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Varieties of relation algebras

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1. Introduction

Relation algebras, as defined by Alfred Tarski, constitute a variety with several very strong structural properties. Most of these properties are consequences of the fact that relation algebras form a discriminator variety (see Werner [1978]). However, we shall make no use of the theory of discriminator varieties, for all the consequences that we need can be derived quite easily from the following three facts, which are easy consequences of Tarski's axioms. First, relation algebras are congruence distributive. Second, for relation algebras, the property of being simple, of being subdirectly irreducible, and of being directly indecomposable are equivalent, and the relation algebras with these properties constitute a universal class. Thirdly, there is an effective way of associating with each open formula ϕ a term ϕ^* in such a way that for simple relation algebras the formulas ϕ and $\phi^* = 1$ are equivalent.

All these facts were known to Tarski as early as the 1940's, and they have been used by him and his collaborators. However, much of the work in this area was done at a time when the general theory of varieties was in its early stages, and many of the ideas, techniques and results from that theory were therefore not available. It is the purpose of this paper to re-examine and extend some of the known facts about relation algebras, making use of these more recent developments. Our primary concern will be varieties of relation algebras. In particular, we shall give simple equational bases for several interesting varieties, and prove a number of theorems about the lattice of all varieties of relation algebras. Among other things it will be shown that this lattice has infinitely many dual atoms, the conjugate varieties of the full relation algebras on finite sets, and that these varieties have simple equational bases (Theorems 7.5 and 7.7).

In order to make this paper more nearly selfcontained, considerable space has been devoted to a summary of known results about relation algebras. It is hoped

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that this brief survey will also serve a useful purpose as a key to the earlier literature, which we feel is not as widely known as it deserves to be.

2. Arithmetic properties of relation algebras

We recall Tarski's definition of a relation algebra, and list without proofs some of the arithmetic consequences of his axioms. These results may be found in Chin, Tarski [1951].

DEFINITION 2.1. *By a relation algebra* we mean an algebra

$$\mathscr{A} = (\mathscr{A}_0, ;, 1', \breve{\ })$$

such that

(i) $\mathscr{A}_0 = (A, +, 0, \cdot, 1, ^-)$ is a Boolean algebra.
(ii) $(A, ;, 1', \breve{\ })$ is an involuted monoid.
(iii) The operations ; and $\breve{\ }$ are distributive over +.
(iv) For all $x, y \in A$, $\breve{x} ; (x ; y)^- \leq y^-$.
We denote by **RA** the class of all relation algebras.

In greater detail, the condition (ii) means that, for all $x, y, z \in A$,

$$x ; (y ; z) = (x ; y); z, \qquad x ; 1' = 1' ; x = x,$$
$$(x ; y)\breve{} = \breve{y} ; \breve{x}, \qquad \breve{\breve{x}} = x,$$

while (iii) means that

$$(x + y) ; z = (x ; z) + (y ; z), \qquad (x + y)\breve{} = \breve{x} + \breve{y}$$

and, consequently, $x ; (y + z) = (x ; y) + (x ; z)$. Observe that the inclusion (iv) can also be written as an identity, since in any lattice the conditions $u \leq v$, $u + v = v$ and $uv = u$ are equivalent.

The Boolean algebra $\mathscr{A}_0$ is called the *Boolean part* of the relation algebra $\mathscr{A}$, the operations ; and $\breve{\ }$ are called the *relative multiplication* and the *conversion*, the element $1'$ is called the *identity element*, and the element $1'^-$, denoted by $0'$, is called the *diversity element*. When concepts from the theory of Boolean algebras are applied to a relation algebra $\mathscr{A}$, it is understood that they refer to the Boolean part $\mathscr{A}_0$ of $\mathscr{A}$. Thus, by an *atom* of $\mathscr{A}$, or an *ideal* of $\mathscr{A}$, we mean an atom of $\mathscr{A}_0$, or an ideal of $\mathscr{A}_0$, and we say that $\mathscr{A}$ is *atomic*, or is *complete*, if $\mathscr{A}_0$ is atomic, or complete, respectively.

DEFINITION 2.2. An element a of a relation algebra is called

(i) an *equivalence element* if $a\,;a = a = a^{\smile}$,
(ii) a *left ideal element* if $1\,;a = a$,
(iii) a *right ideal element* if $a\,;1 = a$,
(iv) an *ideal element* if $1\,;a\,;1 = a$,
(v) a *functional element* if $a^{\smile}\,;a \leq 1'$.

THEOREM 2.3. *The following properties hold in every relation algebra:*

(i) $0^{\smile} = 0,\ 1^{\smile} = 1,\ 1'^{\smile} = 1',\ 0'^{\smile} = 0'$.
(ii) $(xy)^{\smile} = x^{\smile}y^{\smile},\ x^{-\smile} = x^{\smile -}$.
(iii) $0;\ x = x;\ 0 = 0,\ 1;\ 1 = 1$.
(iv) *If the join* $u = \sum x_i$ *exists, then the joins* $\sum x_i^{\smile}, \sum (x_i\,;y)$ *and* $\sum (y\,;x_i)$ *also exist, and they are equal to* $u^{\smile}$, $u\,;y$ *and* $y\,;u$, *respectively.*
(v) *The conditions*

$$(x\,;y)z = 0, \qquad (x^{\smile}\,;z)y = 0, \qquad (z\,;y^{\smile})x = 0$$

are equivalent.
(vi) $(x\,;y)z \leq x\,;((x^{\smile}\,;z)y)$.
(vii) $x \leq x\,;x^{\smile}\,;x$.
(viii) $x^{\smile} = \sum (y,\ y\,;x \leq 0')$
(ix) *If* $x \leq 1'$, *then* x *is an equivalence element.*
(x) *If* x *is an equivalence element, then so is* $x^-\,;x^-$.
(xi) $0'\,;0'$ *is an equivalence element.*
(xii) *If* $x \leq 1'$ *and* $y \leq 1'$, *then* $x\,;y = xy$.
(xiii) *If* $x \leq 1'$ *and* $y \leq 1'$, *then*

$$(xy)\,;z = (x\,;z)(y\,;z), \qquad z\,;(xy) = (z\,;x)(z\,;y).$$

(xiv) $1\,;x\,;1$, $1\,;x$ *and* $x\,;1$ *are, respectively, an ideal element, a left ideal element, and a right ideal element.*
(xv) *If* x *and* y *are ideal elements, then so are* $x+y$, xy *and* x^-. *The corresponding statements for left ideal elements and right ideal elements are also true.*
(xvi) *Every ideal element is an equivalence element.*
(xvii) *If* x *is an ideal element, then* $x\,;y = y\,;x = xy$.
(xviii) *If* x *is an ideal element, then* $x(y\,;z) = (xy)\,;z$.

THEOREM 2.4. *An algebra*

$$\mathscr{A} = (\mathscr{A}_0, ;, 1', ^{\smile})$$

is a relation algebra iff

(i) $\mathscr{A}_0 = (A, +, 0, \cdot, 1, ^-)$ *is a Boolean algebra.*
(ii) $(A, ;, 1')$ *is a monoid.*
(iii) *For all* $z, y, z \in A$, *the conditions*

$$(x ; y)z = 0, \qquad (x^{\smile} ; z)y = 0, \qquad (z ; y^{\smile})x = 0$$

are equivalent.

DEFINITION 2.5. A relation algebra $\mathscr{A}$ is said to be

(i) *Boolean* if $x ; y = xy$ for all $x, y \in A$.
(ii) *symmetric* if $x^{\smile} = x$ for all $x \in A$
(iii) *commutative* if $x ; y = y : x$ for all $x, y \in A$.
(iv) *integral* if $x ; y \neq 0$ for all $x, y \in A$ with $x \neq 0 \neq y$.

Clearly every symmetric relation algebra is commutative. Boolean relation algebras can be alternatively characterized by the condition that $1' = 1$, and from this it is seen that a Boolean relation algebra is nothing more than a Boolean algebra with the operations ;, $1'$ and $^{\smile}$ defined "trivially" by the formulas

$$x ; y = xy, \qquad 1' = 1, \qquad x^{\smile} = x.$$

We have chosen to define a relation algebra to be a structure with eight primitive operations, three binary, two unary, and three nullary, although a much smaller set of operations would have sufficed. As regards the interdependence of the Boolean operations, the situation is well known. The identity element does not have to be listed as a primitive concept, since the semigroup $(A, ;)$ has at most one neutral element. Less obviously, 2.3(viii) shows that the operation $^{\smile}$ could also be omitted. It should be noted that neither $1'$ nor $^{\smile}$ can be equationally defined in terms of the other operations, and that the omission of either, or both, would lead to a class of algebras that is not a variety. However, when this aspect of **RA** is not important, we sometimes speak of a relation algebra $(\mathscr{A}_0, ;, ^{\smile})$, or $(\mathscr{A}_0, ;, 1')$, or $(\mathscr{A}_0, ;)$.

