Algebra Universalis, 13 (1981) 171–201 0002-5240/81/002171-31$01.50+0.20/0

Residually small varieties of semigroups

RALPH MCKENZIE[1]

Although much is known about the residual character of varieties of algebras, interesting unsolved questions can still be formulated. A question posed by Robert Quackenbush in 1971 (in [7]) has received more attention with each passing year. Quackenbush asked whether there exists any finite universal algebra A such that the variety it generates, $V(A)$, is residually finite but not residually $<n$ for any positive integer n. In this paper, a hypothetical algebra with these properties will be called a Quackenbush algebra. Instead of writing "for some positive integer n the variety $\mathcal{V}$ is residually $<n$" we shall write "$\mathcal{V}$ is residually $\ll\omega$." All the concepts used in this introduction will be defined in Section 0.

Our paper is a response to Quackenbush's question, as are the papers [1], [2], [4], and [9]. We prove here that no Quackenbush algebra is to be found in the variety of semigroups. That it cannot be a group was proved in [9]. That in fact it cannot belong to any variety all of whose algebras have modular congruence lattices will be proved in [4]. (Taylor [9] proved a slightly weaker result earlier.)

Quackenbush's question has now been answered in two extreme cases. In the one extreme we have varieties with very well-behaved (modular) congruence lattices, for example, the variety of groups and the variety of rings. At the other extreme lies the variety of semigroups. It is crucial to our arguments in this paper that if a semigroup is not a group, and in fact is not very close to being a rather well-structured union of groups, then it generates a variety in which the congruence relations are so ill-behaved that large subdirectly irreducible algebras are easily constructed. (The generated variety is not even residually small.)

To be much more precise, our main result, whose proof occupies the bulk of the paper, can be formulated as follows (a condensation of Theorems 1, 4, and 5). Any subdirectly irreducible semigroup that generates a residually small variety and has more than three elements either (1) is a group of finite exponent; or (2) is obtained from such a group by adding a zero element; or (3) can be constructed as the disjoint union of a subgroup G of finite exponent and a zero sub-semigroup

[1] Research supported by U.S. National Science Foundation grant number MCS 77-22913.
Presented by W. Taylor. Received October 22, 1979. Accepted for publication in final form June 2, 1980.

U, where, (say) $G \cdot U = U$ and $U \cdot G = (0)$ (or reversely), and the products $G \cdot U$ are defined by a faithful representation of G as a group of permutations of $U-(0)$. From this we achieve a reduction of any question concerning the residual character of a residually small semigroup variety $\mathcal{V}$ to two questions regarding a certain variety $G(\mathcal{V})$ of groups of bounded exponent. The one question concerns the subdirectly irreducible groups in $G(\mathcal{V})$, while the other has to do with the groups $G \in G(\mathcal{V})$ whose lattice of subgroups has a proper strictly meet irreducible member H which contains no non-trivial normal subgroup of G.

The question whether $V(A)$ is residually small (A is an arbitrary semigroup) is similarly reduced, in Theorem 29 of Section 4, to questions about the variety generated by a certain group correlated with A.

Two final remarks to finish the introduction: The results for groups and rings (and more widely, algebras in any congruence modular variety) which were obtained in response to Quackenbush's question are theoretically stronger than what we have been able to prove about semigroups. Any congruence modular variety $V(A)$ generated by a finite algebra A must be residually $\ll\omega$ if it is even residually small. (If A is a group, this happens just in case all the Sylow subgroups of A are abelian – see [4].) We have so far only been able to prove that any variety $V(S)$ generated by a finite semigroup S, which is not residually $\ll\omega$ must contain at least one infinite subdirectly irreducible semigroup. (See the open questions in §6, particularly question 2.)

The second remark is that Quackenbush's original question is still open. There is only one published example of a locally finite, residually finite variety of universal algebras that is not residually $\ll\omega$. This is in Baldwin–Berman [2]; the variety has infinitely many basic operations and, of course, is not generated by any single finite algebra.

Added in April 1980

This manuscript was written in April 1979. The author was blissfully unaware at that time that anyone else was working on the same problems. The referee has now brought to my attention the work of Golubov–Sapir ("Varieties of finitely approximable semigroups"; see Soviet Math. Doklady *20* (1979), no 6, pp. 828–832). Detailed proofs of their results have not yet appeared. From their abstract, however, I draw the following conclusions. They have obtained a characterization of residually small, finitely generated semigroup varieties, in terms of identities, and also of generators. It looks as though one can make a direct translation between their results and those of this paper, in the finitely generated case. The difference between our work and theirs apparently is that we concentrate on the subdirectly irreducible semigroups and do not require the varieties to be finitely generated.

§0. Orientation

An *algebra* is a system consisting of a non-empty set and a list of finitary operations over that set. A *variety* is a class of similar algebras closed under the formation of subalgebras, homomorphic images, and direct products (of any number of factors). For more detail on these basic concepts and any others used in this paper we refer the reader to Grätzer's book [5], especially its appendix on varieties written by Walter Taylor.

A *semigroup* is an algebra with just one basic operation, a binary operation which satisfies the associative identity.

According to a fundamental result of Garrett Birkhoff, a class of algebras is a variety iff it can be defined, like the class of groups and the class of semigroups, as the class of all algebras (similar in type of operations to a given algebra) satisfying a certain set of identities.

A *congruence relation* of an algebra A is a subalgebra of ${}^2A$ that is also an equivalence relation over A. The congruence relations of A form a lattice Con A. With any $R \in \operatorname{Con} A$ we can form the factor algebra A/R and the natural homomorphism π_R from A onto A/R. Here $R = \{(x, y) \in {}^2A : \pi_R(x) = \pi_R(y)\}$; moreover, any set $R \subseteq {}^2A$ is a congruence relation of A iff it is the kernel, in this sense, of some homomorphism with domain A.

An algebra A is called *subdirectly irreducible* (*s.i.* for short) if its congruence lattice has an atom which is a subset of every congruence relation except the identity relation. The importance of subdirectly irreducible algebras hinges on a fact observed by Birkhoff: every algebra A is isomorphic to a subalgebra of a direct product of s.i. homomorphic images of A (which belong to any variety to which A belongs). Thus if a variety possesses only a few s.i. algebras, then all of its members can be found as subalgebras of products of those few.

We use small Greek letters like κ, λ, μ to denote cardinal numbers, and Roman letters like $i, j, k, \ldots, n$ to denote nonnegative integers (finite cardinal numbers). The least infinite cardinal number is $\omega = \{0, 1, 2, 3, \ldots\}$.

A variety is called *residually* $<\kappa$ (here κ is any cardinal, possibly finite) if every s.i. algebra in it is of cardinality less than κ. We shall use in this paper the abbreviation "res$<\kappa$" for this concept. Then "res $\leq \kappa$" is the equivalent of "res $<\kappa^+$" where κ^+ is the cardinal successor of κ. A variety $\mathcal{V}$ is called *res. small* if $\mathcal{V}$ is res $<\kappa$ for some κ. $\mathcal{V}$ is called *res. finite* if it is res $<\omega$, and called *res* $\ll \kappa$ if $\mathcal{V}$ is res $<\lambda$ for some $\lambda < \kappa$. $\mathcal{V}$ is *locally finite* iff every finitely generated algebra in

$\mathcal{V}$ is finite. The smallest variety containing an algebra A is $V(A) = \mathrm{HSP}(A)$, the class of homomorphic images of subalgebras of direct powers of A.

A direct power of A will be denoted ${}^I A$, or ${}^\kappa A$ – the universe of this algebra is the set of all functions from the set I (or, respectively, the set κ) into the universe of A. Exponents on the right will be used notationally for other purposes. For instance, as $|A|$ will denote the cardinality of the universe of the algebra A, we will write $|{}^\kappa A| = |A|^\kappa$ implicitly determining λ^k as notation for cardinal exponentiation. (Take the above as definition of λ^κ.) Also we use B^m, where B is a subset of a semigroup, to denote the set of all m-fold products of members of B.

§1. Definitions and main results

The central results stated in this section, and proved in Sections 2 and 3, are Theorems 1, 4, and 5.

The operation of a semigroup will be written as multiplication. Thus $x \cdot y$, and sometimes xy, stand for the product of x and y, and x^n denotes the n-th power of x. A semigroup will be called a *group of exponent* $n(n > 1)$ if it satisfies the laws $x^{n+1} \approx x$, $x^n \approx y^n$; a semigroup $A = \langle A, \cdot \rangle$ satisfying these laws becomes a group $\langle A, \cdot, {}^{-1} \rangle$ if we define $x^{-1} = x^{n-1}$. A semigroup will be called a *semigroup of exponent* n if it satisfies the law $x^n \approx x^{2n}$.

My search for the properties of a semigroup which would force it to generate a residually small variety led me to progressively stronger necessary conditions in the form of identities. Finally it became possible to combine these conditions into a simple statement. For each integer $n > 1$, I define three varieties $\mathcal{V}_j^{(n)} (j = 1, 2, 3)$.

DEFINITION E. (1) $\mathcal{V}_1^{(n)}$ is the class of all semigroups satisfying the laws (i) $(xy)^{n+1} \approx xy$, (ii) $x^n yz \approx x^n yx^n z$, and (iii) $xyz^n \approx xz^n yz^n$.

(2) $\mathcal{V}_2^{(n)}$ is the class of all semigroups satisfying the laws (i) $x^{n+1} y \approx xy$, and (ii) $x^n y^n z \approx y^n x^n z$.

(3) $\mathcal{V}_3^{(n)}$ is the class of all semigroups satisfying the laws (i) $xy^{n+1} \approx xy$, and (ii) $xy^n z^n \approx xz^n y^n$.

THEOREM 1. *Every residually small semigroup variety is contained in some one of the varieties* $\mathcal{V}_j^{(n)}$.

Notice that $\mathcal{V}_1^{(n)}$ is self-dual, that is if $A = \langle A, \cdot \rangle \in \mathcal{V}_1^{(n)}$ then $A^\partial = \langle A, \cdot^\partial \rangle \in \mathcal{V}_1^{(n)}$ where $x \cdot^\partial y = y \cdot x$. Also $\mathcal{V}_3^{(n)}$ is the dual of $\mathcal{V}_2^{(n)}$. Notice also that each of the varieties $\mathcal{V}_j^{(n)}$ consists of exponent n semigroups. In fact

PROPOSITION P. *The following laws hold identically in* $\mathcal{V}_j^{(n)}$ $(j=1,2,3)$:

(1) $x^n \approx x^{2n}$

(2) $x^2 \approx x^{n+2}$

(3) $x^n \cdot y^n \approx (x^n \cdot y^n)^2$

(4) $x^n \cdot y^n \approx (x^{n+1} \cdot y^{n+1})^n$

(5) $(x^{n+1} \cdot y^{n+1})^{n+1} \approx x^{n+1} \cdot y^{n+1}$

(6) $x^n y^n z^n u^n \approx x^n z^n y^n u^n$

(7) $x^n yz \approx x^n y x^n z$

(8) $xyz^n \approx xz^n yz^n$.

This proposition, which is hardly obvious, will be proved at the end of this section after we first describe the subdirectly irreducible semigroups in the varieties $\mathcal{V}_j^{(n)}$.

By Z_2 we denote the 2-element semigroup consisting of the integers 0, 1 and actual multiplication. A *zero semigroup* is one in which all products are equal. $\not{Z}_2$ is a 2-element zero semigroup. A *left(right) zero semigroup* is one satisfying the law $x \cdot y \approx x$ (or $x \cdot y \approx y$, respectively). L_2, R_2 denote 2-element left zero (respectively, right zero) semigroups. The semigroup resulting from adding a (new) zero element 0 to a given semigroup A is denoted $A^{(0)}$. Thus the universe of $A^{(0)}$ is $A \cup \{0\}$ (it is assumed that $0 \notin A$) and in $A^{(0)}$, $x \cdot y = 0$ if either of x and y is 0, and otherwide $x \cdot y$ is the product of x and y in A.

A further construction is needed.

DEFINITION 2. Let G be a group and α be a representation of G over a non-void set U, such that G, U, and $\{0\}$ are disjoint sets. (A representation is a group homomorphism of G into the group of permutations of U.) We define $R(G, U, \alpha)$ to be the semigroup with universe $G \cup U \cup \{0\}$ and operation defined as follows ($g, h \in G$; $u, v \in U$):

$g \cdot u = \alpha_g(u)$,

$g \cdot h =$ the product in G,

all other products are 0.

