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The lattice of varieties of modal algebras is not strongly atomic

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To Professor Ph. Dwinger
On the occasion of his 65th birthday

It is well-known that any lattice Λ of all subvarieties of a given variety of algebras is atomic, i.e., if $\underline{K} \in \Lambda$ such that $\underline{K} \neq \underline{T}$ where $\underline{T}$ is the variety consisting of one-element algebras only, then there is a $\underline{K}' \in \Lambda$ such that $\underline{K}'$ covers $\underline{T}$ and $\underline{K}' \leq \underline{K}$. A surprising consequence of Jónsson's theorem [4] is that if $\underline{K}$ is a variety of algebras whose congruences form a distributive lattice and which is furthermore generated by its finite algebras then for any $\underline{K}_1 < \underline{K}$ such that $\underline{K}_1$ belongs to the lattice $\Lambda(\underline{K})$ of subvarieties of $\underline{K}$ there is a $\underline{K}_2 \in \Lambda(\underline{K})$ which covers $\underline{K}_1$. The question arises if the lattice L of subvarieties of a congruence distributive variety even has the following stronger property: if $a, b \in L$ such that $a < b$ then there is a $c \in L$ such that c covers a and $c \leq b$. A lattice with this property is called *strongly atomic.* The question if the lattice of varieties of lattices is strongly atomic has remained unanswered up to now (cf. [6]); in the present paper we will show that the lattice $\Lambda(\underline{M})$ of subvarieties of the variety of modal algebras $\underline{M}$ is not strongly atomic by presenting varieties of modal algebras $\underline{K}_1$ and $\underline{K}_2$ such that the interval $[\underline{K}_1, \underline{K}_2]$ is isomorphic to the chain $1+\omega^*$. Note that $\underline{K}_1$ cannot be finitely based since it is easy to verify that any such variety $\underline{K}_1$ does have a cover in any proper interval $[\underline{K}_1, \underline{K}_2]$. Furthermore, any lattice of subvarieties of a variety is weakly atomic, i.e., any proper interval $[\underline{K}_1, \underline{K}_2]$ in the lattice contains $\underline{K}_3$, $\underline{K}_4$ such that $\underline{K}_4$ covers $\underline{K}_3$, (cf. [6]). The fact that $\Lambda(\underline{M})$ is not strongly atomic only strengthens the impression of complexity of the lattice, conveyed by the results of [1] where the existence was shown of a variety of modal algebras having $2^{\aleph_0}$ covers and of a variety generated by a finite algebra which has, besides infinitely many covers generated by a finite algebra, also covers which are not generated by any finite algebra.

Having introduced the needed preliminaries in section 1, we develop, in section 2, a technique which provides a way of visualizing ultraproducts of modal algebras. As a by-product we prove that not every finite subdirectly irreducible modal algebra is a splitting algebra. This may be contrasted with the result in [2]

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which says that every finite subdirectly irreducible interior algebra does split the lattice of varieties of interior algebras. In section 3 we present the announced interval.

1. Preliminaries

A *modal algebra* is an algebra $(A, (+, \cdot, ', ^0, 0, 1,))$ such that $(A, (+, \cdot, ', 0, 1))$ is a Boolean algebra, $+, \cdot$ denoting Boolean sum and product respectively, $'$ denoting complement and 0, 1 the smallest and largest element respectively. The operator ^{0} is a unary operation satisfying $1^0 = 1$ and $(x \cdot y)^0 = x^0 \cdot y^0$. The variety of modal algebras is denoted by $\underline{M}$ and the lattice of varieties of modal algebras by $\Lambda(\underline{M})$. If $A \in \underline{M}$, $S \subseteq A$ then $[S]$ will denote the subalgebra of A generated by S, and if a, $b \in A$ then $[a) = \{x \in A \mid x \geqslant a\}$, $(a] = \{x \in A \mid x \leqslant a\}$ and $[a, b] = \{x \in A \mid a \leqslant x \leqslant b\}$. And if $\underline{K} \subseteq \underline{M}$, then $I(\underline{K})$, $H(\underline{K})$, $S(\underline{K})$ and $P(\underline{K})$ have the usual meaning; $V(\underline{K})$ stands for the variety generated by $\underline{K}$ and $P_U(\underline{K})$ for the class of ultraproducts of families of algebras in $\underline{K}$. We write $H(A)$, $S(A)$ etc. if $\underline{K} = \{A\}$ for some $A \in \underline{M}$. The class of subdirectly irreducibles belonging to a class of algebras $\underline{K}$ is denoted by $\underline{K}_{SI}$. Since $\underline{M}$ is congruence distributive, for any $\underline{K} \in \underline{M}$ we have $V(\underline{K})_{SI} \subseteq HSP_U(\underline{K})$ by Jónsson's results ([4]). For further notions and results of universal algebra we refer to [3]; more on modal algebras may be found in [1].