By the *right* and *left residuals* of an element x over an element y, – in symbols $y \backslash x$ and x/y, – we mean the largest elements u and v such that

$$y ; u \leq x \quad \text{and} \quad v ; y \leq x.$$

Such elements exist in every relation algebra; indeed, they are given by the formulas

$$y\backslash x = (y^{\smile}; x^{-})^{-}, \qquad x/y = (x^{-}; y^{\smile})^{-}.$$

The two operations of residuation can also be taken as primitive notion in place of the conversion. This is done in Birkhoff [1948] and [1967], where relation algebras are fitted into the general framework of residuated lattices. The converse is then defined by either one of the equivalent formulas

$$x^{\smile} = 0'/x^{-}, \qquad x^{\smile} = x^{-}\backslash 0'.$$

3. Construction of relation algebras

We review here and comment on several of the methods that have been used to construct relation algebras.

I. *The full algebra of binary relations.* The algebra $\mathfrak{R}(X)$ of all binary relations on a set X is the most important example of a relation algebra. The Boolean part of this algebra is the set-field consisting of all subsets of X^2, the relative product of two relations R and S, and the converse of R, are defined by the fomulas

$$R; S = \{(x, y) : xRzSy \text{ for some } z\},$$
$$R^{\smile} = \{(x, y) : yRx\},$$

and the identity element is the identity relation on X. This algebra was characterized in J. C. C. McKinsey [1940]. After the general notion of a relation algebra had been introduced, a simpler characterization was given in Jónssons, Tarski [1952, Theorem 4.30].

THEOREM 3.1. *A relation algebra $\mathcal{A}$ is isomorphic to $\mathfrak{R}(X)$ for some set X iff $\mathcal{A}$ is complete and atomic and $p; 1; p^{\smile} \leq 1'$ and $1; p; 1 = 1$ for every atom p of $\mathcal{A}_0$.*

II. *Operations on relation algebras.* For a class $\mathbf{K}$ of algebras, let $\mathbf{P}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$ and $\mathbf{S}(\mathbf{K})$ be the classes consisting of all algebras that are isomorphic, respectively, to direct products, epimorphic images, and subalgebras of algebras in $\mathbf{K}$. Since $\mathbf{RA}$ is a variety, we have

$$\mathbf{P}(\mathbf{RA}) = \mathbf{H}(\mathbf{RA}) = \mathbf{S}(\mathbf{RA}) = \mathbf{RA}.$$

DEFINITION 3.2. By a *concrete* relation algebra we mean a subalgebra of a direct product of full algebras of binary relations. By a *representation* of a relation algebra $\mathscr{A}$ we mean an isomorphism from $\mathscr{A}$ to a concrete relation algebra, and we say that $\mathscr{A}$ is *representable* if such an isomorphism exists. We denote by **RRA** the class of all representable relation algebras.

If e is an equivalence element of a relation algebra $\mathscr{A}$, then the sets

$$Ae = (e] = \{x \in A : x \leq e\},$$
$$e ; A ; e = \{e ; x ; e : x \in A\} = \{y \in A : y = e ; y ; e\}$$

are closed under relative multiplication and conversion, as well as under the Boolean operations of join and meet. In fact, each of the two sets is a relation algebra under these operations. We denote these algebras by $\mathscr{A}e$ and $e ; \mathscr{A} ; e$. In $\mathscr{A}e$, the unit element is e, the complement of an element x is $x^{-}e$, and the identity element is $1'e$, while the corresponding elements of $e ; \mathscr{A} ; e$ are $e ; 1 ; e$, $x^{-}(e ; 1 ; e)$, and e. The algebra $\mathscr{A}e$ is called a *relative subalgebra* of $\mathscr{A}$, although it is not a subalgebra unless $e = 1$. Similarly, $e ; \mathscr{A} ; e$ is not a subalgebra of $\mathscr{A}$ unless $e = 1'$. If $\mathscr{A}$ is representable, then so are $\mathscr{A}e$ and $e ; \mathscr{A} ; e$. This follows from the fact that if E is an equivalence element of $\mathscr{R}(X)$, i.e., an equivalence relation on a subset of X, then $\mathscr{R}(X)E$ is isomorphic to the direct product of the algebras $\mathscr{R}(Y)$ with Y running through the blocks of E, and $E ; \mathscr{R}(X) ; E$ is isomorphic to $\mathscr{R}(Z)$, where Z is the set of all blocks of E.

III. *Diagrams.* A finite relation algebra or, more generally, a complete and atomic relation algebra, is completely determined by the action of the relative multiplication on the set P of all atoms. In other words, it suffices to construct a table giving the values of $p ; q$ for $p, q \in P$, for any such partial operation can be uniquely extended to an operation on the whole algebra, subject to the condition that it be completely distributive over Boolean joins. In order for the resulting algebra to be a relation algebra, the partial operation must obviously be subjected to some conditions, and these conditions can be more conveniently formulated if we assume that the conversion has also been specified, as a permutation of P, and that the element $1'$ has been given, as a join of atoms.

THEOREM 3.3. *Suppose $\mathscr{A} = (\mathscr{A}_0, ;, 1', \breve{\ })$ is a complete and atomic Boolean algebra with operators, with $1'$ a distinguished element, and ; and $\breve{\ }$ operations of rank 2 and 1, respectively, that are completely distributive over Boolean joins. Let P be the set of all atoms of $\mathscr{A}_0$. Then $\mathscr{A}$ is a relation algebra iff the following conditions hold for all $p, q, r \in P$:*

$$p\breve{\ } \in P, \qquad p ; (q ; r) \leq (p ; q) ; r, \qquad 1' ; p = p$$
$$p \leq q ; r \quad \text{implies} \quad p\breve{\ } \leq r\breve{\ } ; q\breve{\ } \quad \text{and} \quad q \leq p ; r\breve{\ }.$$

Proof. The displayed conditions are obviously necessary. Conversely, suppose these conditions are satisfied. We first note that, for all $p \in P$,

$$p^{\smile\smile} = p. \tag{1}$$

Indeed, since $p = 1';p$, there exists an atom $r \leq 1'$ such that $p = r;p$. Hence $r \leq p;p^{\smile}$, and consequently $p \leq r;p^{\smile\smile} \leq p^{\smile\smile}$. Since both p and $p^{\smile\smile}$ are atoms, this implies that $p^{\smile\smile} = p$. It is now easy to show that, for all $p, q, r \in P$,

$$p \leq q;r \quad \text{iff} \quad p^{\smile} \leq r^{\smile};q^{\smile}, \tag{2}$$

$$p \leq q;r \quad \text{iff} \quad q \leq p;r^{\smile}, \tag{3}$$

$$p \leq q;r \quad \text{iff} \quad r \leq q^{\smile};p, \tag{4}$$

$$(p;q)^{\smile} = q^{\smile};p^{\smile}. \tag{5}$$

From (1) and (5) it follows that the equations $a^{\smile\smile} = a$ and $(a;b)^{\smile} = b^{\smile};a^{\smile}$ hold for arbitrary elements a and b, and from (3) and (4) we infer that the conditions

$$(a;b)c = 0, \qquad (a^{\smile};c)b = 0, \qquad (c;b^{\smile})a = 0$$

are equivalent. Since the inclusion $a;(b;c) \leq (a;b);c$ holds whenever a, b and c are atoms, it holds for arbitrary elements of the algebra, and the opposite inclusion is obtained by replacing a, b and c by $c^{\smile}$, $b^{\smile}$ and $a^{\smile}$. Finally, noting that $1'$ is a left identity element, we infer that $1'^{\smile}$ is a right identity element, and conclude that $1' = 1'^{\smile}$ is an identity element.

By 2.4, $\mathscr{A}$ is therefore a relation algebra.

Again suppose we are presented with a table for $p;q$, with p and q running through the atoms of a complete and atomic Boolean algebra. In order to determine whether the resulting Boolean algebra with operators is a relation algebra, we must first determine the potential identity element $1'$ and the conversion operation. The identity element, if one exists, must be the join of all atoms p such that $p;q \leq q$ for all $q \in P$, and the converse of an atom p must be the unique atom r such that $r;p^{-} \leq 0'$. As an illustration, consider McKenzie's [1970] example of a non-representable relations algebra. This algebra has four atoms, a, b, c and d, and their relative products are given by the following table:

;	a	b	c	d
a	a	b	c	d
b	b	b	1	$b+d$
c	c	1	c	$c+d$
d	d	$b+d$	$c+d$	d^{-}

In this case $1' = a$ and

$$a^\smile = a, \qquad b^\smile = c, \qquad c^\smile = b, \qquad d^\smile = d.$$

IV. *Algebras of complexes.* If R is a relation of rank $n+1$ on a set U, then we can form an operation R^Δ of rank n on the family of all subsets of U by letting $R^\Delta(X_0, X_1, \ldots, X_{n-1})$ be the set of all $q \in U$ such that $(p_0, p_1, \ldots, p_{n-1}, q) \in R$ for some $p_i \in X_i$ $(i \in n)$. By the *algebra of complexes* of a relational structure $\mathfrak{U} = (U, R_i, (i \in I))$, – in symbols $\mathscr{C}(\mathfrak{U})$, – we mean the algebra $(\mathscr{C}_0(U), R_i^\Delta, (i \in I))$ where $\mathscr{C}_0(U)$ is the Boolean algebra of all subsets of U. Obviously the operations R_i^Δ are completely distributive over Boolean joins, i.e., over set-theoretic unions.