We define $L(G, U, \alpha)$ analogously, except that α has to be representation of G^∂ over U (i.e. $\alpha_{g \cdot h}(u) = \alpha_h(\alpha_g(u))$), and we put $u \cdot g = \alpha_g(u)$ and $g \cdot u = u \cdot v = 0$.

When the representation α is obvious or implicit we simplify our notation to $R(G, U)$ and $L(G, U)$. This is the case if G or U has only one element.

PROPOSITION 3. *$R(1, m)$ is subdirectly irreducible iff $m = 1$. When $|G| \geqslant 2$ the following conditions are jointly necessary and sufficient for $R(G, U, \alpha)$ to be subdirectly irreducible.*

(1) *α is faithful ($g \neq h$ implies $\alpha_g \neq \alpha_h$).*

(2) *$\alpha(G)$ acts transitively on U.*

(3) *For some $u \in U$ (equivalently, by (1), (2) for every $u \in U$), $\mathrm{Stab}(u) = \{g \in G : \alpha_g(u) = u\}$ is a completely meet irreducible member of the lattice of all subgroups of G.*

This proposition is proved at the end of this section.

The next two theorems show the power of Theorem 1. The varieties $\mathcal{V}_3^{(n)}$ are not mentioned as they are dual to the $\mathcal{V}_2^{(n)}$. For the correct statement of the results for $\mathcal{V}_3^{(n)}$, replace $R(G, U, \alpha)$ by $L(G^\partial, U, \alpha)$ and dualize all other mentioned semigroups.

THEOREM 4. (1) *$\mathcal{V}_1^{(n)}$ is generated as a variety by the semigroups $\not{Z}_2, L_2^{(0)}, R_2^{(0)}$ and by the subvariety $\mathcal{G}^{(n)}$ consisting of all groups of exponent n.*

(2) *The s.i. members of $\mathcal{V}_1^{(n)}$ are the following: G and $G^{(0)}$ where $G \in \mathcal{G}^{(n)}$ is s.i., $Z_2, \not{Z}_2, L_2, R_2, L_2^{(0)}, R_2^{(0)}$.*

(3) *A variety $\mathcal{V} \subseteq \mathcal{V}_1^{(n)}$ is residually small iff $\mathcal{V} \cap \mathcal{G}^{(n)}$ is residually small.*

THEOREM 5. (1) *$\mathcal{V}_2^{(n)}$ is generated as a variety by the semigroups $R_2^{(0)}$, $R(1, 1)$, and the subvariety $\mathcal{G}^{(n)}$ of exponent n groups.* (2) *The s.i. members of $\mathcal{V}_2^{(n)}$ are the following: G and $G^{(0)}$ where $G \in \mathcal{G}^{(n)}$ is s.i.; $R(G, U, \alpha)$ where $G \in \mathcal{G}^{(n)}$ and the conditions of Proposition 3 hold; $Z_2, \not{Z}_2, R_2, R_2^{(0)}$.* (3) *A variety $\mathcal{V} \subseteq \mathcal{V}_2^{(n)}$ is residually small iff: $\mathcal{V} \cap \mathcal{G}^{(n)}$ is residually small and, further, if $\mathcal{V} \nsubseteq \mathcal{V}_1^{(n)}$ then the groups $G \in \mathcal{V} \cap \mathcal{G}^{(n)}$ which admit a representation with the properties Prop. 3(1–3) are bounded in cardinality.*

Proof of Proposition P. This proof and the next one are not crucial to understanding the later arguments, and may be skipped over.

First, assume that identities E1 (of definition E) hold in a semigroup A. Substitute x for y, z in E1(i, ii) to obtain $x^{2n+2} \approx x^2$ and $x^{n+2} \approx x^{2n+2}$, from which P1, P2 follow directly. Then by E1(ii) and P1, $x^n y^n x^n y^n = x^n y^n y^n = x^n y^n$, which is P3. P5 is a substitution instance of (i) and P7, P8 are just (ii), (iii). Noting that P3 yields $x^n \cdot y^n = (x^n \cdot y^n)^n$, we derive P6 from P1–P3, P7, P8 (abbreviating $e = x^n$, $f = y^n$, $g = z^n$, $h = u^n$): $efgh = efghefgh$, and from the latter expression, using P7, P8, one removes in turn the first f, the first h, the second e, the second g, to obtain $egfh$.

We now show that P4 follows from P1–P3 and P5–P8. Again denote x^n by e, y^n by f, $(x^{n+1}\cdot y^{n+1})^n$ by g. We have $e^2=e$, $f^2=f$, $g^2=g$. Now $(fy^{n-1}ex^{n-1}f)\cdot(x^{n+1}y^{n+1})^2=(fy^{n-1}ey^{n+1})(x^{n+1}y^{n+1})=(fy^{n-1}ey^{n+1})(ex^{n+1}y^{n+1})=fy^{2n}ex^{n+1}y^{n+1}=fx^{n+1}y^{n+1}$, using P7, P8. Thus if $u=(fy^{n-1}ex^{n-1}f)^{n-1}$, then $u\cdot g=f\cdot x^{n+1}\cdot y^{n+1}$ and $ex^{n-1}ug=ex^{n-1}fx^{n+1}\cdot fy=e^3fy=ey^{n+1}$. Taking n-th powers, we get $ef=ey^{n\cdot(n+1)}=(ey^{n+1})^n$ (by P7) $=v\cdot g$ for a certain v. Therefore $ef=ef\cdot g=efgf=egf=g$. The equality $ef=g$ is just P4.

Now assume that the identities E2 defining $\mathcal{V}_2^{(n)}$ hold in a semigroup A. That P2 holds in A, and therefore P1 as well, follows by substitution of y for x in E2(i). P3 follows immediately from P1 and E2(ii). For P5, $x^{n+1}y^{n+1}=(x^{n+1}y^{n+1})\cdot y^n=(x^{n+1}\cdot y^{n+1})^{n+1}y^n=(x^{n+1}\cdot y^{n+1})^{n+1}$ using P1 and E2(i). For P7, $x^nyz=(x^ny)^{n+1}z=(x^ny)x^n(x^ny)^nz=(x^ny)(x^ny)^nx^nz=x^nyx^nz$ using P1 and E2(i, ii). For P8, $zu^n=(zu^n)^{n+1}\cdot u^n=(zu^n)^n\cdot u^nzu^n=u^n(zu^n)^n\cdot z\cdot u^n=u^n(zu^n)^{n+1}\cdot u^n=u^nzu^n$. Now P6, P4 follow as in the argument for $\mathcal{V}_1^{(n)}$. □

Proof of Proposition 3. The correctness of the proposition when $|G|=1$ being trivial, we consider the case $|G|>1$. Assume that $R(G,U,\alpha)=S$ is subdirectly irreducible. A block of U is a set $B=\{\alpha_g(u):g\in G\}$ where $u\in U$. We define some relations over S (x, y range over S).

$$x\theta_\alpha y \leftrightarrow x=y\vee(x,y\in G\wedge \alpha_x=\alpha_y)$$

$$x\bar{\theta}_U y \leftrightarrow x=y\vee(x,y\in U)$$

$$x\theta_B y \leftrightarrow x=y\vee(x,y\in B\cup\{0\})$$

where B is a block of U. It is clear that each of these relations is a congruence of S. Since $\theta_B>\Delta$ (the identity relation on S) and $\theta_B\wedge\theta_C=\Delta$ for distinct (and therefore disjoint) blocks B, C, it follows from the s.i. character of S that there is only one block $B=U$ – i.e. $\alpha(G)$ is transitive over U. Since $\theta_U\wedge\theta_\alpha=\Delta$ (U is a block), then $\theta_\alpha=\Delta$ – i.e. α is faithful. Then since G has more than one element, so does U, and it follows that the unique smallest congruence $\psi>\Delta$ of S is a subset of $\bar{\theta}_U$.

Let $u_0\in U$, $g_0\in G$ be such that $\alpha_{g_0}(u_0)\neq u_0$. Define $F=\mathrm{Stab}(u_0)=\{g\in G: \alpha_g(u_0)=u_0\}$. We conclude the proof that s.i. ⇒ (1–3) and also establish that (1–3) ⇒ s.i., by proving the following claim.

CLAIM. Assuming that (1), (2) hold, the pair $(u_0,\alpha_{g_0}(u_0))$ belongs to every congruence of S except Δ iff g_0 belongs to every subgroup of G that properly includes F.

To prove it suppose first that $u_0\theta g_0\cdot u_0$ whenever $\theta>\Delta$ is a congruence. If H

is a subgroup of G properly extending F, then

$$\Delta \cup \{(g \cdot u_0, g \cdot h \cdot u_0) : h \in H, g \in G\} = \theta$$

is a congruence relation on S, bigger than Δ. Hence $(u_0, g_0 \cdot u_0) = (g \cdot u_0, g \cdot h \cdot u_0)$ for some $(g, h) \in G \times H$. Here $g, g_0^{-1} \cdot g \cdot h \in F$ hence $g_0 \in H$.

Conversely, suppose that g_0 does belong to every subgroup properly extending F. Then, given $\theta > \Delta$, we can define $H = \{g \in G : u_0 \theta g \cdot u_0\}$. It is clearly a group and $H \supseteq F$. We must investigate θ. If $\theta \cap {}^2G \nsubseteq \Delta$ then, where $(g, h) \in \theta - \Delta$, we have $(g \cdot u, h \cdot u) \in \theta \cap {}^2U - \Delta$ for some u, by faithfulness of α. If $g \theta u$ for some $g \in G$, $u \in U$ then $g \cdot u \theta u \cdot u = 0$ and $U = G \cdot g \cdot u \subseteq 0/\theta$ hence ${}^2U \subseteq \theta$. The same conclusion follows if $0 \theta x$ for some $x \neq 0$. Putting all this together, we conclude that there exists $(u, v) \in \theta - \Delta$, $u, v \in U$. By transitivity of $\alpha(G)$ we have $u_0 \theta g \cdot u_0 \neq u_0$ for some g. Consequently, $H > F$, and so $g_0 \in H$. So, finally, $u_0 \theta g_0 u_0$. □

§2. Proving Theorems 4 and 5

We begin by introducing a few concepts and items of notation used in the remainder of the paper. The least element of the lattice of congruences, Con A, will always be denoted by Δ – it is the identity relation over A. The least element $\theta > \Delta$ of Con A, where A is subdirectly irreducible, will be called the *monolith* of A, and will usually be denoted by β.

An algebra A is s.i. iff it is *(a, b)-irreducible* for some $a \neq b$ in A – that is, iff the following are equivalent for any congruence $\theta \in \mathrm{Con}\, A$: $\theta > \Delta$, $(a, b) \in \theta$. A is (a, b)-irreducible iff $a \neq b$ and $(a, b) \in \beta$, the monolith.

An *ideal* in A is a non-empty set $J \subseteq A$ such that $J \cdot A \cup A \cdot J \subseteq J$, i.e. $x \cdot y \in J$ if either of x, y is in J. With any ideal J we can form the *ideal congruence* $\Delta \cup {}^2J = \{(x, y) \in {}^2A : x = y \text{ or } x, y \in J\}$. The factor semigroup may be denoted simply A/J.

In general, the least congruence θ of A that includes a given set of ordered pairs (c_i, d_i), $i = 1, \ldots, n$, will be denoted $C_g((c_i, d_i), 1 \leq i \leq n)$; a pair (x, y) is in this congruence iff $x = y$ or there is a finite path of elements $x_0, \ldots, x_k$ with $x = x_0$, $y = x_k$, and for each $j < k$ there are elements r_j, s_j and some $i = 1, \ldots, n$ such that $\{x_j, x_{j+1}\}$ equals one of $\{c_i, d_i\}$, $\{r_j c_i, r_j d_i\}$, $\{c_i \cdot s_j, d_i \cdot s_j\}$, or $\{r_j \cdot c_i \cdot s_j, r_j \cdot d_i \cdot s_j\}$.

Now we make a few remarks on what has to be proved. It is easy to check that all of the semigroups mentioned in Theorem 4 belong to $\mathcal{V}_1^{(n)}$ (i.e. satisfy the laws E1); that Z_2, L_2, R_2 are subalgebras of $L_2^{(0)}$ or of $R_2^{(0)}$; and that $G^{(0)}$ is a homomorphic image of $G \times Z_2$ for any group G. In other words, the semigroups listed in 4(2) generate the same variety as those listed in 4(1), and it is a subvariety of $\mathcal{V}_1^{(n)}$. Thereby, 4(1) and 4(2) are reduced to the assertion that every s.i. member of $\mathcal{V}_1^{(n)}$ is listed in 4(2). We also observe that 4(3) is an easy consequence of 4(1) and 4(2).

