *The following are equivalent for a topological space S.*
(1) *For every subset A of S, I_A is a regular operator.*
(2) *For all $p \in S$, $p \in \langle p \rangle^{¤}$, where $\langle p \rangle = \bigcap\{X \in O(S) : p \in X\}$.*

Proof. By Lemma 6.6(ii), (iv), $I_{S-\{p\}}$ is regular iff

$$(S-\{p\})^{¤} = \bar{p}^{¤'¤}.$$

Now

$$\begin{aligned}\bar{p}^{¤'¤} &= (\langle p \rangle \cap \{p\}^-)^{¤'¤}\\ &= (\langle p \rangle^{¤} \cap \{p\}^-)'^{¤}\\ &= (\langle p \rangle^{¤'} \cup (S-\{p\})^0)^{¤} \supseteq (S-\{p\})^{¤}.\end{aligned}$$

(1)→(2). From the above, $\forall p \in S$, $(\langle p \rangle^{¤'} \cup (S-\{p\})^0)^{¤} = (S-\{p\})^{¤}$.
Hence $\langle p \rangle^{¤'¤} \cup (S-\{p\})^{¤} = (S-\{p\})^{¤}$ so that $\langle p \rangle^{¤'¤} \subseteq (S-\{p\})^{¤}$. Hence $p \notin \langle p \rangle^{¤'¤}$.
By Lemma 6.7, all closed sets in S^* are open.
Hence $\langle p \rangle^{¤'¤} = \langle p \rangle^{¤'}$. Hence $p \notin \langle p \rangle^{¤'}$ and so $p \in \langle p \rangle^{¤}$.
(2)→(1). If $p \notin \langle p \rangle^{¤'}$, then $p \notin (\langle p \rangle^{¤'} \cup (S-\{p\})^0)^{¤}$.
Hence $(\langle p \rangle^{¤'} \cup (S-\{p\})^0)^{¤} \subseteq (S-\{p\})^{¤}$ and thus $(\langle p \rangle^{¤'} \cup (S-\{p\})^0)^{¤} = (S-\{p\})^{¤}$.
Hence $I_{S-\{p\}}$ is regular.
Hence, $\forall p \in S$, $I_{S-\{p\}}$ is regular.
It now follows from Lemma 6.8 that all I_A operators are regular.

If $p \in \langle p \rangle^{¤}$, then there exists open U and closed C in S such that $p \in U \cap C \subseteq \langle p \rangle$ and clearly we may take $C = \{p\}^-$. Hence the condition $p \in \langle p \rangle^{¤}$ is equivalent to the following.

There exists $U \in O(S)$, $p \in V$ such that, for any $W \in O(S)$ with $p \in W$, we have $V \cap \{p\}' \subseteq W$.

DEFINITION. A topological space S will be called a T_M space if and only if for any point $p \in S$, there exists an open neighbourhood V of p such that for any open neighbourhood W of p, we have $V \cap \{p\}^- \subseteq W$.

THEOREM 6.10. *The following are equivalent for a topological space S.*
(1) *S is a T_M space.*
(2) *Every I_A operator on S is regular.*

T_M would appear to be new. It is weaker than either T_D or R_0 and is unrelated to T_0. Since any finite topology is T_M it is not really appropriate to call T_M a separation axiom.

Final remarks

A major problem is the determination of the structure of $M(H)$ in terms of the structure of H. For example, when is $M(H)$ compact? If $M(M(H))$ is compact, then $M(H)$, (and hence $M(M(H))$), are finite boolean algebras which suggests that $M(M(H))$ has a yet tighter structure than $M(H)$. If we define $M^{n+1}(H) = M(M^n(H))$ and $M^0(H) = H$, when is $M^n(H)$ a boolean algebra? Other related questions obviously suggest themselves. It is our view that their solution should lead to a better understanding of the structure of Heyting algebras in general.

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