In order to obtain algebras of complexes that are relation algebras, we should consider relational structures $\mathfrak{U} = (U, R, S, E)$ where R is a ternary relation, S is a binary relation (in fact a unary operation), and E is a unary relation (a subset of U). If we wish to omit the identity element or the conversion, or both, from the list of primitive notions, we can also consider structures (U, R, S), or (U, R, E), or (U, R). The necessary and sufficient conditions for $\mathscr{C}(\mathfrak{U})$ to be a relation algebra can be most easily formulated by treating R as a multivalued binary operation. However, these conditions then become essentially just a reformulation of the conditions in 3.3, and there is therefore no need to state them. On the other hand, these conditions become particularly elegant in the case when $\mathfrak{U}$ is a partial algebra or an algebra.

THEOREM 3.4. *The algebra of complexes of a partial groupoid $\mathfrak{U}$ is a relation algebra iff $\mathfrak{U}$ is a category in which every morphism is an isomorphism.*

COROLLARY 3.5. *The algebra of complexes of a groupoid $\mathfrak{U}$ is a relation algebra iff $\mathfrak{U}$ is a group.*

For the proofs of these results see Jónsson, Tarski [1952], Section 5. The backward implication in Corollary 3.5 was discovered independently by G. Birkhoff and J. C. C. McKinsey, c.f. Birkhoff [1948], p. 212 and Jónsson, Tarski [1948]. The converse and Theorem 3.4 are due to Jónsson and Tarski.

V. *Relations algebras from modular lattices and projective geometries.* In Jónsson [1959], non-Arguesian projective geometries were used to construct non-representable integral relation algebras. In Lyndon [1961], this idea was greatly improved upon, and a relation algebra was associated with each projective geometry that does not contain a line with exactly three points. In Maddux [1978], related methods were used to associate a relation algebra with every modular lattice having a zero element. Both Lyndon's algebras and the ones constructed by Maddux can be viewed as the algebras of complexes of certain relational structures.

THEOREM 3.6 (Maddux [a]). *Suppose* $(M, \wedge, \vee)$ *is a modular lattice with a zero element, and let* R *be the set of all* $(p, q, r) \in M^3$ *such that*

$$p \vee q = q \vee r = r \vee p.$$

Then $\mathscr{C}(M, R)$ *is a symmetric integral relation algebra.*

THEOREM 3.7 (Lyndon [1961]). *Let* $(M, \vee, \wedge)$ *be the lattice of all subspaces of a projective geometry* G, *let* P *be the subset of* M *consisting of the zero element and all the atoms of* M, *and let* R *be the set of all* $(p, q, r) \in P^3$ *such that*

$$p \vee q = q \vee r = r \vee p.$$

If no line of G *has exactly three points, then* $\mathscr{C}(P, R)$ *is a symmetric relation algebra, and if every line of* G *has at least four points, then* $\mathscr{C}(P, R)$ *is integral.*

If we modify the construction in Theorem 3.7 by letting P consist of just the points of G, then R becomes the relation of collineation, i.e., $(p, q, r) \in R$ iff either $p = q = r$, or else p, q and r are distinct but collinear points. The algebra of complexes, $\mathscr{C}(P, R)$, is in this case "almost" a relation algebra: The operation ; (i.e., R^{Δ}) is associative, even if G has lines containing exactly three points. In fact, the associativity of ; is exactly the projective axiom. Also, because of the symmetry of R, the conditions

$$(a\,;b)c = 0, \qquad (c\,;b)a = 0, \qquad (a\,;c)b = 0$$

are equivalent. However, there is in this case no identity element.

4. Simple relation algebras

Theorems 4.5 and 4.8 below are of fundamental importance for the study of varieties of relation algebras. Theorem 4.5, which was first announced in Jónsson and Tarski [1948], is due to J. C. C. McKinsey and Tarski, while Theorem 4.8 was proved in Tarski [1941].

If $\mathscr{A}_0$ is the Boolean part of a relation algebra $\mathscr{A}$, then the congruence lattice of $\mathscr{A}$ is of course a sublattice of the congruence lattice of $\mathscr{A}_0$, $\operatorname{Con} \mathscr{A} \subseteq \operatorname{Con} \mathscr{A}_0$. The familiar isomorphism $\theta \to 0/\theta$ from $\operatorname{Con} \mathscr{A}_0$ to the lattice of all ideals of $\mathscr{A}_0$

therefore maps the members of Con $\mathcal{A}$ onto certain ideals of $\mathcal{A}_0$. We call these ideals congruence ideals of $\mathcal{A}$.

DEFINITION 4.1. An ideal N of a relation algebra $\mathcal{A}$ is called a *congruence ideal* of $\mathcal{A}$ if $N = 0/\theta$ for some congruence relation θ on $\mathcal{A}$.

THEOREM 4.2. *For an ideal N of a relation algebra $\mathcal{A}$, the following conditions are equivalent:*

(i) *N is a congruence ideal.*
(ii) *N is a ideal of the semigroup* $(A, ;)$.
(iii) $1;x;1 \in N$ *for all* $x \in N$.

The condition (ii) means that $A;N \subseteq N$ and $N;A \subseteq N$, i.e., that $x;y \in N$ and $y;x \in N$ whenever $x \in A$ and $y \in N$. If N is a congruence ideal, then the associated congruence relation θ is of course defined by the condition

$$x\theta y \quad \text{iff} \quad x \oplus y \in N,$$

where $\oplus$ denotes the symmetric difference, i.e., $x \oplus y = xy^- + x^- y$.

COROLLARY 4.3. *A principal ideal* $(a]$ *of a relation algebra $\mathcal{A}$ is a congruence ideal iff a is an ideal element.*

THEOREM 4.4. *If a is an ideal element of a relation algebra $\mathcal{A}$, then the map* $x \rightarrow (ax, a^- x)$ *is an isomorphism from $\mathcal{A}$ to* $\mathcal{A}a \times \mathcal{A}a^-$.

THEOREM 4.5. *For a non-trivial relation algebra $\mathcal{A}$, the following conditions are equivalent:*

(i) *$\mathcal{A}$ is simple.*
(ii) *$\mathcal{A}$ is subdirectly irreducible.*
(iii) *$\mathcal{A}$ is directly indecomposable.*
(iv) *$\mathcal{A}$ has exactly two ideal elements.*
(v) $1;x;1 = 1$ *whenever* $0 \neq x \in A$.

COROLLARY 4.6. *Every relation algebra is isomorphic to a subdirect product of simple relation algebras.*

COROLLARY 4.7. *Every subalgebra of a simple relation algebra is simple.*

THEOREM 4.8. *One can effectively correlate with every open formula ϕ in the* *language of relation algebra a term ϕ^* in the same language in such a way that for every simple relation algebra $\mathcal{A}$,*

$$\mathcal{A} \models \phi \leftrightarrow \phi^* = 1.$$

There will be occasions when we need an explicit form for ϕ^*, and we therefore recall how this term can be constructed. We may assume that all the atomic subformulas of ϕ are of the form $w = 1$, and that the only connectives that occur in ϕ are $\wedge$ and $\neg$. We then proceed recursively as follows:

If ϕ is $w = 1$, let ϕ^* be w.
If ϕ is $\phi_0 \wedge \phi_1$, let ϕ^* be $\phi_0^* \cdot \phi_1^*$.
If ϕ is $\neg\psi$, let ϕ^* be $1;(\psi^{*-});1$.

It is a routine matter to check that ϕ^* has the required properties. The crucial case is the one in which ϕ is $\neg\psi$. The argument for this case is based on 4.5(v): For a given assignment to satisfy $\neg\psi$ means that ψ is not satisfied, hence that the value of ψ^* is not 1, hence that the value of ψ^{*-} is not 0, and finally that the value of $1;(\psi^{*-});1$ is 1.