The situation as regards Theorem 5 is similar. It boils down to showing that every s.i. semigroup in $\mathcal{V}_2^{(n)}$ has been listed in Statement 5(2).

With these preliminaries out of the way we begin the arguments. For the remainder of this section n denotes a fixed integer larger than 1. We define $\mathcal{V}^{(n)}$ to be the variety of semigroups defined by the identities (1–8) of Proposition P, thus $\mathcal{V}^{(n)} \supseteq \mathcal{V}_1^{(n)} \cup \mathcal{V}_2^{(n)}$.

DEFINITION 6. Given a semigroup A we denote by $I(A)$ the set of idempotent elements of A (that is $x \in I(A)$ iff $x^2 = x$). For $e \in I(A)$, A_e denotes the largest subgroup of A containing e (thus $x \in A_e$ iff $e \cdot x \cdot e = x$ and $y \cdot x = e = x \cdot z$ for some $y, z \in A$). We put $A_G = \bigcup \{A_e : e \in I(A)\}$.

LEMMA 7. (*Let* $A \in \mathcal{V}^{(n)}$.)
(1) $I(A) = \{x^n : x \in A\}$.
(2) $A_e = \{x^{n+1} : x^n = e\}$ *for* $e \in I(A)$.
(3) $A_G = \{x^{n+1} : x \in A\} = \{x : x^{n+1} = x\} \supseteq \{x^2 : x \in A\}$.
(4) $I(A)$ *and* A_G *are subalgebras of* A.
(5) $A_e \cdot A_f \subseteq A_{e \cdot f}$ *for* $e, f \in I(A)$.

This lemma is an obvious reformulation of the significance of identities P1–P5.

LEMMA 8. *Let* $A \in V^{(n)}$ *be* (a, b)*-irreducible where* $a, b \in A_e$ *and* $e = e^2$. *Then* $A = G$ *or* $A = G^{(0)}$ *for some s.i. group* $G \in \mathcal{G}^{(n)}$.

Proof. Recall that the concept of (a, b)-irreducibility was defined in the first paragraph of this section. Define $x\theta y$ iff $exe = eye$. By P7 and P8, θ is a congruence relation on A. Since $(a, b) \notin \theta$, then $\theta = \Delta$. Since $x\theta e \cdot x \cdot e$ it follows that e is a two-sides identity element for A. Redenote e by 1. Suppose that $f \in I(A)$ and $f \neq 1$. Either the ideal congruence $^2(A \cdot f \cdot A) \cup \Delta$ is Δ, or else $a, b \in A \cdot f \cdot A$ and consequently $1 \in A \cdot f \cdot A$. In the latter case, $1 = u \cdot f \cdot v$, $1 = u \cdot f \cdot v \cdot 1 \cdot 1 = u \cdot f \cdot v \cdot f \cdot 1$ (by P7) $= u' \cdot f$. Thus $1 = u' \cdot f \cdot f = 1 \cdot f = f$, a contradiction. We conclude that $A \cdot f \cdot A = \{f\}$. There can be at most one idempotent f with this property, hence either $I(A) = \{1\}$ or $I(A) = \{1, 0\}$ where 0 is a true zero element.

If $I(A) = \{1\}$ then every $x \in A$ satisfies $x^n = 1$, $x^{n+1} = x$, so $A = A_1$. Assume now that $I(A) = \{0, 1\}$, and that $A > \{0\} \cup A_1$. Then there is $u \in A$, $u \neq 0$, $u^n = 0$.

(Since $u^n \in \{0, 1\}$.) As before, we have that 1 is in the ideal generated by u. Thus $1 = u$, $x \cdot u$, $u \cdot y$, or $x \cdot u \cdot y$. All cases reduce to $1 = x \cdot u \cdot y$ as $1^2 = 1$. Then $x \cdot u = x \cdot u \cdot x \cdot u \cdot y$. Inductively, we get $x \cdot u = (x \cdot u)^n \cdot y^{n-1}$. Now $1 \neq 0$ implies $x \cdot u \neq 0$ implies $(x \cdot u)^n \neq 0$ (by the last formula) implies $(x \cdot u)^n = 1$. Thus $u = (x \cdot u)^{n-1} \cdot x \cdot u^2$ and $u^2 \neq 0$. But then by P2, $u^n \neq 0$, a contradiction. □

LEMMA 9. *(Let $A \in \mathcal{V}_j^{(n)}$, $j = 1$ or 2.) The least congruence θ on A such that A/θ satisfies $x^2 \approx x^3$ has the following properties* (1) *if $x\theta y$ and $x \in A_G$, then $y \in A_G$;* (2) *for $x, y \in A_G$, $x\theta y$ iff $x^n = y^n$.*

Proof. By Lemma 7, θ is the congruence generated by collapsing each maximal subgroup of A to a point. Two elements u, v of A are θ-related iff there is a path $u = x_0, \ldots, x_k = v$ such that for $i < k$, $(x_i, x_{i+1}) = (c \cdot g_i \cdot d, c \cdot h_i \cdot d)$ for some $c, d \in A$ where $g_i, h_i \in A_e$ for some $e \in I(A)$. We can prove the lemma by showing that when $c, d \in A$, $x, y \in A_e$, $c \cdot x \cdot d \in A_f$, then $c \cdot y \cdot d \in A_f$.

Suppose that $c \cdot x \cdot d \in A_f$ and $x, y \in A_e$. Assume first that $A \in \mathcal{V}_1^{(n)}$. Then $A \cdot A \subseteq A_G$ by E1(i), hence $c \cdot y \cdot d \in A_g$ where

$$\begin{aligned} g &= (c \cdot y \cdot d)^n = (ce \cdot y \cdot ed)^n \\ &= (ce)^n y^n (ed)^n \quad \text{by E1(i), P4} \\ &= (ce)^n x^n (ed)^n \\ &= (cxd)^n = f. \end{aligned}$$

Now assume that $A \in \mathcal{V}_2^{(n)}$. Then

$$\begin{aligned} ced &= ce(ce)^n ed \quad \text{by E2(i)} \\ &= ce(ce \cdot x)^n d \quad \text{by Lemma 7(2, 5) and E2(i)} \\ &\quad (ce = ce \cdot e = (ce)^{n+1} \cdot e = (ce)^{n+1}) \\ &= ce(ce \cdot x)^{n-1} cxd. \end{aligned}$$

This last formula puts $ced \in A_g$ where $g = (ce)^n (ce)^n x^n (cxd)^n = (ce)^n f$ again by Lemma 7. But $cxd = ce \cdot x \cdot d = (ce)^{n+1} xd$ implies $f = (ce)^n f$, so $ced \in A_f$. Then

$$\begin{aligned} cyd &= ce \cdot yd = ce(ce)^n yd \\ &= ce(ce \cdot y^{n-1})^n yd \quad \text{by Lemma 7} \\ &= ce(cy^{n-1})^{n-1} cy^n d \\ &= ce(ce \cdot y^{n-1})^{n-1} ced \end{aligned}$$

which puts $cyd \in A_g$ where now

$$g = (ce)^n \cdot (ce)^n \cdot e \cdot (ced)^n$$
$$= (ce)^n \cdot f = f. \quad \square$$

COROLLARY 10. (*Let* $A \in \mathcal{V}_j^{(n)}$, $j = 1$ *or* 2.) *If A is s.i. with monolith* β *then one of the following holds:* (1) $A = G$ *or* $G^{(0)}$ *where G is a group.* (2) *A satisfies* $x^2 \approx x^3$. (3) $\beta \subseteq {}^2(A - A_G) \cup \Delta$.

Proof. If (2) fails then for the θ of Lemma 9, $\theta \geqslant \beta$. Then by Lemma 9 either $\beta \cap {}^2A_e \nsubseteq \Delta$ for some idempotent e, and we get (1) by Lemma 8, or else $\beta \cap {}^2A_G \subseteq \Delta$ and (3) follows by Lemma 9 $(\theta \subseteq {}^2A_G \cup {}^2(A - A_G))$. $\square$

LEMMA 11. $\mathcal{V}_1^{(n)}$ *has no s.i. members satisfying alternative* (3) *of Corollary* 10. *Every s.i. member of* $\mathcal{V}_2^{(n)}$ *satisfying* 10(3) *is of the form* $R(G, U, \alpha)$.

Proof. Let $A \in \mathcal{V}_j^{(n)}$ ($j = 1$ or 2) be s.i. with monolith $\beta \subseteq \Delta \cup {}^2(A - A_G)$.

(1) A has a zero element 0 and, where

$$\bar{U} = \{x : x^2 = 0\}, \qquad \beta \subseteq \Delta \cup {}^2\bar{U}. \qquad \text{Moreover } A \notin \mathcal{V}_1^{(n)}.$$

Indeed, take any $(a, b) \in \beta - \Delta$. Then $(a^n, b^n) \in \beta \cap {}^2A_G$ implies $a^n = b^n = e$ say. Suppose that $A \cdot e \cdot A > \{e\}$. Then the ideal congruence for the ideal $A \cdot e \cdot A$ includes β so $a, b \in A \cdot e \cdot A$. Say $b = u \cdot e \cdot v$. Then ($\mathcal{V}_1^{(n)}$ or $\mathcal{V}_2^{(n)}$) $u \cdot e = (ue)^{n+1}$, so $b = (u \cdot e)^n b = (ue)^n eb = (ue)^n be = be$. Similarly, $a = a \cdot e$. Then $a, b \in A_e$, contradiction. We conclude that A has a zero element 0, namely $0 = e$. Moreover, $a^2 = a^{n+2} = 0 \cdot a^2 = 0$, $b^2 = 0$. Now if A were in $\mathcal{V}_1^{(n)}$ we would have the ideal $A \cdot A \subseteq A_G$ (E1(i)) and so the ideal congruence would be disjoint from β, giving $A \cdot A = \{0\}$. Since $\mathcal{Z}_2$ is the only s.i. zero semigroup, we would have $A \cong \mathcal{Z}_2$. However, $\mathcal{Z}_2$ does not satisfy 10(3).

(2) $x \in \bar{U}$ iff $x \cdot y = 0$ for all $y \in A$.

This follows from the fact that $A \in \mathcal{V}_2^{(n)}$ (by E2(i)).

(3) $I(A) = \{0, e\}$ where $e \neq 0$ and

$$e \cdot x = x \quad \text{for all} \quad x \in A.$$

To show this we first establish that where $(a, b) \in \beta$, $a \neq b$, we have $e \cdot a = a$, $e \cdot b = b$ for every idempotent $e \neq 0$. In fact the ideal $A \cdot e \cdot A$ contains at least two elements, 0 and e, hence $a, b \in A \cdot e \cdot A$. Then, say $a = u \cdot e \cdot v$, giving $a = (ue)^n euev = e(ue)^n uev$ by E2(ii), and $a = e \cdot a$. Likewise $b = e \cdot b$.

Next, fixing a, b, e as above, we show that $e \cdot x = x$ for all x. Because A satisfies P7, the relation

$$\theta = \{(x, y): ex = ey\}$$

is easily seen to be a congruence of A. Now if $e \cdot x \neq x$ then $x\theta e \cdot x$ and hence $\beta \subseteq \theta$ and we get $a\theta b$, which contradicts $e \cdot a = a$, $e \cdot b = b$.

Next, suppose that there are two idempotents $e \neq 0 \neq f \neq e$. Both e and f act as left identity elements. The relation

$$xJy \leftrightarrow x = y \vee (x, y \in A_G \wedge (\forall z)(x \cdot z = y \cdot z))$$

is certainly a congruence, and eJf. Thus $\beta \subseteq J$. This is impossible, because $\beta - \Delta \subseteq {}^2(\mathrm{A} - \mathrm{A_G})$.

Finally, suppose that 0 is the only idempotent. Then $x \cdot y = x^{n+1}y = 0$ for all x and y, giving us the contradictory conclusion that $A \cong \not{Z}_2$, once again.

