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Generating the algebraic theory of $C(X)$

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Let **T** be the (infinitary) algebraic theory consisting of all continuous maps between cubes $I^n \to I^m$ ($I=[0, 1]$, m and n any cardinals). Thus the set of n-ary operations of **T** is Hom (I^n, I); the laws are determined by the generic model, I itself. In this note I show (1) that **T** is generated by an explicit set of five simple operations (one of which is infinitary), and (2) that the models of **T** are precisely the algebras Hom (X, I) for compact Hausdorff spaces X.

The result (2) was first proved by Duskin [1]. The present proof is no longer and uses much less category theory.

This note is a condensation of Zbigniew Semadeni's notes of my lectures in the Banach Center, Warsaw, 1974. I am very grateful to him for the invitation, and for the lecture notes. The result, with citation of Semadeni's notes, has since appeared as an exercise in Manes' book [3, p. 194]. A fuller record seems desirable because more work remains to be done. A number of other dualities ride piggyback on this Gelfand–Stone duality (cf. [2]). For the theory **T** itself, what are the laws?

One can show (let's leave this among the exercises) that a compact topological **T**-algebra is a power of I. This may be compared with a result of Taylor [4] on topological algebras whose n-ary operations are given by Hom (X^n, X) for rather general H-spaces X, for finite n only; but Taylor uses a narrower notion of topological X-algebra. Still there should be more connection than a mere resemblance.

To begin work, let us coordinatize I as $[-1, 1]$; then Hom (X, I) is precisely the unit ball of the Banach algebra $C(X)$. The five operations are 0-ary 1 (the constant function with value 1), unary $-$: $I \to I$ $(-(x) = -x)$ and $\bar{2}$ $(\bar{2}(x) = (2x \wedge 1) \vee$

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$(-1))$, binary m $(m(x, y) = xy)$ and $\aleph_0$-ary σ $(\sigma(x_1, x_2, \ldots) = \sum 2^{-1}x_i)$. And the first thing to do is to deduce some other needed operations from these. Averaging $a(x, y) = (x+y)/2$ is $\sigma(x, y, y, \ldots)$; then $0x$ is $a(x, -x)$, and for any scalar λ in $(0, 1)$, λ is $\sum \varepsilon_i 2^{-i}$ with certain coefficients $\varepsilon_i \in \{0, 1\}$ and λx is $\sigma(\varepsilon_1 x, \varepsilon_2 x, \ldots)$.

(One can reduce the five operations to four since $-\sigma$ generates $-$ and σ. I don't know how far one can reduce.)

We'll now show that for any cardinal n, in Hom (I^n, I) the closure K of the set of coordinate projections p_i under our five operations is all of Hom (I^n, I). In the ring of real-valued functions $C(I^n)$ let RK be the set of scalar multiples λf of functions f in K. It will suffice, because of the Stone–Weierstrass theorem, to prove that

(a) RK is a uniformly closed separating subalgebra of $C(I^n)$ containing 1 – thus $C(I^n)$ – and

(b) the unit ball of RK is K.

At least RK is a linear subspace of $C(I^n)$. Any pair of its elements can be written as λf and $\kappa\lambda g$ with $\lambda \in R$, $\kappa \in I$, f and $g \in K$, and then $\lambda f \pm \kappa\lambda g$ is $2\lambda a(f, \pm\kappa g) \in RK$; and clearly RK is non-empty and closed under scalar multiplication.

Now (b). The functions in K have norm ≤ 1 and so lie in the unit ball. Conversely, if λf has norm ≤ 1, $\lambda \in R$ and $f \in K$, there is a positive integer e such that $2^{-e}\lambda \in I$; and $\lambda f = 2^e(2^{-e}\lambda f) = \tilde{2}^e(2^{-e}\lambda f) \in K$.

For (a), RK contains (in K) 1 and the projections p_i, which separate points. The product of λf and μg in RK is $\lambda\mu m(f, g) \in RK$. And given a norm-convergent sequence (f_j) in RK with limit $f \in C(I^n)$, we may assume (using scalar multiplication) $\|f_j\| \leq 2^{-1}$ and (choosing a rapidly convergent subsequence) $\|f_{j+1} - f_j\| \leq 2^{-j-1}$. Then $f = f_1 + \sum (f_{j+1} - f_j) = \sum g_j = \sum 2^{-j} 2^j g_j = \sigma(\tilde{2}g_1, \tilde{2}^2 g_2, \ldots) \in RK$. These operations generate the theory, as claimed.

Now the theory $\mathbf{T}$ is by definition (I recall; see e.g. [3]) substantially the dual of the full subcategory of *Cpt* on all the objects I^n. Recall, the free $\mathbf{T}$-algebras are just the algebras Hom (I^n, I); and every $\mathbf{T}$-algebra is a quotient of a free one. Thus for any $\mathbf{T}$-algebra A we have a surjective homomorphism q : Hom $(I^n, I) \to A$. Let J be the kernel of q, and in $C(I^n) = R$ Hom (I^n, I) consider RJ. By Gelfand–Stone theory, when we show

(c) RJ is a closed ideal in $C(I^n)$,

we'll have that for some closed subset H, RJ is the set of functions vanishing on H; $C(I^n)/RJ$ is $C(H)$, and A is its unit ball Hom (H, I).

RJ is a linear subspace, as before. If $f \in J$ and $g \in$ Hom (I^n, I), products $\lambda f \cdot \mu g$ are $\lambda\mu m(f, g)$, and $q(m(f, g)) = m(q(f), q(g)) = 0$, so $m(f, g) \in J$. Finally, if (f_j) in RJ converges in norm to f, we may assume as before that $2f_j \in J$ and $2^{j+1}(f_{j+1} - f_j) \in J$; so $f = \sum 2^{-j} 2^j g_j = \sigma(\tilde{2}g_1, \tilde{2}^2 g_2, \ldots) \in RJ$.

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