In many cases this method does not yield the simplest translation of an open formula into an equation (or an inclusion). E.g., if ϕ is an implication

$$u_0 = 0 \wedge u_1 = 0 \wedge \cdots \wedge u_n = 0 \rightarrow v = 0$$

or, equivalently,

$$u_0 + u_1 + \cdots + u_n = 0 \rightarrow v = 0,$$

then ϕ is satisfied in a simple relation algebra iff the inclusion

$$v \leq 1;(u_0 + u_1 + \cdots + u_n);1$$

is satisfied. To take a more specific example, consider relation algebras $\mathcal{A}$ whose diversity element is either 0 or an atom. This means that the formula

$$0'x = 0 \vee 0'x^- = 0$$

holds in $\mathcal{A}$. If $\mathcal{A}$ is simple, then this formula holds just in case the identity

$$(0'x);1;(0'x^-) = 0$$

holds.

Since relation algebras are Boolean algebras with certain additional operations, **RA** is both congruence distributive and congruence permutable. We recall some of the most important consequences of the congruence distributivity, and then combine these facts with Theorems 4.5 and 4.8 to obtain the fundamental correspondence between varieties of relation algebras and universal classes of simple relation algebras.

Recall that, for a class **K** of algebras, **P**(**K**), **H**(**K**) and **S**(**K**) are the classes consisting of all algebras that are isomorphic, respectively, to direct products, epimorphic images and subalgebras of members of **K**. We shall also denote by **Ps** (**K**) and **Pu** (**K**) the classes consisting of all algebras that are isomorphic, respectively, to subdirect products and to ultraproducts of members of **K**, and by **Si** (**K**) the class of all subdirectly irriducible members of **K**. Finally, we denote by **Var** (**K**) the variety generated by **K**.

For any class **K** of algebras,

$$\mathbf{Var}\,(\mathbf{K}) = \mathbf{HSP}(\mathbf{K}),$$

but if **K** is contained in a congruence distributive variety, then we also have

$$\mathbf{Var}\,(\mathbf{K}) = \mathbf{Ps}\,\mathbf{HS}\,\mathbf{Pu}\,(\mathbf{K}),$$

$$\mathbf{Si}\,\mathbf{Var}\,(\mathbf{K}) \subseteq \mathbf{HS}\,\mathbf{Pu}\,(\mathbf{K}).$$

Three special cases are particularly important:

If **K** is a finite set of finite algebras, contained in a congruence distributive variety, then

$$\mathbf{Var}\,(\mathbf{K}) = \mathbf{Ps}\,\mathbf{HS}(\mathbf{K}),$$

$$\mathbf{Si}\,\mathbf{Var}\,(\mathbf{K}) \subseteq \mathbf{HS}(\mathbf{K}).$$

If **K** is a positive universal subclass of a congruence distributive variety, then

$$\mathbf{Var}\,(\mathbf{K}) = \mathbf{Ps}\,(\mathbf{K}),$$

$$\mathbf{Si}\,\mathbf{Var}\,(\mathbf{K}) \subseteq \mathbf{K}.$$

If **U** and **V** are subvarieties of a congruence distributive variety, then **U**+**V**, the lattice join of **U** and **V**, consists of all algebras that are isomorphic to a subdirect product of a member of **U** and a member of **V**. In particular,

$$\mathbf{Si}\,(\mathbf{U}+\mathbf{V}) = \mathbf{Si}\,(\mathbf{U}) \cup \mathbf{Si}\,(\mathbf{V}).$$

We now return to relation algebras.

DEFINITION 4.9. We denote by Λ the lattice of all subvarieties of **RA**.

Of course Λ is a dually algebraic lattice and, since **RA** is congruence distributive, Λ is distributive. Recall also that the subdirectly irreducible members of **RA** are simple.

THEOREM 4.10. *The correspondence* $\mathbf{V} \to \mathbf{Si}(\mathbf{V})$ *is an isomorphism from* Λ *to the lattice of all universal sublcasses of* $\mathbf{Si}(\mathbf{RA})$. *The inverse isomorphism is the correspondence* $\mathbf{K} \to \mathbf{Var}(\mathbf{K})$.

Proof. By Theorem 4.5, $\mathbf{Si}(\mathbf{V})$ is a universal class for every subvariety $\mathbf{V}$ of **RA**, for $\mathbf{Si}(\mathbf{V})$ consists of all those members of $\mathbf{V}$ that satisfy the open formulas $0 \neq 1$ and $x \neq 0 \to 1;x;1 = 1$. The equality $\mathbf{Var}\,\mathbf{Si}(\mathbf{V}) = \mathbf{V}$ and the inclusion $\mathbf{K} \subseteq \mathbf{Si}\,\mathbf{Var}(\mathbf{K})$ hold for any variety $\mathbf{V}$ and any class $\mathbf{K}$ of subdirectly irreducible algebras. To complete the proof, it therefore suffices to show that the above inclusion is in fact an equality when $\mathbf{K}$ is a universal class of subdirectly irreducible relation algebras.

By Theorem 4.8, there exists a set Σ of identities such that $\mathbf{K}$ is precisely the class of all those members of $\mathbf{Si}(\mathbf{RA})$ that are models of Σ. In other words, $\mathbf{K}$ is the class of all members of **RA** that are models of Σ and of the two formulas $0 \neq 1$ and $x = 0 \vee 1;x;1 = 1$. If we omit the formula $0 \neq 1$, the members of **RA** that are models of the remaining formulas form a positive, universal class $\mathbf{K}'$ with $\mathbf{K} \subseteq \mathbf{K}'$. Consequently, $\mathbf{Si}\,\mathbf{Var}(\mathbf{K}) \subseteq \mathbf{K}'$. But the only member of $\mathbf{K}'$ that is not in $\mathbf{K}$ is the trival relation relation algebra. Hence $\mathbf{Si}\,\mathbf{Var}(\mathbf{K}) \subseteq \mathbf{K}$.

THEOREM 4.11. **RRA** *is a variety.*

This result was first proved in Tarski [1955] by a rather indirect argument. The proof outlined here was suggested by R. McKenzie.

Let $\mathbf{K}$ be the class of all simple, representable relation algebra. Then $\mathcal{A} \in \mathbf{K}$ iff $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{R}(X)$ for some non-empty set X. Observe that

$$\mathbf{RRA} = \mathbf{Ps}(\mathbf{K}) \subseteq \mathbf{Var}(\mathbf{K}).$$

To prove the theorem it therefore suffices to show that $\mathbf{S}(\mathbf{K})$, $\mathbf{H}(\mathbf{K})$ and $\mathbf{Pu}(\mathbf{K})$ are subclasses of **RRA**. Actually, $\mathbf{S}(\mathbf{K})$ is equal to $\mathbf{K}$, and $\mathbf{H}(\mathbf{K})$ is simply $\mathbf{K}$ with the one-element algebra adjoined (because all the members of $\mathbf{K}$ are simple.) We claim that $\mathbf{Pu}(\mathbf{K}) = \mathbf{K}$. To prove this it suffices to show that if $\mathcal{A}$ is an ultraproduct of full algebras of relations $\mathfrak{R}(X_i)$, $i \in I$, modulo an ultrafilter $\mathcal{U}$ on I, then $\mathcal{A}$ is isomorphic to a subalgebra of $\mathfrak{R}(Y)$ for some set Y. We take for Y the ultra-product of the sets X_i. With a member $F/\mathcal{U}$ of $\mathcal{A}$, where $F(i) \subseteq X_i^2$ for $i \in I$, we associate the relation F^α on Y determined by the condition that $(x/\mathcal{U})F^\alpha(y/\mathcal{U})$ iff

$$\{i \in I : x(i)F(i)y(i)\} \in \mathcal{U}.$$

It is a routine argument to show that the map $F/\mathcal{U} \to F^\alpha$ is well defined, and is an embedding of $\mathcal{A}$ into $\mathfrak{R}(Y)$.

5. The atoms and the center of Λ

The relation algebra $\mathfrak{R}(n)$ satisfies the identity $0';0' = 1$ if $n \geq 3$, but for $n = 2$ and $n = 1$ we have $0';0' = 1'$ and $0';0' = 0$, respectively. It is convenient to have names for the varieties determined by these three equations.

DEFINITION 5.1. We denote by $\mathbf{B}_1$, $\mathbf{B}_2$, and $\mathbf{B}_3$ the varieties consisting of all relation algebras that satisfy, respectively, the identities $0';0' = 0$, $0';0' = 1'$, and $0';0' = 1$.

Incidentally, $0';0' = 0$ is equivalent to $0' = 0$, and hence to $1' = 1$. In fact, if $(0';0')1' = 0$, then $0' = 0$ by 2.3(v). Thus $\mathbf{B}_1$ is precisely the variety of all Boolean relation algebras.

The next theorem was proved in Jónsson, Tarski [1952].

THEOREM 5.2. *Every relation algebra is isomorphic to a direct product* $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3$ *where* $\mathcal{A}_i \in \mathbf{B}_i$ *for* $i = 1, 2, 3$.