(4) $A = A_e \dot{\cup} U \dot{\cup} \{0\}$ (disjoint union) where

$$U = \bar{U} - \{0\}, \qquad A_e \cdot U \subseteq U, \qquad U \cdot A = \{0\}.$$

It is clear that the union is disjoint. To show that it is A, suppose that $x \notin \bar{U}$. Then $x^n = e$, as $x^n = 0$ implies $x^2 = x^{n+2} = 0$. So we have $x^{n+1} = x$ by (3), and $x^n = e$, giving $x \in A_e$.

That $U \cdot A = \{0\}$ is a consequence of E2(i).

The only thing remaining, to establish (4), is to show that $g \in A_e$ and $u \in \bar{U}$, $u \neq 0$, imply $g \cdot u \neq 0$ (clearly $g \cdot u \in \bar{U}$ by (2). This follows by the calculation $u = e \cdot u = g^{n-1} \cdot g \cdot u$, which establishes also

(5) For $g \in A_e$ the function $\alpha_g \in {}^U U$, $\alpha_g(u) = g \cdot u$, is a permutation of U. α: $A_e \to {}^U U$ is a homomorphism of A_e into the symmetric group on U.

Now checking Definition 2 we see that $A = R(A_e, U, \alpha)$. The proof of Lemma 11 is complete. □

In light of the last two results, what remains in the proof of Theorems 4 and 5 is to consider s.i. semigroups in $\mathcal{V}_j^{(n)}$ which satisfy the law $x^2 \approx x^3$.

LEMMA 12. *The only s.i. members of* $\mathcal{V}_2^{(n)}$ *satisfying* $x^2 \approx x^3$ *are* $Z_2, \not{Z}_2, R_2, R_2^{(0)}$ *and* $R(1, 1)$.

Proof. Take A s.i. in $\mathcal{V}_2^{(n)}$ with monolith β and satisfying $x^2 \approx x^3$. Notice that $A_G = I(A)$ and that A Satisfies $xyz \approx yxz$ (since $xyz = x^{n+1}y^{n+1}z = x^2y^2z = x^ny^nz = y^nx^nz = yxz$) and $xy \approx x^2y$.

(1) When $x \neq x^2$ in A then x^2 is a (the) zero element.

For the proof, let $a \neq b$, $a\beta b$ and $x \neq x^2$, $A \cdot x^2 \cdot A > \{x^2\}$. The relation defined by $r\theta s$ iff $r = s$, or else $r, s \in A^{(1)} \cdot x$ and $r \cdot t = s \cdot t$ for all t is a congruence. $(a, b) \in \theta$ and consequently: $a, b \in \{x\} \cup A \cdot x$, and $a \cdot t = b \cdot t$ for all $t \in A$. We also have $a, b \in A \cdot x^2 \cdot A$. Thus $e \cdot a = a$ for some $e \in I(A)$ (by E2(i)) and we can actually write $a = u \cdot x = u_0 \cdot x^2 \cdot u_1$, $b = v \cdot x = v_0 \cdot x^2 \cdot v_1$. Then $x \cdot a = x \cdot u_0 x^2 \cdot u_1 = u_0 \cdot x^3 \cdot u_1 = u_0 x^2 \cdot u_1 = a$, $x \cdot b = b$, $a = x \cdot a = x \cdot u \cdot x = u \cdot x \cdot x = a \cdot x$, $b = b \cdot x$. But $a \cdot t = b \cdot t$ for all t so $a = b$, a contradiction.

(2) When $e \in I(A)$ and is not a zero element, then $e \cdot x = x$ for all x.

Let a, b be as above. The relation $\{(u, v): e \cdot u = e \cdot v\}$ is congruence. Hence unless e is a left identity element we have $e \cdot a = e \cdot b$. But if e is not the zero element then $a, b \in A \cdot e \cdot A = e \cdot A$, implying $e \cdot a = a$, $e \cdot b = b$.

By (1), (2) we can write $A = S \dot\cup T$ where S is the set of idempotents that are not zeros, $s \cdot x = x$ whenever $s \in S$ and $x \in A$, and moreover if $T \neq \emptyset$, then A has a zero $0 \in T$ and $t \cdot x = t^2 \cdot x = 0$ for all $t \in T$ and $x \in A$. If $x, y \in S$ or $x, y \in T$, then it follows that ${}^2\{x, y\} \cup \Delta$ is a congruence. Hence by subdirect irreducibility, $|A| \leq 3$, and $|S|, |T| \leq 2$. Checking through the possibilities establishes the lemma. □

LEMMA 13. *The only s.i. members of* $\mathcal{V}_1^{(n)}$ *satisfying* $x^2 \approx x^3$ *are* Z_2, $\not{Z}_2$, L_2, R_2, $L_2^{(0)}$ *and* $R_2^{(0)}$.

Proof. Take A s.i. in $\mathcal{V}_1^{(n)}$ with monolith β and satisfying $x^2 \approx x^3$. Again $I(A) = A_G$, and since $A \in \mathcal{V}_1^{(n)}$, $A_G = A \cdot A$. If $|A \cdot A| = 1$, then we have $\not{Z}_2$. Otherwise $\beta \subseteq {}^2(A \cdot A) \cup \Delta$, hence there are idempotents $u \neq v$, $(u, v) \in \beta$.

(1) A satisfies $x \cdot y \approx x^2 \cdot y^2 \approx x^n \cdot y^n$.

Since

$$xy = (xy)^2 xy = (xy)^2 x (xy)^2 y = (xy) x (xy) y = xyx^2y^2 = xx^2yx^2y^2$$
$$= x^2yx^2y^2 = x^2y^3 = x^2y^2.$$

Here we have used E1(i–iii) as appropriate with n-th powers replaced by squares.

(2) A satisfies $x \approx x^2$.

It follows from 10(2) and (1) above that the elements x and x^2 satisfy $x \cdot t = x^2 \cdot t$ and $t \cdot x = t \cdot x^2$ for all t. Hence ${}^2\{x, x^2\} \cup \Delta$ is a congruence. Thus if $x \neq x^2$ then $\{u, v\} = \{x, x^2\}$. But $u = u^2$, $v = v^2$.

(3) Every element e in A is either a left identity, a right identity, or a zero element.

Suppose not. Then the relations $\{(x, y): ex = ey\}$, $\{(x, y): x \cdot e = y \cdot e\}$ are nontrivial congruences (by P7, P8), hence $eu = ev$, $ue = ve$. But also $u, v \in A \cdot e \cdot A$. Say $u = r \cdot e \cdot s$. Then $u = r \cdot e \cdot u = u \cdot e \cdot u$ $(r \cdot e,\ e \cdot u \in I(A))$. Likewise $v = v \cdot e \cdot v$. But $u \cdot e \cdot u = u \cdot e \cdot v = v \cdot e \cdot v$.

Now we have $A = L \cup R \cup Z$ where L is the set of left identity elements, R the set of right identity elements and $Z = \emptyset$ or $Z = \{0\}$.

(4) Either $L \subseteq R$ or $R \subseteq L$.

Suppose to the contrary that $e \in L - R$ and $f \in R - L$. Say $x_0 \cdot e \neq x_0$. Now $x_0 \cdot e \cdot t = x_0 \cdot t$ for all t. Hence (with u, v as before, $(u, v) \in \beta$) we have $u \cdot t = v \cdot t$ for all t. Likewise (from f), $t \cdot u = t \cdot v$ for all t. Then $u = u \cdot u = v \cdot u = v \cdot v = v$. This is a contradiction.

The conclusion of the lemma follows directly from (3) and (4). □

§3. Proving Theorem 1.

Throughout this section $\mathcal{V}$ denotes a fixed residually small variety of semigroups. The notation $I(A)$, A_e, A_G from Definition 6 will be carried into this section. A variety will called *of finite exponent* if it satisfies some law $x^n \approx x^{2n}$ $(n > 1)$. The least such n will be called its *exponent*. A variety (or a semigroup) is *nilpotent of class* $\leqslant m$ if it satisfies the law $x_1 \cdot \cdots \cdot x_m \approx y_1 \cdot \cdots \cdot y_m$. It is *nil-m* if it is nilpotent of class $\leqslant m$ and not nilpotent of class $\leqslant m-1$.

We use the notation S^2 (and more generally S^m, $m \geqslant 1$) for the range of the operation of the semigroup S (for the set of elements of S which can be represented as m-fold products). Notice that S is nil-m iff S^m has just one element while S^{m-1} has more than one, and that S^m is always an ideal of S. The notation ${}^{\kappa}S$, where κ is any cardinal, denotes the κ-th direct power of S, whose elements are all the functions from κ into S.

For constructing s.i. semigroups, we use the fact that when $\theta \in \operatorname{Con} S$ and $(a, b) \notin \theta$ $(a, b \in S)$, then the set of congruences Γ of S such that $(a, b) \notin \Gamma$ and $\theta \subseteq \Gamma$ has maximal members, and for any such maximal Γ, S/Γ is s.i., in fact is $(a/\Gamma, b/\Gamma)$-irreducible.

The construction of Definition 2 will be needed in this section in a slightly more general form.

DEFINITION 14. Let α be any homorphism from a semigroup S into the semigroup ${}^{U}U$ where S, U, and $\{0\}$ are disjoint sets. $R(S, U, \alpha)$ is formed as in Definition 2, so that S is a subsemigroup, $s \cdot u = \alpha_s(u)$ for $s \in S, u \in U$, and $x \cdot y = 0$ if $y = 0$ or $x \notin S$. If $\beta: S^{\partial} \to {}^{U}U$ homorphically then $L(S, U, \beta)$ is formed analogously but with $u \cdot s = \beta_u(s)$.

The arguments of this section take the following form. We produce relative splittings of the lattice of semigroup varieties – that is pairs $\mathfrak{A}$, Σ where $\mathfrak{A}$ is a set of semigroups and Σ is a set of identities with the property that every semigroup variety either includes some member of $\mathfrak{A}$ or else satisfies some law in Σ. For this to be useful in the present situation we need to be able to show that for each $S \in \mathfrak{A}$, the variety $V(S)$ is not residually small. It then follows that our residually small variety $\mathcal{V}$ must satisfy one of the laws in Σ.

LEMMA 15. *$\mathcal{V}$ is of finite exponent and contains no nilpotent semigroups except zero semigroups.*

Proof. If $\mathcal{V}$ were not of finite exponent then the $\mathcal{V}$-free semigroup on one generator, $F_{\mathcal{V}}(1)$, would be isomorphic to the free semigroup on one generator. Factoring by the ideal congruence $^2J \cup \Delta$ where $J = \{x^n : n \geq 3\}$ and x is the generator, we get a nil-3 semigroup in $\mathcal{V}$.

Now suppose that $\mathcal{V}$ contains a semigroup S which is nil-m, $m \geq 3$. Then S has a zero element and in fact $S^m = \{0\}$. Put $S_1 = \{x \in S : x \cdot S = S \cdot x = \{0\}\}$. Then $\{0\} < S_1 < S$. There is $a \in S_1$, $a \neq 0$, and $b \in S - S_1$ with $b \cdot S \cup S \cdot b \subseteq S_1$.

For any cardinal κ let

$$S^{(\kappa)} = \{f \in {}^{\kappa}S : f(\alpha) \neq 0 \text{ for at most one } \alpha < \kappa\}.$$

Then $S^{(\kappa)}$ is a subalgebra of the κ-th direct power of S, and we define a congruence on it. Put

$$f\theta_\kappa g \leftrightarrow (\exists \alpha, \beta < \kappa)(f(\alpha), g(\beta) \in S_1 - \{0\}) \vee f = g.$$

Define

$$\bar{0} = \langle 0 : \alpha < \kappa \rangle / \theta_\kappa, \; \bar{a} = (\langle a : \alpha = 0 \rangle \cup \langle 0 : \alpha > 0 \rangle)/\theta_\kappa,$$

$\bar{h}_\beta = (\langle b : \alpha = \beta \rangle \cup \langle 0 : \alpha \neq \beta \rangle)/\theta_\kappa$ for $\beta < \kappa$. Now $\{\bar{h}_\beta : \beta < \kappa\}$ is a set of κ distinct elements of $A = S^{(\kappa)}/\theta_\kappa$, and $\bar{0} \neq \bar{a}$ in A. But any congruence Ψ of A that identifies any two $\bar{h}_\beta$, $\bar{h}_\delta$, $\beta \neq \delta$ will also identify $\bar{0}$ with $\bar{a}$. (There is $\bar{f} \in A$ such that either $\bar{f} \cdot \bar{h}_\beta = \bar{a}$, or $\bar{h}_\beta \cdot \bar{f} = \bar{a}$, say $\bar{f} \cdot \bar{h}_\beta = \bar{a}$, while $\bar{f} \cdot \bar{h}_\delta = \bar{0} = \bar{h}_\delta \cdot \bar{f}$.) Taking Ψ to be a maximal member of the set of congruences that don't identify $\bar{a}$ with $\bar{0}$, A/Ψ is s.i. and has at least κ elements.