COROLLARY 5.3. *Every simple relation algebra belongs to one of the varieties* $\mathbf{B}_i$. *Hence,* $\mathbf{RA} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$.

COROLLARY 5.4. *The smallest subalgebra* $\mathcal{A}'$ *of a non-trivial relation algebra* $\mathcal{A}$ *has order* 2, 4, 8, 16 *or* 32. *If* $\mathcal{A} \in \mathbf{B}_1$, *then the order of* $\mathcal{A}'$ *is* 2, *but if* $\mathcal{A} \in \mathbf{B}_2$, *or* $\mathcal{A} \in \mathbf{B}_3$, *then the order of* $\mathcal{A}'$ *is* 4.

This follows from the fact that the set $\{0, 1, 1', 0'\}$ is a subuniverse of $\mathcal{A}$ provided $0';0'$ is a member of the set. This happens just in case $\mathcal{A}$ is in one of the varieties $\mathbf{B}_i$. If $\mathcal{A}$ is in $\mathbf{B}_1$, then $0' = 0$ and $1' = 1$, but if $\mathcal{A}$ is in $\mathbf{B}_2$ or $\mathbf{B}_3$, then 0, $0'$, 1 and $1'$ are easily seen to be distinct.

For $\mathscr{A} \in \mathbf{B}_i$, $i = 1, 2, 3$, the smallest subalgebra $\mathscr{A}'$ of $\mathscr{A}$ is determined up to isomorphism by the fact that $0';0'$ is equal to 0, $1'$ and 1, respectively. We need names for these three algebras, and for the varieties that they generate.

DEFINITION 5.5. For $n = 1, 2, 3$, we denote by $\mathscr{C}_n$ the smallest subalgebra of $\mathfrak{R}(n)$, and we denote by $\mathbf{A}_n$ the variety generated by $\mathscr{C}_n$.

COROLLARY 5.6. *Λ has precisely three atoms:* $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$.

This readily follows from the fact that every simple relation algebra has a subalgebra isomorphic to one of the algebras $\mathscr{C}_n$. This corollary was proved in Tarski [1956] by a different argument.

THEOREM 5.7. *The center of Λ is a Boolean algebra of order* 8 *whose atoms are* $\mathbf{B}_1$, $\mathbf{B}_2$ *and* $\mathbf{B}_3$.

Proof. By 5.3, $\mathbf{RA} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3$. Furthermore, any non-trivial member of $\mathbf{B}_i$ has a subalgebra isomorphic to $\mathscr{C}_i$, and therefore cannot belong to $\mathbf{B}_j$ for $j \neq i$. Hence each of the three varieties $\mathbf{B}_i$ has the join of the other two as a complement, and they therefore generate a Boolean algebra of order 8, contained in the center of Λ. This Boolean algebra must in fact be the whole center of Λ, for otherwise Λ would contain four pairwise disjoint non-zero members, contrary to the fact that every non-zero member of Λ contains one of the algebras $\mathscr{C}_i$ as a member.

6. Examples of equational bases

The varities $\mathbf{A}_i$ and $\mathbf{B}_i$, $i = 1, 2, 3$, generate a sublattice of Λ or order 18. Four members of this sublattice, $\mathbf{RA}$, $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$, were defined as models of certain sets of identities, so we already have equational bases for these varieties. A fifth member is the trivial variety. We are going to find one or more equational bases for each of the remaining thirteen varieties, as well as some alternative bases for $\mathbf{B}_1$. First we need to take a closer look at $\mathbf{B}_1$ and $\mathbf{B}_2$.

THEOREM 6.1. $\mathbf{A}_1 = \mathbf{B}_1$.

Proof. By the remark following Definition 2.5, $\mathbf{B}_1$ is simply the variety of all Boolean relation algebras. In other words, a member $\mathscr{A} = (\mathscr{A}_0, ;, 1', \breve{\ })$ of $\mathbf{B}_1$ is obtained from the Boolean algebra $\mathscr{A}_0$ by letting $x ; y = xy$, $1' = 1$ and $x\breve{\ } = x$. It follows that $\operatorname{Con}\mathcal{A} = \operatorname{Con}\mathcal{A}_0$, and hence that $\mathcal{A}$ is simple iff $\mathcal{A}_0$ is. Thus the only simple member of $\mathbf{B}_1$ is $\mathscr{E}_1$, which belongs to $\mathbf{A}_1$. Consequently $\mathbf{B}_1 = \mathbf{A}_1$.

The facts concerning $\mathbf{B}_2$ are somewhat less trivial. The following result from Jónsson, Tarski [1952] is needed.

THEOREM 6.2. *Every relation algebra in which the unit element is the join of finitely many functional elements is representable.*

A more general representation result can be found in Maddux [1978].

COROLLARY 6.3. $\mathbf{B}_1 + \mathbf{B}_2 \le \mathbf{RRA}$.

Proof. In any member of $\mathbf{B}_1 + \mathbf{B}_2$, both $1'$ and $0'$ are functional elements.

COROLLARY 6.4. *An algebra* $\mathcal{A} \in \mathbf{B}_2$ *is simple iff* $\mathcal{A} \cong \mathscr{E}_2$ *or* $\mathcal{A} \cong \mathscr{R}(2)$.

Proof. A simple, representable relation algebra $\mathcal{A}$ is isomorphic to a subalgebra of $\mathscr{R}(\kappa)$ for some cardinal κ. If $\mathcal{A}$ belongs to $\mathbf{B}_2$, then it satisfies the condition $0' ; 0' = 1'$. Hence $\mathscr{R}(\kappa)$ must also satisfy the condition, which implies that $\kappa = 2$. From this the conclusion follows, since the only proper subalgebra of $\mathscr{R}(2)$ is $\mathscr{E}_2$.

COROLLARY 6.5. $\mathbf{B}_2$ *covers* $\mathbf{A}_2$ *in* Λ.

Since all the varieties under consideration are subvarieties of $\mathbf{RA}$, we speak of equational bases for these varieties when we actually mean equational bases modulo $\mathbf{RA}$. For brevity we write

$$\tau(x) = (0'x) ; 1 ; (0'x^{-}).$$

The prevalence of this term is explained by the observation, made in Section 4, that a simple relation algebra satisfies the identity $\tau(x) = 0$ iff its diversity element is either an atom or the zero element.

THEOREM 6.6. *The varieties listed below have the equational bases indicated. Different equational bases for the same variety are separated by a semicolon.*

$\mathbf{A}_1$	$1' = 1$; $x ; y = xy$; $x ; x = x$.
$\mathbf{A}_2$	$0' ; 0' = 1'$ *and* $x ; y = y ; x$;
	$0' ; 0' = 1'$ *and* $x^{\smile} = x$;
	$0' ; 0' = 1'$ *and* $x ; x ; x = x$;
	$0' ; 0' = 1'$ *and* $\tau(x) = 0$.

$\mathbf{A}_3$	$0';0'=1$ *and* $\tau(x)=0$.
$\mathbf{A}_1+\mathbf{A}_2$	$0';0'\le 1'$ *and* $x;y=y;x$;
	$x;x;x=x$; $\quad x;x^{\smile};x=x$.
$\mathbf{A}_1+\mathbf{A}_3$	$0'\le 0';0'$ *and* $\tau(x)=0$.
$\mathbf{A}_2+\mathbf{A}_3$	$1'\le 0';0'$ *and* $\tau(x)=0$.
$\mathbf{A}_1+\mathbf{A}_2+\mathbf{A}_3$	$\tau(x)=0$.
$\mathbf{B}_2+\mathbf{A}_3$	$1'\le 0';0'$ *and* $\tau(x)\le 1;(0';0')^-$.
$\mathbf{A}_2+\mathbf{B}_3$	$1'\le 0';0'$ *and* $x;y\le(y;x)+(0';0')$.
$\mathbf{B}_2+\mathbf{B}_3$	$1'\le 0';0'$.
$\mathbf{A}_1+\mathbf{B}_2$	$0';0'\le 1'$.
$\mathbf{A}_1+\mathbf{B}_3$	$0'\le 0';0'$.
$\mathbf{A}_1+\mathbf{A}_2+\mathbf{B}_3$	$x;y\le(y;x)+(0';0')$; $\quad x^{\smile}\le x;(0';0')$.
$\mathbf{A}_1+\mathbf{B}_2+\mathbf{A}_3$	$\tau(x)\le 1;(0';0')^-$.

Proof. As an equational basis for $\mathbf{A}_1$ we can take any identity, or set of identities, that holds in $\mathscr{E}_1$ but fails in $\mathscr{E}_2$ and $\mathscr{E}_3$. Each of the three identities listed fulfils this requirement. Similarly, for $\mathbf{A}_2$ we can take any set of identities that holds in $\mathscr{E}_2$ but fail is $\mathscr{E}_1$, $\mathscr{E}_3$ and $\mathscr{R}(2)$. The identity $0';0'=1'$ excludes $\mathscr{E}_1$ and $\mathscr{E}_3$, while each of the four identities

$$x\cdot y=y\cdot x, \qquad x^{\smile}=x, \qquad x;x;x=x, \qquad \tau(x)=0$$

excludes $\mathscr{R}(2)$. For $\mathbf{A}_1+\mathbf{A}_2$ we need identities that hold in $\mathscr{E}_1$ and $\mathscr{E}_2$ but fail in $\mathscr{E}_3$ and $\mathscr{R}(2)$. The identities $0';0'\le 1'$ and $x;y=y;x$ jointly fulfil these requirements, and so does each of the identities $x;x;x=x$ and $x;x^{\smile};x=x$.

For most of the remaining varieties, this type of argument does not work, for we do not know all their covers.

As was observed earlier, a simple relation algebra satisfies the identity $\tau(x)=0$ iff its diversity element is either an atom or the zero element. The algebras $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ have this property, and we claim that they are the only such algebras. To prove this it suffices to show that if the diversity element of a simple relation algebra is an atom, then so is the identity element. Suppose, to the contrary, that $1'=x+y$ with $xy=0$ and $x\ne 0\ne y$. Then $(x;0')+(y;0')=0'$ and, by 2.3(xiii), $(x;0')(y;0')=0$. Consequently, either $x;0'$ or $y;0'$ is 0, say $x;0'=0$. Thus $(x;0')0'=0$, hence $(0';0')x=0$. This shows that $1'\not\le 0';0'$, and the only simple relation algebra with this property is $\mathscr{E}_1$.

The variety generated by $\mathscr{E}_1$, $\mathscr{E}_2$ and $\mathscr{E}_3$ is $\mathbf{A}_1+\mathbf{A}_2+\mathbf{A}_3$, and the equation $\tau(x)=0$ therefore constitutes an equational basis for this variety. To obtain equational bases for $\mathbf{A}_2+\mathbf{A}_3$, or $\mathbf{A}_1+\mathbf{A}_3$, or $\mathbf{A}_3$, we need only add identities that exclude $\mathscr{E}_1$, or $\mathscr{E}_2$, or both. Such identities are $1'\le 0';0'$, $0'\le 0';0'$, and $0';0'=1$, respectively.

The variety $\mathbf{A}_1+\mathbf{B}_2=\mathbf{B}_1+\mathbf{B}_2$ is the complement of $\mathbf{B}_3$. The inclusion $0';0'\leq 1'$ is an equational basis for $\mathbf{B}_1+\mathbf{B}_2$ because it holds in $\mathbf{B}_1$ and $\mathbf{B}_2$, but fails in $\mathscr{E}_3$, and therefore in every non-trivial member of $\mathbf{B}_3$.

The inclusion $\tau(x)\leq 1;(0';0')^-$ holds in $\mathbf{A}_1$ and in $\mathbf{B}_2$, where $1;(0';0')^-=1$, and it also holds in $\mathbf{A}_3$, for that variety satisfies the identity $\tau(x)=0$. On the other hand, if $\mathscr{A}$ is a simple relation algebra that does not belong to $\mathbf{A}_1+\mathbf{B}_2+\mathbf{A}_3$, then $\mathscr{A}\in\mathbf{B}_3$, and the diversity element of $\mathscr{A}$ is not an atom. The inclusion $\tau(x)\leq 1;(0';0')^-$ therefore fails in $\mathscr{A}$, for the right hand side is identically zero, but the left hand side is not. This inclusion therefore constitutes an equational basis for $\mathbf{A}_2+\mathbf{B}_2+\mathbf{A}_3$.

Each of the inclusions $x;y\leq(y;x)+(0';0')$ and $x^{\smile}\leq x+(0';0')$ is an equational basis for $\mathbf{A}_1+\mathbf{A}_2+\mathbf{B}_3$, for both inclusions fail in $\mathscr{R}(2)$, but in every simple relation algebra not isomorphic to $\mathscr{R}(2)$ we have either $0';0'=1$ or $x;y=y;x$ and $x^{\smile}=x$.

The inclusion $1'\leq 0';0'$ is easily seen to be an equational basis for $\mathbf{B}_2+\mathbf{B}_3$. The equational bases for $\mathbf{B}_2+\mathbf{A}_3$ and $\mathbf{A}_2+\mathbf{B}_3$ are obtained by observing that

$$\mathbf{B}_2+\mathbf{A}_3=(\mathbf{A}_1+\mathbf{B}_2+\mathbf{A}_3)\cap(\mathbf{B}_2+\mathbf{B}_3),$$
$$\mathbf{A}_2+\mathbf{B}_3=(\mathbf{A}_1+\mathbf{A}_2+\mathbf{B}_3)\cap(\mathbf{B}_2+\mathbf{B}_3).$$

7. Splitting algebras and their conjugate varieties

The notion of a *splitting algebra* arose in McKenzie [1972] in the context of lattice varities, but the concept obviously applies to arbitrary varieties of algebras. A subdirectly irreducible algebra $\mathscr{A}$ in a variety $\mathbf{V}_0$ is said to be splitting (in $\mathbf{V}_0$) if the subvarieties of $\mathbf{V}_0$ that do not have $\mathscr{A}$ as a member form a principal ideal in the lattice of all subvarieties of $\mathbf{V}_0$. The variety generating this ideal is then the largest subvariety of $\mathbf{V}_0$ that does not have $\mathscr{A}$ as a member. It is called the *conjugate variety* of $\mathscr{A}$. Since $\mathscr{A}$ is not a member of its conjugate variety $\mathbf{V}$, there must exist an identity that holds in $\mathbf{V}$ but not in $\mathscr{A}$, and because of the maximality of $\mathbf{V}$, this identity must form a basis for $\mathbf{V}$ modulo $\mathbf{V}_0$. This identity, which is called the *conjugate identity* of $\mathscr{A}$, is therefore unique up to equivalence modulo the identities that hold in $\mathbf{V}_0$.

THEOREM 7.1. *Every finite, simple relation algebra $\mathscr{A}$ is splitting. A simple relation algebra $\mathscr{B}$ belongs to the conjugate variety of $\mathscr{A}$ iff $\mathscr{A}$ is not monomorphic to $\mathscr{B}$.*

Proof. Let $\mathbf{K}$ be the class of all simple relation algebras $\mathscr{B}$ such that $\mathscr{A}$ is not monomorphic to $\mathscr{B}$. Since $\mathscr{A}$ is finite, $\mathbf{K}$ is a universal class, whence by Theorem

4.10, **Si Var** (**K**) = **K**. In particular, $\mathcal{A} \notin$ **Var** (**K**). If **U** is any subvariety of **RA** that is not included in **Var** (**K**), then there must exist a simple relation algebra $\mathcal{B}$ that belongs to **U** but not to **K**. It follows that $\mathcal{A}$ is monomorphic to $\mathcal{B}$, and therefore a member of **U**. This shows that $\mathcal{A}$ is splitting, with **Var** (**K**) as its conjugate variety.

DEFINITION 7.2. For any non-zero cardinal κ, we let **RA** (κ) be the variety generated by $\mathfrak{R}(\kappa)$, and for κ finite, we let **RA*** (κ) be the conjugate variety of $\mathfrak{R}(\kappa)$.

LEMMA 7.3. *For any non-zero element p of a simple relation algebra* $\mathcal{A}$, *if*

$$p;1;p^{\smile} \leq 1' \quad \textit{and} \quad p^{\smile};1;p \leq 1', \tag{i}$$

then p is an atom of $\mathcal{A}$.

Proof. The first inclusion in (i) can be written $(p;1;p^{\smile})0' = 0$, and by two applications of 2.3(v) we obtain first $(p^{\smile};0')(1;p^{\smile}) = 0$, then $(p^{\smile};0';p)1 = 0$, i.e., $p^{\smile};0';p = 0$. Now suppose $p = a + b$, with $ab = 0$. Then $a^{\smile};0';b = 0$. Also, $(a;1')b = 0$, which by 2.3(v) yields $(a^{\smile};b)1' = 0$, i.e., $a^{\smile};b \leq 0'$. Consequently,

$$a^{\smile};1;b = (a^{\smile};0';b) + (a^{\smile};1';b) \leq 0',$$

and together with the second part of (i) this yields $a^{\smile};1;b = 0$. Since $\mathcal{A}$ was assumed to be simple, we conclude that $a = 0$ or $b = 0$. Thus p is an atom of $\mathcal{A}$.