The above considerations contradict the assumption that $\mathcal{V}$ is residually small. □

From here on we let n denote the (fixed) exponent of $\mathcal{V}$ (Lemma 15). Thus $n > 1$ by definition.

LEMMA 16. *$\mathcal{V}$ satisfies* P1, P2 *and one at least of the laws* E1(i), E2(i), E3(i).

Proof. We have P1 already. For P2, i.e. $x^2 \approx x^{n+2}$, look again at $F_{\mathcal{V}}(1)/J$, $J = \{x^n : n \geq 3\}$. By the last lemma this semigroup is a zero semigroup. This implies that $x^2 \in J$. So $\mathcal{V}$ satisfies an identity $x^2 \approx x^{2+k}$, $k \geq 1$. Then also $x^2 = x^{2+n \cdot k} = x^{2+n}$ in $F_{\mathcal{V}}(1)$, hence $x^2 \approx x^{2+n}$ holds in $\mathcal{V}$.

Now we look at $F = F_{\mathcal{V}}(2) = F_{\mathcal{V}}(x, y)$. Take $J = F^3$. By Lemma 15, F/J is nilpotent class 2, hence $x \cdot y \in J$. So we have for some word $w = w(x, y)$ of length at least 3, that $\mathcal{V}$ satisfies

$$x \cdot y \approx w(x, y). \tag{1}$$

If w begins with x^2 then from P2, P1 we derive $x \cdot y = x^n \cdot x \cdot y$ in F, i.e. $\mathcal{V}$ satisfies E2(i). If w begins with y^2 then, similarly $\mathcal{V}$ satisfies $x \cdot y \approx y^n(xy) \approx (y^n x)y \approx x^n(y^n x) \cdot y$, the last equation by substitution and replacement from the first. Then again $x \cdot y = x^n \cdot x \cdot y$ in F. Similarly, if w ends in x^2 or y^2 we get E3(i).

If w begins with $y \cdot x$ then we derive $x \cdot y \approx y \cdot x \cdot \alpha(x, y) \approx x \cdot y \cdot \alpha(y, x) \cdot \alpha(x, y)$, which brings us the case that w begins with $x \cdot y$, and ends with neither x^2 nor y^2.

Thus w is $xy^{2+k}x\beta$ or $xyxy\beta$ or $xyx^2\beta$ or xyx where β may be empty, except in the third case. In the first case, by substitution,

$$\begin{aligned} xy &\approx xy^{1+k} \cdot yx \cdot \beta(x, y) \approx xy^{1+k} \cdot yx^{2+k}y\beta(y, x) \cdot \beta(x, y) \\ &\approx \gamma \cdot x \cdot y \cdot \delta \end{aligned}$$

where $\delta = \varnothing$ is possible but $\gamma \neq \varnothing$. Then

$$xy \approx \gamma^n \cdot xy \cdot \delta^n$$

is derivable, and then from P1, $x \cdot y \approx \gamma^n \cdot xy$. Now plugging in γ^n for x and xy for y in the original identity (1), we get

$$\begin{aligned} xy &\approx \gamma^n xy \approx \gamma^n (xy)^{2+k} \cdots \\ &\approx (xy)^{2+k} \cdots \end{aligned}$$

from which $xy \approx (xy)^{n+1}$ (or E1(i)) follows with the help of P1, P2.

In the case $w \equiv (xy)^2\beta$, E1(i) is immediate. In case $w \equiv xyx^2\beta$ we reduce to the preceding case: $xy \approx xyx^2\beta(x, y) \approx xyxy^2\beta(y, x)x\beta(x, y)$. In the case $w \equiv xyx$ we have $xy \approx x(yx) \approx x(yxy)$ giving E1(i) again. □

DEFINITION 17. Let p be a prime integer. $S_{4,p}$ is the semigroup with presentation

$$(e, f: e^2 = e, f^2 = f, (ef)^p e = e, (fe)^p f = f).$$

In other words, it is the factor algebra of the free semigroup on two generators e, f modulo the congruence generated by the four listed ordered pairs of words.

LEMMA 18. *$V(S_{4,p})$ is not residually small. As $\mathcal{V}$ contains no $S_{4,p}$ and no nil-m semigroup ($m > 2$), it must satisfy the law* P3.

Proof. $S_{4,p}$ has four distinct idempotents, $e, f, (ef)^p = g, (fe)^p = h$. It is the union of four cyclic p-groups A_e, A_f, A_g, A_h generated by efe, fef, ef, and fe respectively. A word represents a member of A_e iff it begins and ends with e, represents a member of A_g iff it begins with e and ends with f, and so on. $S_{4,p}$ has $4 \cdot p$ elements.

Given a cardinal $\kappa \geq \omega$ let $U \subseteq {}^{\kappa}S_{4,p}$ consist of all functions f such that $f(0) = e, \{\beta : f(\beta) \neq e\}$ is finite, and (therefore) eventually $f(\alpha) = e$. Define

$$f\theta g \text{ iff for all } \beta, \quad (f(\beta))^p = (g(\beta))^p,$$

and

$$\prod_{\alpha<\kappa} f(\alpha) = \prod_{\alpha<\kappa} g(\alpha).$$

These products are to be formed in the following manner. Since $f \in U$, we can partition κ into disjoint convex sets $C_0 < C_1 < \cdots < C_m$ such that f is constant on C_i, and $|C_i| = 1$ if $f(C_i) \neq \{e\}$, and $f(C_i) = f(C_{i+1})$ implies $f(C_i) \neq \{e\}$. The parition is unique. Taking $f(C_i) = \{x_i\}$, then

$$\prod_{\alpha<\kappa} f(\alpha) \overset{\text{def}}{=} \prod_{0}^{m} x_i \quad \text{in } S_{4,p}.$$

Now U is clearly a subalgebra of ${}^{\kappa}S_{4,p}$ and θ is clearly an equivalence relation on U. In fact, θ is a congruence and U/θ is a s.i. semigroup of cardinality κ, proving that $V(S_{4,p})$ is not residually small. We omit the proof. The reader who wishes to construct it should first look at the analogous situation where κ is a finite cardinal and the products $\prod_{\alpha<\kappa} f(\alpha)$ can be more directly manipulated.

To prove that $\mathcal{V}$ satisfies P3, we assume not. Since it does satisfy P1, P2, we have $A \in \mathcal{V}$ and two idempotent elements $e, f \in A$ such that $e \cdot f$ is not idempotent. Using that A satisfies P1, P2 and some one of E1(i), E2(i), E3(i) we can show that some $S_{4,p}$ is a homomorphic image of the subsemigroup $\langle e, f\rangle \subseteq A$ generated by e and f.

In more detail, we can assume without losing generality that A is generated by e, f and is subdirectly irreducible. A must satisfy $x^{n+1} \approx x$. (Because $A \cdot A = A$ if

E1(i) holds, and because every element of A is in the form $u \cdot x \cdot v$ where $u^2 = u$, $v^2 = v$, in case E2(i) or E3(i) holds.)

Any two words built out of e, f that have equal value in A must begin and end the same way. Otherwise, by multiplying appropriately and taking n-th powers we would be led to either $(efe)^n = (ef)^n$ or $(efe)^n = (fe)^n$ in A. (For instance, if $f(ef)^\kappa = (ef)^l\kappa$, $l \geq 1$ then $(efe)^n = (efe)^{l \cdot n} = ((ef)^l e)^n = (f(ef)^\kappa e)^n = (fe)^{(\kappa+1)\cdot n} = (fe)^n$.) Now $(efe)^n = (ef)^n$ implies $(ef)^n = (ef)^n e = (ef)^n f$ implies $(ef)^n = (ef)^n ef = ef$ implies $ef = (ef)^2$. Similarly, $(efe)^n = (fe)^n$ implies $fefe = fe$, which implies $ef = (ef)^{n+1} = e(fe)^n f = efef$.

Therefore A is the union of four disjoint parts: $T_e = \{e\} \cup \{(efe)^k : k \geq 1\}$, $T_f = \{f\} \cup \{(fef)^k : k \geq 1\}$, $T_g = \{(ef)^k : k \geq 1\}$, $T_h = \{(fe)^k : k \geq 1\}$. T_g, T_h are cyclic groups ($g = (ef)^n$, $h = (fe)^n$); T_e, T_f are semigroups each of which is the union of a cyclic group and an idempotent which may or may not be in the group. The relation

$$^2\{e, (efe)^n\} \cup {}^2\{f, (fef)^n\} \cup \Delta$$

is a congruence on A that doesn't identify ef with $(ef)^2$. (e and $(efe)^n$ are fixed by multiplication by e and identified by multiplcation with f.) Hence we can assume that $e = (efe)^n$, $f = (fef)^n$.

The four cyclic groups of which A is the union are now isomorphic: If for instance $(efe)^k = (efe)^n = e$ then $(fef)^{k+1} = f(efe)^k f = fef$ giving $(fef)^k = (fef)^n = f$. Also $(fe)^{k+1} = (fef)^k e = fe$ giving $(fe)^k = (fe)^n$. The structure of A is transparent. Since it is s.i., the order of the four groups is a prime p which divides n, and $A \cong S_{4,p}$. □

DEFINITION 19. $S^{(1)}$ denotes the semigroup obtained by adjoining an identity element to S. L_2 and R_2 denote the 2-element left and right zero semigroups.

LEMMA 20. *$L_2^{(1)}$, $R_2^{(1)}$ generate non-residually small varieties. Consequently, every idempotent semigroup in $\mathcal{V}$ satisfies the law $xyzu \approx xzyu$. Equivalently (since* P1, P3 *hold in $\mathcal{V}$) the law* P6: *$x^n y^n z^n u^n \approx x^n z^n y^n u^n$ holds in $\mathcal{V}$.*

Proof. $B = R_2^{(1)}$ is a 3-element idempotent semigroup, $B = \{1, a, b\}$ with $a \cdot b = b$, $b \cdot a = a$, $1 \cdot x = x \cdot 1 = x$. It is easily seen that every identity holding in B is a consequence of the associative law, $x^2 \approx x$, and $xyx \approx yx$ – which hold in B. Thus we can construct a semigroup S in $V(B)$ by giving a presentation. Let κ be a cardinal and S_κ be the homomorphic image of the free semigroup generated by letters x_α, y_α, $z_\alpha (\alpha < \kappa)$ modulo the congruence θ generated by all instances of the laws $x^2 \approx x$, $xyx \approx yx$, and by the relations:

$$x_\alpha y_\alpha = x_\alpha z_\alpha,$$

$$x_\alpha y_\beta = x_\delta y_\gamma, \qquad x_\alpha z_\beta = x_\delta z_\gamma \quad (\alpha < \beta, \delta < \gamma)$$

Any congruence Ψ on S_κ that has $x_\alpha \Psi x_\beta$ for some $\alpha < \beta$ has $x_0 \cdot y_1 \Psi x_0 \cdot z_1$, because $x_0 y_1 = x_\alpha y_\beta \Psi x_\beta y_\beta = x_\beta z_\beta \psi x_\alpha z_\beta = x_0 z_1$. We can show rather easily that $x_0 y_1 \neq x_0 z_1$ in S. Thus S_κ has a κ-element s.i. homorphic image.

A proof that $(x_0 \cdot y_1, x_0 \cdot z_1) \notin \theta$ goes as follows. Let Q be the set of all words (elements of the free semigroup) which can be written as $a \cdot x_\alpha \cdot w$ where a is a possibly empty word, w is a word in which no x_γ occurs, w ends in some y_γ, and the indices of the letters occuring in $x_\alpha \cdot w$ are not all equal. Clearly $x_0 \cdot y_1 \in Q$ but $x_0 \cdot z_1 \notin Q$. By checking all possibilities, it can be shown that whenever $q \in Q$ and q' is obtained from q by replacing an occurrence of some r in q by s, where $r = s$ is one of the relations generating θ, then $q' \in Q$. (Here $r = s$ may, in particular be an instance of either of the laws $x^2 \approx x$ and $xyx \approx yx$.) From a well-known characterization of θ, it follows that Q is a union of θ-equivalence classes.