LEMMA 7.4. *For any positive integer n,* $\mathfrak{R}(n)$ *is not isomorphic to a proper subalgebra of a simple relation algebra.*

Proof. Suppose f is a monomorphism from $\mathfrak{R}(n)$ to a simple relation algebra $\mathcal{A}$. Since the atoms of $\mathfrak{R}(n)$ satisfy the condition 7.3(i), their images in $\mathcal{A}$ must be atoms. From this the conclusion follows, for if a monomorphism from a finite Boolean algebra to a Boolean algebra takes atoms into atoms, then it is an isomorphism.

THEOREM 7.5. *The varieties* **RA*** (n) *with n a positive integer are pairwise distinct coatoms of* Λ.

Proof. By 7.4, **RA*** (n) contains every simple relation algebra except $\mathfrak{R}(n)$. From this theorem follows.

LEMMA 7.6. *For every positive integer n, a simple relation algebra $\mathscr{A}$ is isomorphic to $\mathfrak{R}(n)$ iff there exists $a \in A$ with*

$$a\,;a \leq a, \qquad a + a^{\smile} = 0', \qquad a^n = 0, \qquad a^{n-1} \neq 0$$

Proof. The forward implication holds because the less than relation on the set $n = \{0, 1, \ldots, n-1\}$ satisfies the displayed formulas.

Now suppose $a \in A$ has the indicated properties. By 7.4 it suffices to show that $\mathscr{A}$ has a sublattice isomorphic to $\mathfrak{R}(n)$. Thus we wish to find n^2 elements $p_{i,j}$, $i, j \in n$, with the following properties:

$$1 = \sum_{i,j \in n} p_{i,j} \tag{1}$$

$$1' = \sum_{i \in n} p_{i,i}. \tag{2}$$

$$p_{i,j}p_{k,m} = 0 \quad \text{for} \quad (i, j) \neq (k, m). \tag{3}$$

$$p_{i,j}{}^{\smile} = p_{j,i}. \tag{4}$$

$$p_{i,j}\,; p_{j,k} = p_{i,k}. \tag{5}$$

$$p_{i,j}\,; p_{k,m} = 0 \quad \text{for} \quad j \neq k. \tag{6}$$

Motivated by the properties of the less than relation, we define

$$b_k = (1\,; a^k)1' \quad \text{for} \quad k = 0, 1, \ldots, n,$$
$$c_k = b_k(b_{k+1})^- \quad \text{for} \quad k = 0, 1, \ldots, n-1,$$
$$p_{i,j} = c_i\,; 1\,; c_j \quad \text{for} \quad i, j = 0, 1, \ldots, n-1.$$

From the fact that $a^2 \leq a$ and $a^n = 0$ it follows that

$$1' = b_0 \geq b_1 \geq \cdots \geq b_n = 0 \tag{7}$$

and, consequently,

$$1' = c_0 + c_1 + \cdots + c_{n-1}, \tag{8}$$

$$c_i c_j = 0 \quad \text{for} \quad i \neq j. \tag{9}$$

From the fact that $c_i \leq 1'$ for all $i \in n$, we infer by 2.3(xiii) that

$$p_{i,j}p_{k,m} \leq (c_i c_k\,; 1)(1\,; c_j c_m),$$

whence (3) follows by (9). Furthermore,

$$\sum_{i,j\in n} p_{i,j} = \Big(\sum_{i\in n} c_i\Big);1;\Big(\sum_{j\in n} c_j\Big) = 1';1;1' = 1,$$

proving (1). The formulas (4) and (6) are obvious.

Observe that $p_{i,j};p_{j,k} = c_i;1;c_j;1;c_k$. The formula (5) will therefore follow from the simplicity of $\mathscr{A}$ if we can show that $c_j \neq 0$. I.e., it suffices to show that the elements b_i form a strictly decreasing sequence. By 2.3(vi),

$$1;b_k = 1;(1;a^k)1' \geq (1;1')a^k = a^k,$$

hence $1;b_k \geq 1;a^k$. The opposite inclusion is obvious, and we therefore have $1;b_k = 1;a^k$. Thus, for $k<n$, we cannot have $b_k = b_{k+1}$, for this would imply $1;a^k = 1;a^{k+1}$, hence $1;a^{n-1} = 1;a^n = 0$. The formula (5) therefore holds.

To complete the proof of the lemma, it only remains to verify (2). This will be done by showing that $p_{i,i} = c_i$, i.e., that $c_i;1;c_i = c_i$. Obviously, $c_i;1';c_i = c_i$, and the conclusion will therefore follow if we show that $c_i;0';c_i = 0$. Since $0' = a + a^{\smile}$, this is equivalent to the conjunction of the two equations $c_i;a;c_i = 0$ and $c_i;a^{\smile};c_i = 0$, and by symmetry it suffices to prove one of these equations, say the former. Writing this equation in the form $(c_i;a;c_i)1 = 0$, and applying 2.3(v), we obtain an equivalent equation

$$(c_i;a)(1;c_i) = 0. \tag{10}$$

By 2.3(xiv), (xv), the element $d_i = (1;a^i)(1;a^{i+1})^-$ is a left ideal element. Hence, by 2.3(vi),

$$1;(d_i1') = 1;((1;d_i)1') \geq (1;1')d_i = d_i,$$

and consequently $1;(d_i1') = d_i$, i.e.,

$$1;c_i = (1;a^i)(1;a^{i+1})^-.$$

To prove (10) it therefore suffices to show that

$$(c_i;a)(1;a^{i+1})^- = 0.$$

This last equation obviously holds, for

$$c_i;a \leq 1;a^i;a = 1;a^{i+1}.$$

The proof of the lemma is now complete.

THEOREM 7.7. *For any positive integer n, $\mathcal{R}(n)$ has the conjugate identity*

$$x^{n-1} \leq 1;((x;x)x^- + 0'x^-x^{\smile -} + x^n);1. \tag{i}$$

Proof. A simple relation algebra $\mathcal{A}$ belongs to the conjugate variety of $\mathcal{R}(n)$ iff, for every element a of A, one of the following four conditions fails:

$$a;a \leq a, \qquad a + a^\smile \geq 0', \qquad a^{n-1} \neq 0, \qquad a^n = 0. \tag{1}$$

Here we have replaced the equality $a + a^\smile = 0'$ in 7.6 by an inclusion $a + a^\smile \geq 0'$. This is justified by the observation that any element a that satisfies the inclusion but not the equality necessarily fails to satify the fourth condition, $a^n = 0$.

To say that at least one of the conditions in (1) fails, is equivalent to saying that

$$a^{n-1} \neq 0 \quad \text{implies} \quad (a;a)a^- + 0'a^-a^{\smile -} + a^n \neq 0,$$

and since $\mathcal{A}$ is simple, this holds iff a satisfies (i).

The conjugate varieties of $\mathcal{R}(1)$ and $\mathcal{R}(2)$ are $\mathbf{B}_2 + \mathbf{B}_3$ and $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{B}_3$. In the preceding section it was shown that these varieties have the equational bases $1' \leq 0';0'$ and $x;y \leq (y;x) + (0';0')$, respectively, while the present result yields the bases

$$1' \leq 1;((x;x)x^- + 0'x^-x^{\smile -} + x);1,$$
$$x \leq 1;(x;x + 0'x^-x^{\smile -});1.$$

The new basis for $\mathbf{B}_2 + \mathbf{B}_3$ easily reduces to $1' \leq 1;0';1$, and it is not hard to show that this is equivalent to $1' \leq 0';0'$. It does not appear to be as easy to prove directly that the two bases for $\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{B}_3$ do in fact define the same variety.

THEOREM 7.8. *There exists $2^{\aleph_0}$ varieties of symmetric, representable relation algebras.*

Proof. For any odd prime p, let $\mathcal{A}_p$ be the Lyndon algebra of a projective line G_p of order p, and for any set S of odd primes let $\mathbf{V}_S$ be the variety generated by the algebras $\mathcal{A}_p$ with $p \in S$. Since the algebras $\mathcal{A}_p$ are symmetric and representable, so are the members of $\mathbf{V}_S$. It therefore suffices to show that the correspondence $S \to \mathbf{V}_S$ is one-to-one or, equivalently, that $\mathcal{A}_q \in \mathbf{V}_S$ implies $q \in S$.

The atoms of $\mathscr{A}_q$ are the identity element and the points of G_q. The points r of G_q are characterized by the properties $r \leq 0'$ and $r;r = r + 1'$. Consequently, any monomorphism from $\mathscr{A}_q$ to $\mathscr{A}_p$ must take atoms into atoms, for the points of G_q must go into points of G_p, and of course the identity element of $\mathscr{A}_q$ goes into the identity element of $\mathscr{A}_p$. It follows that no such isomorphism exists unless $p = q$. Consequently, if $q \notin S$, then $\mathbf{V}_S$ is contained in the conjugate variety of $\mathscr{A}_q$, and thus $\mathscr{A}_q \notin V_S$, as was to be shown.

Observe that the varieties in the above theorem also satisfy the identity $x^2 = x^3$.

If $\mathscr{A}$ is a finite, non-representable, simple relation algebra, then the conjugate identity of $\mathscr{A}$ obviously holds in **RRA**, but not in **RA**. However, these identities tend to be rather involved, and often simpler, but weaker, identities can be obtained by more ad hoc methods. This is illustrated in the next theorem, which was discovered by analyzing the proof of the non-representability of McKenzie's algebra described in part III of Section 3.

THEOREM 7.9. *Let* $\rho(x) = (x^{-}x^{\smile -} ; x^{-}x^{\smile -})^{-}$. *Then the inclusion*

$$(x ; x^{\smile})(x^{\smile} ; x)\rho(x) \leq 1 ; (x ; x)(x^{-} + \rho(x)) ; 1$$

holds in **RRA** *but fails in McKenzie*'s *non-representable relation algebra.*

Proof. The inclusion fails in McKenzie's algebra if we take $x = 1' + p$, where p is one of the two atoms with $p^{\smile} \neq p$.

In order to show that the inclusion holds in every representable relation algebra it suffices to show that it holds in $\mathfrak{R}(X)$. We want to show that if x is a transitive relation, i.e., if $(x ; x)x^{-} = 0$, and if

$$(x ; x^{\smile})(x^{\smile} ; x)\rho(x) \neq 0, \tag{1}$$

then $(x ; x)\rho(x) \neq 0$. Adopting the terminology normally used for partial orders, we observe that $\rho(x)$ consists of all ordered pairs (α, β) with the property that every member ε of X is comparable with either α or β, i.e., satisfies one of the four conditions

$$\varepsilon x \alpha, \qquad \alpha x \varepsilon, \qquad \varepsilon x \beta, \qquad \beta x \varepsilon. \tag{2}$$

Our hypothesis (1) thus means that there exist elements $\alpha, \beta \in X$ such that one of the conditions in (2) holds for every $\varepsilon \in X$ and furthermore, for some $\gamma, \delta \in X$, the four conditions

$$\alpha x\gamma, \qquad \beta x\gamma, \qquad \delta x\alpha, \qquad \delta x\beta$$

hold. Clearly (δ, γ) belongs to $x;x$, and to show that (δ, γ) also belongs to $\rho(x)$, we observe that if, for a given $\varepsilon \in X$, either $\varepsilon x\alpha$ or $\varepsilon x\beta$, then $\varepsilon x\gamma$, but if $\alpha x\varepsilon$ or $\beta x\varepsilon$, then $\delta x\varepsilon$.

8. The varieties $\mathbf{RA}(\kappa)$

Obviously **RRA** is the join of the varieties $\mathbf{RA}(\kappa)$, with κ running through all the non-zero cardinals. We shall prove that all the varieties $\mathbf{RA}(\kappa)$ with κ infinite are equal. This is certainly not surprising, but it does not appear to be completely trivial. From this it follows that

$$\mathbf{RRA} = \mathbf{RA}(1) + \mathbf{RA}(2) + \cdots + \mathbf{RA}(\aleph_0),$$

and it will be shown that this representation is irredundant.

THEOREM 8.1. *For any infinite cardinals κ and λ, $\mathbf{RA}(\kappa) = \mathbf{RA}(\lambda)$.*

Proof. It is clearly sufficient to show that if an open formula $\phi(\xi_1, \xi_2, \ldots, \xi_n)$ in the language L of relation algebras is satisfiable in $\mathfrak{R}(X)$ for some infinite set X, then ϕ is satisfiable in $\mathfrak{R}(X)$ for every infinite set X. In order to prove this, we are going to associate with each such formula ϕ a sentence ϕ^α in a language L^α with binary predicates $P_1, P_2, \ldots, P_n$ in such a way that, for any set X, a sequence of binary relations $R_1, R_2, \ldots, R_n$ on X satisfies ϕ in $\mathfrak{R}(X)$ iff ϕ^α hold in the structure

$$\mathscr{X} = (X, R_1, R_2, \ldots, R_n). \tag{1}$$

First we associate with each term t in L a formula $t^\alpha(\eta_1, \eta_2)$ in L^α, containing no free variables distinct from η_1 and η_2, in such a way that the truth set of $t^\alpha(\eta_1, \eta_2)$ in $\mathscr{X}$ is the relation $t(R_1, R_2, \ldots, R_n)$ in $\mathfrak{R}(X)$. If t is 0, or 1, or 1′, or a variable ξ_i, we take t^α to be $\neg(\eta_1 = \eta_1)$, or $\eta_1 = \eta_1$, or $\eta_1 = \eta_2$, or $P_i(\eta_1, \eta_2)$, respectively, and for composite terms t we define t^α recursively by

$$(s+t)^\alpha = s^\alpha \vee t^\alpha, \qquad (st)^\alpha = r^\alpha \wedge t^\alpha, \qquad (s^-)^\alpha = \neg s^\alpha,$$
$$(s\,;t)^\alpha = (\exists \eta_3)(s^\alpha(\eta_1, \eta_3) \wedge t^\alpha(\eta_3, \eta_2)),$$
$$(t^\smallsmile)^\alpha(\eta_1, \eta_2) = t^\alpha(\eta_2, \eta_1).$$

If ϕ is an atomic formula $s=t$, we let ϕ^α be the sentence

$$(\forall\eta_1)(\forall\eta_2)(s^\alpha \leftrightarrow t^\alpha),$$

and we extend the definition to other formulas in such a way that the logical connectives are preserved. An easy recursive argument shows that t^α and ϕ^α have the required properties.

Suppose now that X and Y are infinite sets, of cardinalities κ and λ, respectively, and suppose the open formula $\phi(\xi_1, \xi_2, \ldots, \xi_n)$ in L is satisfied in $\mathfrak{R}(X)$ by a sequence $(R_1, R_2, \ldots, R_n)$. Then the structure $\mathscr{X}$ in (1) is a model of ϕ^α. Since ϕ^α has a model in one infinite power it has, by the Löwenheim-Skolem-Tarski Theorem, a model in every infinite power. In particular, ϕ^α has a model

$$\mathscr{Y}=(Y, S_1, S_2, \ldots, S_n)$$

whose universe is Y, and from this it follows that ϕ is satisfied in $\mathfrak{R}(Y)$ by the sequence $(S_1, S_2, \ldots, S_n)$.

The next result was suggested by G. Birkhoff.

LEMMA 8.2. *The identity*

$$(x^\smile ; x)0' \leq 1;((x;x^\smile)\oplus 1');1 \tag{i}$$

holds in every finite relation algebra, but fails in $\mathfrak{R}(X)$ *if* X *is infinite.*

Proof. It is well known that in a finite monoid every left or right inverse of an element a is a two-sided inverse of a. Hence, in a finite relation algebra,

$$x;y=1' \quad \text{implies} \quad y;x=1'.$$

If ϕ denotes this implication, then ϕ^* can be taken to be

$$(y;x)\oplus 1' \leq 1;((x;y)\oplus 1');1. \tag{1}$$

Thus (1) holds in every finite relation algebra, and (i) is but a weaker form of (1). For an infinite set X, (i) fails in $\mathfrak{R}(X)$ whenever $x^\smile$ is a surjective map that is not injective.

COROLLARY 8.3. *Neither one of the varieties* **RA** *and* **RRA** *is generated by its finite members.*

THEOREM 8.4. **RRA** *is the irredundant join of the varieties* $\mathbf{RA}(\kappa)$ *with* $1 \leq \kappa \leq \aleph_0$.

Proof. By 8.1, **RRA** is the join of the indicated varieties. In this representation $\mathbf{RA}(\aleph_0)$ is not redundant because of 8.3, while $\mathbf{RA}(n)$, with n finite, cannot be omitted because $\mathbf{RA}(\kappa) \subseteq \mathbf{RA}^*(n)$ for $\kappa \neq n$.

It is worth noting that the join of the varieties $\mathbf{RA}(n)$ with n finite does not even include all the finite members of **RRA**. This is so because there exist finite relation algebras that are representable but cannot be represented over a finite set. A simple example is the relation algebra generated by a totally ordering relation that is dense and has neither a first nor a last element. This algebra has three atoms, $1'$, a and $a^\smile$, with

$$a\,;a=a, \qquad a+a^\smile=0', \qquad a\,;a^\smile=1, \qquad a^\smile;a=1.$$

The conjugate identity of this algebra therefore fails in $\mathfrak{R}(n)$ for n finite. The inclusion

$$x0' \leq 1\,;((x\,;x)\oplus(x0'))\,;1$$

is a simplified, but weakened, version of this identity which also fails in this algebra but holds in all the algebras $\mathfrak{R}(n)$.

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