The above paragraphs show that $R_2^{(1)} \notin \mathcal{V}$ and, by duality, $L_2^{(1)} \notin \mathcal{V}$. The idempotent semigroups in $\mathcal{V}$ form a subvariety $\mathcal{V}^i \subseteq \mathcal{V}$, and $\mathcal{V}^i$ must satisfy an identity that doesn't hold in $R_2^{(1)}$, and likewise for $L_2^{(1)}$. Now the identities of $R_2^{(1)}$ are exactly those $w_0 \approx w_1$ that satisfy: (a) *balanced* – w_0 and w_1 contain the same variables; (b) the rightmost occurences of the letters in w_0 have the same order from left to right as they do in w_1.

Every idempotent variety not containing $R_2^{(1)}$ satisfies

$$xzyz \approx xzyzxz. \tag{1}$$

For the proof of this, suppose that $w_0 \approx w_1$ does not hold in $R_2^{(1)}$. We have to show that together with $x \approx x^2$ this identity implies (1). If it is not balanced, say the letter x occurs in w_0 and not in w_1, replace all letters but x by y, multiply both sides of both words by y, and simplify by idempotence to derive $yxy \approx y$. This implies (1). If $w_0 \approx w_1$ is balanced but fails (b), then there are letters x and y such that

$$w_0 = \alpha_0 \cdot x \cdot \beta_0 \cdot y \cdot \gamma_0$$

$$w_1 = \alpha_1 \cdot y \cdot \beta_1 \cdot x \cdot \gamma_1$$

(any of $\alpha_i, \beta_i, \gamma_i$ may be empty) where γ_0, γ_1 contain no x or y, β_0 contains no x, β_1

contains no y. Replace all other letters by z, multiply by x on the left, replace x and y by zxz, zyz respectively, and simplify by idempotence to obtain $zxzyz \approx zxzyzxz$. Then multiply by x on the left to get (1).

Similarly, $\mathcal{V}^i$ must satisfy

$$zxzy \approx zyzxzy \quad \text{(the dual of (1)).} \tag{2}$$

From (1) and (2) we can derive the law for the idempotent semigroups in $\mathcal{V}$ required by this lemma. First $xyxz \cdot x = xzxyxz \cdot x$ (by (2)) $= xzx \cdot yxzx = xzx \cdot yxzxyx$ (by (1)) $= xzxy \cdot xzxy \cdot x = xzxyx$. We have derived $xyxzx \approx xzxyx$. Using this several times, $xyzx = xyxyzxzx = xzxyzxyx = x \cdot zxy \cdot zxy \cdot x = xzxyx$. Thus we have

$$xyxzx \approx xyzx \approx xzyx. \tag{3}$$

Finally, $xyzu = xyzuxyzu = xyxz \cdot ux \cdot uyzu = xzyx \cdot ux\,uzyu = xzyu$. □

LEMMA 21. *The semigroup* $\langle\{0, 1, 2, 3\}, \cdot\rangle$, *in which* $0 \cdot 3 = 2$, $0 \cdot y = 1$ *for* $y \neq 3$, *and* $x \cdot y = x$ *otherwise, generates a non-residually small variety. A non-tautologous identity holds in this semigroup iff both words have length* $\geqslant 2$ *and agree in their first two places (on the left).*

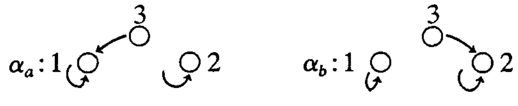
Proof. Call the semigroup T (and verify that it is a semigroup). Take a cardinal κ and let U be the set of functions $f \in {}^{\kappa}T$ such that $f(\alpha) = 1$ for all but at most one $\alpha < \kappa$. Let

$$\theta = \Delta \cup \{(f, g) : 2 \in \text{ran}(f) \cap \text{ran}(g)\}.$$

Verify that θ is a congruence on U and that any congruence $\Psi \geqslant \theta$ which identifies two functions f, g with $0 \in \text{ran}(f) \cap \text{ran}(g)$ must also identify the function $\langle 1 : \alpha < \kappa\rangle$ with all the members of the nontrivial θ-block. Thus U/θ has a s.i. homomorphic image of cardinal κ.

The statement about the identities of T is easily verified. □

DEFINITION 21. We construct two more semigroups, using Definition 14. Let $U = \{1, 2, 3\}$ and let R_2, (L_2) be the right zero (left zero) semigroup with 2-element universe $\{a, b\}$ disjoint from $\{0, 1, 2, 3\}$. Define $\alpha : R_2 \to {}^{U}U$ by



Define $T_0 = R(R_2, U, \alpha)$ and $T_1 = R(L_2, 1)$.

LEMMA 22. (1) *Neither of* $V(T_i)$, $i=0,1$, *is residually small.*

(2) *A law* $\varepsilon_0 \approx \varepsilon_1$ *holds in* T_0 iff *for some letter* x, $\varepsilon_i = \gamma_i \cdot x$ *where* γ_0, γ_1 *have the same letters (hence* γ_0 *is empty iff* γ_1 *is), and if* γ_0 *is not empty and* x *does not occur in* γ_0 *then* γ_0 *and* γ_1 *terminate with the same letter.*

(3) *A non-tautologous law* $\varepsilon_0 \approx \varepsilon_1$ *holds in* T_1 *iff it is balanced and for some letters* x, y_0, y_1, $\varepsilon_i = x \cdot \gamma_i \cdot y_i$ *(*γ_0 *or* γ_1 *may be empty) and either* y_i *occurs in* $x \cdot \gamma_i$ *for each of* $i=0,1$, *or else* $y_0 = y_1$ *and* y_0 *does not occur in* $x \cdot \gamma_0 \cdot \gamma_1$.

Proof. We leave the reader to verify (2) and (3). They are needed in the proof of (1).

To see that $V(T_0)$ is not residually small, let (for any cardinal $\kappa \geq \omega$) F_κ be the free algebra in $V(T_0)$ generated by elements $x_{\alpha\beta}$, y_α $(\alpha<\beta<\kappa)$. Let θ be the congruence on F_κ generated by the relations $x_{\alpha\beta}y_\alpha = x_{\gamma\delta}y_\gamma$ and $x_{\alpha\beta}y_\beta = x_{\gamma\delta}y_\delta$ for $\alpha<\beta<\kappa$, $\gamma<\delta<\kappa$. Using the characterization (2) of the identities of T_0, it's quite easy to show that the set of all elements of F_κ of the form $w \cdot x_{\alpha\beta}y_\alpha$, where w is a possibly empty word containing no letters $y_\delta (\delta<\kappa)$, is a union of θ-blocks and does not include the element $x_{01}y_1$. Thus $(x_{01}y_0, x_{01}y_1) \notin \theta$. However, every congruence $\Psi \geq \theta$ that has $y_\alpha \Psi y_\beta$ for some $\alpha<\beta$ clearly has $x_{01}y_0 \Psi X_{01}y_1$. So F_κ/θ has a s.i. quotient of cardinality κ.

To see that $V(T_1)$ is not residually small, let now F_κ be the free algebra in $V(T_1)$ generated by $x_{\alpha\beta}$, $y_{\alpha\beta}$, u_α $(\alpha<\beta<\kappa)$, and let θ be the congruence generated by the relations

$$x_{\alpha\beta}u_\alpha = x_{01}u_0, \qquad x_{\alpha\beta}u_\beta = y_{\alpha\beta}u_\beta,$$

$$y_{\alpha\beta}x_{01}u_0 = y_{01}x_{01}u_0 = y_{\alpha\beta}y_{01}x_{01}u_0,$$

$$x_{01}u_0 = x_{\alpha\beta}y_{01}x_{01}u_0.$$

Any $\Psi \geq \theta$ that has $u_\alpha \Psi u_\beta$ for some $\alpha<\beta$ has $x_{01}u_0 \Psi y_{01}x_{01}u_0$. ($y_{01}x_{01}u_0 \theta y_{\alpha\beta}x_{\alpha\beta}u_\alpha \Psi y_{\alpha\beta}y_{\alpha\beta}u_\alpha = y_{\alpha\beta}u_\alpha$ in F_κ by (3), and this element is $\Psi y_{\alpha\beta}u_\beta \times \theta x_{\alpha\beta}u_\beta \Psi x_{\alpha\beta}u_\alpha \theta x_{01}u_0$.) To see that $(x_{01}u_0, y_{01}x_{01}u_0) \notin \theta$ show, using (3), that the set of all elements of F_κ of the form $x_{\alpha\beta} \cdot w \cdot u_\delta$ where $x_{\delta\gamma}$ occurs in $x_{\alpha\beta} \cdot w$ for some γ, and where w contains no letter u_η $(\eta<\kappa)$, is θ-closed and does not include $y_{01}x_{01}u_0$. $\square$

LEMMA 23. *Either* $\mathcal{V} \subseteq \mathcal{V}_2^{(n)}$ *or* $\mathcal{V} \subseteq \mathcal{V}_3^{(n)}$ *or else* $\mathcal{V}$ *satisfies* E1(i).

Proof. Assume $\mathcal{V}$ does not satisfy E1(i). Then for sure $\mathcal{V}$ satisfies E2(i) or E3(i) (Lemma 16). We assume E2(i). We also have P1–P3, P6 (lemmas 16, 18, 20).

All the identities of $\mathcal{V}$ are balanced. Otherwise, by substituting n-th powers of variables we derive the law (using P1, P3) $x^n y^n x^n \approx x^n$ holding in $\mathcal{V}$. Then $xy = x \cdot x^n \cdot y$ (by E2(i)) $= x \cdot x^n (x^n y)^n x^n \cdot y = x^{2n+1} y (x^n y)^n = u \cdot e$ where $e = e^2$; so $(u \cdot e)^{n+1} = (ue)^{n+1} e = uee = ue$ by E2(i) and we have derived E1(i), contrary to assumption.

Now we apply Lemma 22. $\mathcal{V}$ satisfies some balanced identity that is not of the form 22(2). This means either $x \approx x^k$ $(k > 1)$; or $\alpha \cdot x \approx \beta \cdot y$ (x, y, z denote distinct variables in this discussion); or $\alpha \cdot x \approx \beta \cdot x$ where $\alpha, \beta \neq \emptyset$ and x occurs in α, not in β; or $\alpha \cdot y \cdot x \approx \beta \cdot z \cdot x$ where x is not in $\alpha \cdot \beta$ (and by balance y is in β, z is in α). The first alternative is out because it gives E1(i). The second leads to $x^n y^n x^n \approx x^n y^n$ by substituting x^n for x, y^n for y, x^n for every other variable, multiplying by x^n on the left and using P1, P3. The third leads to $y^n x = (y^n x)^k$ $(k > 1)$ and then to $y^n x = (y^n x)^{n+1}$ by P2 – we just replace x by $y^n x$ and every other variable by y^n. However, this takes us back to E1(i): $yx = yy^n x = y(y^n x)(y^n x)^n = yxe$ with $e^2 = e$. So the third is out. The fourth alternative leads to $y^n z^n y^n x \approx y^n z^n x$: replace y by y^n, z and every other variable save x by z^n, and multiply by y^n on the left. We have established:

(1) $\mathcal{V}$ satisfies either (i) $x^n y^n x^n \approx x^n y^n$, or (ii) $y^n z^n y^n x \approx y^n z^n x$.

$\mathcal{V}$ satisfies some balanced identity that is not in the form 22(3). This means either $x \approx x^k$ $(k > 1)$; or $x \cdot \alpha \approx y \cdot \beta$; or $x \cdot \alpha \cdot y \approx x \cdot \beta \cdot y$ where y occurs in α, not in β; or $x \cdot \alpha \cdot y \approx x \cdot \beta \cdot x$ with y not in α; or $x \cdot \alpha \cdot y \approx x \cdot \beta \cdot z$ with y not in α. We have ruled out the first alternative. The second leads straightway to $x^n y^n \approx y^n x^n y^n$. The third leads to $(x^n y)^{n+1} \approx x^n y$ which we have seen is ruled out. The fifth gives the fourth after replacing z by x. The fourth gives $x^n y \approx (x^n y)^k x^n$ $(k \geqslant 1)$ and thence $x^n y \approx x^n y x^n$, which with E2(i) gives $x^n y \approx (x^n y)^{n+1}$ and then E1(i) again. We have established

(2) $\mathcal{V}$ satisfies $x^n y^n \approx y^n x^n y^n$.

Finally, (1)(i) above and (2) give $y^n z^n \approx z^n y^n$ which certainly implies E2(ii). But (1)(ii) and (2) also give E2(ii) immediately. □

LEMMA 24. P1–P3, P6 *and* E1(i) *jointly imply* P4 *and* P7′, P8′:
P7′: $x^n yzu \approx x^n y x^n zu$
P8′: $xyzu^n \approx xyu^n zu^n$.

Proof. For P7′ take $e = x^n$, $g = (zu)^n$. Then

$$\begin{aligned} eyzu &= ey(ey)^n gzu \quad \text{by} \quad \text{E1(i)} \\ &= eyee(ey)^n gzu \\ &= ey(ey)^n egzu \quad \text{by} \quad \text{P6} \\ &= eyezu. \end{aligned}$$

P8′ is by duality.

For P4, put $u=x^{n+1}$, $v=y^{n+1}$, $e=x^n=u^n$, $f=y^n=v^n$, $h=(uv)^n$ which we hope equals ef. Clearly, $eu=u$, $fv=v$, $eh=h=hf$. Using P3, P6, $efh=efhf=ehff=h$, $hef=h$. Now

$$(v^{n-1}u^{n-1})(uv)ef=v^{n-1}evef=v^nef=fef$$

by P8′ as $v^{n-1}=f\cdot v^{n-1}$. Thus

$$\begin{aligned}(v^{n-1}u^{n-1})^n\cdot h&=(v^{n-1}u^{n-1})^n uv(uv)^{n-1}ef\\&=(v^{n-1}u^{n-1})^{n-1}v^{n-1}u^{n-1}uv\cdot ef\cdot(uv)^{n-1}ef\end{aligned}$$

by P3, P8′

$$\begin{aligned}&=(v^{n-1}u^{n-1})^{n-1}fef(uv)^{n-1}fef\quad\text{by above}\\&=(v^{n-1}u^{n-1})^{n-1}(uv)^{n-1}ef\\&\quad\cdot\\&\quad\cdot\\&\quad\cdot\\&=v^{n-1}u^{n-1}uvef\\&=fef.\end{aligned}$$

Comparing first and last lines of the computation gives $fef=fefh=fh$ and then $ef=ef\cdot ef=e\cdot fh=h$. □

LEMMA 25. *If* $\mathcal{V}$ *satisfies* E1(i) *then* $\mathcal{V}\subseteq\mathcal{V}_1^{(n)}$.

Proof. The proof will in fact show that $\mathcal{V}_1^{(n)}$ satisfies E2(i) and E3(i) (which can be derived directly from Theorem 4).

Let $\mathcal{V}$ satisfy E1(i). Then by Lemma 24 and the earlier lemmas it satisfies P1–P6, P7′, P8′. (P5 is a special case of E1(i).) We return to Lemma 21, and use it to prove that $\mathcal{V}$ satisfies

(1) (i) $x^ny^nx\approx x^ny^nx^{n+1}$

(ii) $xy^nx^n\approx x^{n+1}y^nx^n$.

$\mathcal{V}$ has an identity that is not of the type specified by Lemma 21. This means either $x\approx x^{n+1}$ or $xx\cdot\alpha\approx x\cdot y\cdot\beta$. (The alternative $x\cdot\alpha\approx y\cdot\beta$ leads to the latter.) The first possibility implies (1)(i), (ii). The second leads to $x\cdot y\cdot\beta\approx x^{n+1}\cdot y\cdot\beta$; in this replace y by y^nx^2 getting $x\cdot y^n\gamma\approx x^{n+1}y^n\gamma$ where γ is a word in y^n and powers x^k $(k>1)$. By P4, P2, γ^n is (actually, is $\mathcal{V}$-equivalent to) a word in x^n, y^n. So we derive $x\cdot y^n\cdot\gamma^n\cdot x^n\approx x^{n+1}y^n\gamma^n\cdot x^n$ which simplifies to (1)(ii) using P3. By duality of assumptions, we also get (1)(i).

Now we derive E3(i). Put $e=(xy)^n$, and note that $ey=ey^{n+1}$.

$$\begin{aligned}(xy)^2 &= exyxy = exeyxy \quad \text{by P7}'\\ &= exeyy^nxy\\ &= exeyy^nx(yx)^ny\\ &= exeyy^nxy^n(yx)^ny \quad \text{by P7}'\\ &= \alpha\cdot y^{n+1} \quad \text{by (1)(i) above.}\end{aligned}$$

Then $(xy)^2=(xy)^2y^n$, so $xy=(xy)^{n-1}(xy)^2=xy^{n+1}$.

Now we can get E1(ii), i.e. P7.

$$\begin{aligned}x^nyz &= x^nyzz^n = x^nyx^nzz^n \quad \text{by P7}'\\ &= x^nyx^nz \quad \text{by E3(i).}\end{aligned}$$

E1(iii) follows analogously. The proof of Theorem 1 is now complete. (See Lemmas 23 and 25.) □

§4. Review and restatement

By Theorems 1, 4, and 5, if a variety $\mathcal{V}$ of semigroups is residually small then the really interesting subdirect irreducibles in $\mathcal{V}$ are groups, groups with 0, and semigroups of the form $R(G, U, \alpha)$ and $L(G, U, \alpha)$ where G is a group in $\mathcal{V}$ and the conditions of Proposition 3 are realized. To understand the possibilities here, we shall now reformulate the conditions of Proposition 3 to eliminate U and α.

DEFINITION 26. A rep-system is a triple (G, H, u) where G is a group, H is a subgroup of G, $u\in G-H$, and the following hold:

(i) $\bigcap(x\cdot H\cdot x^{-1}: x\in G)=\{1\}$.

(ii) Every subgroup K of G, $H<K\subseteq G$, contains u. Equivalently (if G has finite exponent)

$$(\forall x\in G)\Big(x\notin H\rightarrow \bigvee_{m<\omega}\varnothing_m(x)\Big)$$

where $\varnothing_m(x)=\varnothing_m(x, u, H)$ is the formula

$$(\exists h_1,\dots,h_m, h\in H)(u=x^{h_1}\cdots x^{h_m}\cdot h)$$

and here $x^y=y\cdot x\cdot y^{-1}$ by definition.

For any rep-system $\Sigma = (G, H, u)$, we let $U^\Sigma = \{x \cdot H : x \in G\}$, and we let $\alpha_y^\Sigma(x \cdot H) = y \cdot x \cdot H$.

PROPOSITION 27. *A semigroup of the form $R(G, U, \alpha)$ is subdirectly irreducible iff it is isomorphic to $R(G, U^\Sigma, \alpha^\Sigma)$ for some rep-system $\Sigma = (G, H, u)$.*

Proof. Easy, by Proposition 3: It is known from group theory that every faithful, transitive representation of a group is isomorphic to a representation that has the group acting naturally as translations on the set of left cosets of a certain subgroup. Condition 26(i) is the requirement for faithfulness. Condition 26(ii), for some $u \in G - H$, is the equivalent of Proposition 3, statement 3. □

For the next lemma, recall that we are using $\mathcal{G}^{(n)}$ to denote the variety of exponent n groups, that is, semigroups satisfying the laws $x \approx x^{n+1}$ and $x^n \approx y^n$. It is clear that a semigroup belonging to any one of the varieties $\mathcal{V}_j^{(n)}$ $(j = 1, 2, 3)$ is in $\mathcal{G}^{(n)}$ just in case it is a group. We denote by $G[A]$ the direct product of all of the maximal subgroups of A, where A is any semigroup. Thus, in the notation of Definition 6

$$G[A] = \prod (A_e : e \in I(A)).$$

LEMMA 28. *Let $A \in \mathcal{V}_j^{(n)}$. Then $V(A) \cap \mathcal{G}^{(n)} = V(G[A])$, or in other words, a group G belongs to $V(A)$ iff G satisfies all identities that hold in every subgroup of A.*

Proof. It is clear that $V(G[A]) \subseteq V(A) \cap \mathcal{G}^{(n)}$. To get the other inclusion, we let ε be any identity that holds in all maximal subgroups of A, and we show that the class $\mathcal{K}$ consisting of all $B \in \mathcal{V}_j^{(n)}$ such that every maximal subgroup of B satisfies ε, is a variety. Thus $V(A) \subseteq \mathcal{K}$ and, in particular, $V(A) \cap \mathcal{G}^{(n)}$ satisfies ε.

In proving that $\mathcal{K}$ is a variety, we need only show that $HSP\mathcal{K} = \mathcal{K}$. That $P\mathcal{K} = \mathcal{K}$ is easy, using Lemma 7: every maximal subgroup of $\prod (B_i : i \in I)$, where all $B_i \in \mathcal{V}_j^{(n)}$, is a direct product of maximal subgroups of the B_i. That $S\mathcal{K} = \mathcal{K}$ is also easy: if $B \subseteq C \in \mathcal{V}_j^{(n)}$ and $e \in I(B)$ then $B_e \subseteq C_e$. That $H\mathcal{K} = \mathcal{K}$ is a little harder: Let ϕ map C homomorphically onto B, where $C \in \mathcal{K}$. Let B_e be any maximal subgroup of B, with $e = e^2$ in B. Let ε have the form $w_0(x_1, \ldots, x_k) \approx w_1(x_1, \ldots, x_k)$. Choose orbitrary elements $u_1, \ldots, u_k \in B_e$ and elements $\bar{u}_i \in C$ with $\phi(\bar{u}_i) = u_i$. Put $f = \prod_1^k \bar{u}_i^n$ and put $\bar{v}_j = f \cdot \bar{u}_j^{n+1} \cdot f$. By Proposition P and Lemma 7, we have $f \in I(C)$, $\bar{v}_j \in C_f$ $(j = 1, \ldots, k)$, and moreover $\phi(f) = e$, $\phi(\bar{v}_j) = u_j$

obviously. Since ε holds in C_f, we obtain that

$$\begin{aligned} w_0(u_1, \ldots, u_k) &= \phi(w_0(\bar{v}_1, \ldots, \bar{v}_k)) \\ &= \phi(w_1(\bar{v}_1, \ldots, \bar{v}_k)) \\ &= w_1(u_1, \ldots, u_k). \end{aligned}$$

As $e \in I(B)$ and $u_1, \ldots, u_k \in B_e$ are arbitrary, we have proved that $B \in \mathcal{K}$. □

In the following, main, theorem of this section, the phrase "res. small" can be replaced by "res $<\kappa$," or by "res $\ll \kappa$" for any fixed infinite cardinal κ, with the appropriate reformulation of the third condition of the theorem.

THEOREM 29. *$V(A)$ is res. small (A is a semigroup) iff A has finite exponent, say n, and the following hold:*

(1) *A satisfies one of the systems of identities* E1, E$_2$, E3 *defininig $\mathcal{V}_i^{(n)}$ ($i = 1, 2, 3$).*

(2) *$V(G[A])$ is a res. small variety of groups.*

(3) *If the law $(x \cdot y)^{n+1} \approx x \cdot y$ does not hold in A, then the groups $G \in V(G[A])$ which admit a rep-system (G, H, u) are of bounded cardinality.*

Proof. The necessity of (1) is Theorem 1, of (2) is obvious. For the necessity of (3), notice that if (1) holds but (3) fails then by Theorem 1, A belongs to $\mathcal{V}_2^{(n)}$ or $\mathcal{V}_3^{(n)}$, say $A \in \mathcal{V}_3^{(n)}$. Then by the dual of Theorem 5, $V(A)$ is generated by s.i. semigroups each of which is isomorphic to Z_2, $\not{Z}_2$, L_2, $L_2^{(0)}$ or to G, $G^{(0)}$, or $L(G, U, \alpha)$ for some group G. As A fails to satisfy the law E1(i), some $L(G, U, \alpha)$ must belong to $V(A)$. It is easy to see that $V(L(G, U, \alpha)) = V(G, L(1, 1))$; it can be done by checking the structure of identities. Hence $L(1, 1) \in V(A)$ and $L(G, U, \alpha) \in V(A)$ whenever $G \in V(G[A])$. The necessity of (3) then follows by Proposition 27. (Notice that the concept of rep-system is self-dual.)

The sufficiency of (1–3) is by Theorems 4, 5, Prop. 27, and Lemma 28.

§5. Finite semigroups

In this section we prove

THEOREM 30. *If $V(A)$ is res. finite where A is a finite semigroup, then $V(A)$ is res $\ll \omega$. (That is $V(A)$ has only finitely many non-isomorphic s.i. semigroups.)*

The necessary lemma, which we prove by a model theoretic argument inspired by Taylor's paper [9], is

LEMMA 31. *Assume that F is a finite group and that there is no finite bound on the cardinalities of the rep-systems $\Sigma = (G, H, u)$ in which $G \in V(F)$. Then there exists an infinite group $G \in V(F)$ which admits a rep-system.*

Proof of 30 (*given* 31). Let $V(A)$ be res. finite. Then (1) of Theorem 29 holds, and $V(G[A])$ is res. finite also. By the main result of [4], or of [9], $V(G[A])$ is res $\ll \omega$. By Lemmas 28 and 31 and Proposition 27 and its dual, if A fails E1(i) then there must be a finite bound on the cardinalities of s.i. semigroups $R(G, U, \alpha)$ or $L(G, U, \alpha)$ belonging to $V(A)$. Hence, finally, by Theorems 4, 5 or by the "res $\ll \omega$" version of Theorem 29, $V(A)$ is res $\ll \omega$.

Proof of 31. Let F be a finite group having no finite bound on the size of the *finite* rep-systems (G, H, u), $G \in V(F)$. (If there exists an *infinite* such rep-system, we are done.) Define an *F-system* to be any structure

$$\Sigma = (G, H, u, N_1, \ldots, N_k)$$

such that k is finite; $G \in V(F)$; (G, H, u) is a rep-system (Definition 26);

$$u \in N_1 < N_2 < \cdots < N_k = G;$$

$N_i \lhd G$ (N_i is a normal subgroup of G) for $i = 1, \ldots, k$; and such that

$$1 < [H \cdot N_{i+1} : H \cdot N_i] \leq |F|$$

for $i = 1, \ldots, k-1$, and $1 < [H \cdot N_1 : H] \leq |F|$.

Notice that for any F-system as above, $2^k \leq [G:H] \leq |F|^k$.

LEMMA 32. *Every finite rep-system (G, H, u), $G \in V(F)$, gives rise to an F-system $(G, H, u', N_1, \ldots, N_k)$. Here $|G| \leq |F|^{k \cdot |F|^k}$.*

Proof. It is well known that G is isomorphic to a factor group of a subgroup of a finite direct power of F. Thus we can assume that $G = \bar{G}/\bar{N}$ where $\bar{G} \subseteq {}^mF$ and $\bar{N} \lhd \bar{G}$. We can write $H = \bar{H}/\bar{N}$, $\bar{N} \subseteq \bar{H}$, $u = \bar{u}/\bar{N}$, $\bar{u} \in {}^mF$. Define

$$\bar{T}_i = \{f \in {}^mF : f_j = 1 \text{ for all } j \geq i\}.$$

Let $1 \leqslant t_1 < t_2 < \cdots < t_k \leqslant m$ be the unique sequence of integers such that

$$\begin{aligned}\bar{H} &= \bar{H}\cdot(\bar{G}\cap\bar{T}_0) = \cdots = \bar{H}\cdot(\bar{G}\cap\bar{T}_{t_1-1})\\ &< \bar{H}\cdot(\bar{G}\cap\bar{T}_{t_1}) = \cdots = \bar{H}\cdot(\bar{G}\cap\bar{T}_{t_2-1})\\ &< \bar{H}\cdot(\bar{G}\cap\bar{T}_{t_2})\cdots\bar{H}\cdot(\bar{G}\cap\bar{T}_{t_k-1})\\ &< \bar{H}\cdot(\bar{G}\cap\bar{T}_{t_k}) = \bar{G}.\end{aligned}$$

Define $\bar{N}_j = \bar{N}\cdot(\bar{G}\cap\bar{T}_{t_j})$ for $j = 1, \ldots, k-1$ and $\bar{N}_k = \bar{G}$, and put $N_j = \bar{N}_j/\bar{N}$ for $j = 1, \ldots, k$.

We clearly have that

$$H\cdot N_j = [\bar{H}\cdot(\bar{G}\cap\bar{T}_{t_j})]/\bar{N} \quad (1 \leqslant j \leqslant k)$$

and therefore

$$\{1\} < N_1 < \cdots < N_k = G,$$

the last equality following from the definition.

The index of $H\cdot N_j$ in $H\cdot N_{j+1}$, or $[H\cdot N_{j+1} : H\cdot N_j]$, is the same as $[\bar{H}\cdot(\bar{G}\cap \bar{T}_{t_j+1}) : \bar{H}\cdot(\bar{G}\cap\bar{T}_{(t_j+1)-1})]$ which is no greater than $[\bar{T}_a : \bar{T}_{a-1}] = |F|$ (where $a = t_{j+1}$), and is at least 2. The same calculation holds for $[H\cdot N_1 : H]$.

Since $H\cdot N_1 > H$, we can write $u = h\cdot u'$ with $h \in H$, $u' \in N_1$. Then (G, H, u') is clearly a rep-system.

The bound on the cardinality of G follows from the fact that G is faitfully represented as permutations of a set of cardinality $b = [G:H]$, and $[G:H] \leqslant |F|^k$. □

The next lemma is a consequence of the preceding one and of our assumption about F.

LEMMA 33. *There is no finite bound on the length k of finite F-systems* $\Sigma = (G, H, u, N_1, \ldots, N_k)$.

LEMMA 34. *For an F-system $(G, H, u, N_1, \ldots, N_k)$, any element $x \in N_a - H$ satisfies the formula $\varnothing_{|F|^a}$ from Definition* 26.

Proof. We know that x satisfies $\varnothing_t$ for some positive integer t. Take t minimal and write

$$u = x^{h_1}\cdots x^{h_t}\cdot h \qquad (h_i, h \in H).$$

Notice that $[N_a : N_a \cap H] = [H \cdot N_a : H] \leqslant |F|^a$. Assuming then that $t > |F|^a$, and using that N_a is normal, some two of the products $x^{h_1} \cdots x^{h_i}$ $(i \leqslant t)$ are in the same left co-set of $N_a \cap H$; we have then

$$x^{h_1} \cdots x^{h_j} = x^{h_1} \cdots x^{h_i} \cdot h'$$

where $h' \in H$ and $1 \leqslant i < j \leqslant t$. Then

$$u = x^{h_1} \cdots x^{h' \cdot h_{j+1}} \cdots x^{h' \cdot h_t} \cdot h' \cdot h$$

showing that x satisfies $\varnothing_{t-j+i}$, and giving us a contradiction. □

We are ready to conclude the proof of Lemma 31 with a simple model-theoretic argument. Readers unversed in the rudiments of model theory and logic are referred to Chang–Keisler [3], especially chapters 1 and 2 and the compactness theorem on page 67.

A first order language appropriate to our argument has symbols which can be used to denote the basic operations, $\cdot$ and $^{-1}$, of a group G; it has unary relation symbols H and N_j $(1 \leqslant j < \omega)$ which can be used to denote (the universes of) subgroups of G; and it has constant symbols 1 (for the identity element) and u. Within such a language we can write out a set of sentences T expressing exactly the following:

(T1) G is a group and belongs to $V(F)$. (These sentences are identities.)

(T2) H is a subgroup and the N_j are normal subgroups of G.

(T3) $N_j \subseteq N_{j+1} \nsubseteq H \cdot N_j$ for $1 \leqslant j < \omega$.

(T4) $u \in N_1 - H$.

(T5) For $1 \leqslant j < \omega$, $(\forall x \in N_j - H)\varnothing_{|F|^j}(x)$.

From Lemmas 33 and 34 we see that every finite subset of the set T of sentences has a model. By the compactness theorem, T has a model. Let $(G, H, u, N_j\ (1 \leqslant j < \omega))$ be a model of T.

From this model we now construct an infinite group in $V(F)$ which admits a rep-system. We take

$$G' = H \cdot \left(\bigcup_{n<\omega} N_n\right)$$

$$N' = \bigcap (x \cdot H \cdot x^{-1} : x \in G')$$

$$\mathcal{J} = (\bar{G}, \bar{H}, \bar{u}) = (G'/N', H/N', u/N').$$

The claim is that $\bar{G} \in V(F)$ is infinite and that $\mathcal{J}$ is a rep-system. Some verification have to be made.

It is obvious that $\bigcup_{n<\omega} N_n$ is a normal subgroup of G, and that N' is a normal subgroup of G', and that $N' \leqslant H \leqslant G'$, and $u \in G'$. Thus $\mathcal{J}$ is well-defined. Also $\bar{G} \in V(F)$, clearly and it is infinite because $[G':N'] \geqslant [N_k : N_k \cap H] = [H \cdot N_k : H] \geqslant 2^k$ for each positive integer k (which is a consequence of T3).

Referring back to Definition 26, three things remain to be shown. First, $\bar{u} \notin \bar{H}$, simply because $u \notin H \cdot N' = H$. Second, $\bigcap(\bar{x} \cdot \bar{H} \cdot \bar{x}^{-1} : \bar{x} \in \bar{G}) = \{\bar{1}\}$, which follows easily from the definition of N'. Third, each element $\bar{x} \in \bar{G} - \bar{H}$ must satisfy some formula $\varnothing_m$. Suppose that $\bar{x} = x/N'$ and, without losing generality, allow that $x = h \cdot y$, $y \in N_k$ (look at the definition of G'). Clearly, $y \notin H$. By $T5$, we have $\varnothing_m(y, u, H)$ holding in G, where $m = |F|^k$. An easy calculation shows that $\varnothing_m(y)$ implies $\varnothing_m(h \cdot y)$ if $h \in H$. It follows readily that $\varnothing_m(\bar{x}, \bar{u}, \bar{H})$ holds in $\bar{G}$.

This concludes our proof of Lemma 31. □

§6. Open questions

The class of cardinalities of the members of a class of algebras $\mathcal{K}$ is called the spectrum of $\mathcal{K}$. It is well known that the spectrum of the class of subdirectly irreducible algebras in a variety of semigroups takes the form of a union of a set of positive integers, with a convex class of infinite cardinals. For the class of infinite cardinals there are only four possibilities: the empty set, $\{\omega\}$, $[\omega, 2^\omega]$, the class of all infinite cardinal numbers. (These results, holding more generally for any variety of algebras with countably many basic operations, are proved by model theoretic arguments in [8] and [6].) The possible residual behaviour (spectrum of subdirect irreducibles) of any semigroup variety is limited by the restrictions just stated.

The work reported in this paper gives no information at all on the finite spectrum in those cases where the infinite spectrum is unbounded. In all other cases, our work reduces everything to the following question. Let $\mathcal{G}$ be any residually small variety of groups of finite (bounded) exponent. Let $\mathcal{S}(\mathcal{G})$ denote the class of s.i. groups in $\mathcal{G}$ and let $\mathcal{R}(\mathcal{G})$ denote the class of all rep-systems (G, H, u) where $G \in \mathcal{G}$. The spectrum of subdirect irreducibles for the semigroup varieties associated with $\mathcal{G}$ throughout this paper can be any one of the following, and no other (we ignore cardinals less than 4): (1) Spectrum $(\mathcal{S}(\mathcal{G}))$; (2) $\{\kappa : \kappa$ or $\kappa - 1$ is in Spectrum $(\mathcal{S}(\mathcal{G}))\}$; (3) the union of the class (2) with the class of all cardinals $1 + \kappa + \lambda$ where for some $(G, H, u) \in \mathcal{R}(\mathcal{G})$, we have $\kappa = |G|$ and $\lambda = [G:H]$.

QUESTION 1. What classes are realized as (1), (2), or (3) for arbitrary varieties $\mathcal{G}$ of finite exponent?

This question is no doubt hopeless, but we have some others.

QUESTION 2. Does there exist a finite group F, with abelian Sylow subgroups, such that $\mathcal{R}(V(F))$ is unbounded? – is bounded but includes rep-systems of size ω? – of size 2^{ω}?

QUESTION 3. Is it possible for one, but not both, of $\mathcal{S}(\mathcal{G})$ and $\mathcal{R}(\mathcal{G})$ to be bounded?

PROBLEM 4. Characterize the class of finite groups F such that $\mathcal{R}(V(F))$ is a finite set of finite systems, up to isomorphic systems. (This problem is certainly amenable through the methods of proof used in Section 5. The problem is really to find a very nice characterization.)

QUESTION 5. Does there exist any variety of group or of semigroups which is res $<\omega$ and not res $\ll\omega$? Can it be a locally finite variety?

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University of California
Berkeley, California
U.S.A.