2. Ultraproducts of modal algebras

A *frame* is a pair (W, R) such that W is a set and $R \subseteq W \times W$ is a binary relation on W. We write Rwv instead of $(w, v) \in R$. The Boolean algebra $\mathcal{P}(W)$ of all subsets of W endowed with the operation ^{0} defined by

$$x^0 = \{w \in W \mid \forall v \in W[Rwv \Rightarrow v \in x]\}$$

for any $x \subseteq W$ is easily seen to be a modal algebra. Such an algebra will be called the *Kripke algebra* F^+ derived from the frame $F = (W, R)$. An element $w \in W$ is called reflexive if Rww, otherwise it is called irreflexive. Kripke algebras owe their importance to the following theorem which is an immediate consequence of Jónsson and Tarski [5].

2.1 THEOREM. *For every modal algebra A there is a frame F such that $A \in S(F^+)$.*

Examples of modal algebras are most easily obtained by describing some frame (W, R) and specifying a Boolean algebra of subsets of W closed under the operation ^{0} induced by R. When studying the variety generated by such algebras it is useful to learn how the operations of forming homomorphic images, subalgebras and direct products correspond to operations on frames.

If (W, R) is a frame and $W' \subseteq W$ then $(W', R \mid W')$ is called a *generated subframe* of (W, R) if for all $w \in W'$ the implication $Rwv \Rightarrow v \in W'$ holds. If $A \in S(F^+)$, where $F = (W, R)$, and $F_1 = (W_1, R \mid W_1)$ is a generated subframe of F then $A \mid F_1$ denotes $\{x \cap W_1 \mid x \in A\}$.

2.2 LEMMA. *Let $F = (W, R)$ be a frame, $A \in S(F^+)$ and $F_1 = \{W_1, R \mid W_1)$ a generated subframe of F. Then $A \mid F_1$ is a modal algebra and $A \mid F_1 \in H(A)$.*

Proof. The map $h: A \to F_1^+$ defined by $x \mapsto x \cap W_1$ is clearly a Boolean homomorphism, which furthermore preserves ^{0}. In fact, for any $x \in A$ and $w \in W_1$ we have

$$
\begin{aligned}
w \in h(x)^0 \quad & \text{iff } \forall v \in W_1[Rwv \Rightarrow v \in h(x)] \\
& \text{iff } \forall v \in W[Rwv \Rightarrow v \in x] \\
& \quad \text{(since } F_1 \text{ is a generated subframe)} \\
& \text{iff } w \in h(x^0).
\end{aligned}
$$

Hence $h[A] = A \mid F_1$ is a modal algebra and $A \mid F_1 \in H(A)$.

If $A \in S(F^+)$ and $A_1 \in S(A)$ then we can use the same frame F, although sometimes a simpler frame F_1 can be found such that $A \in S(F_1^+)$ by identifying elements of F. And if $A_i \in S(F_i^+)$, $i \in I$ then it is not difficult to see that $\prod_{i \in I} A_i \in S(\bigoplus_{i \in I} F_i)$, where $\bigoplus_{i \in I} F_i = (\bigcup_{i \in I} W_i, R)$, the union being disjoint, with Rwv iff $\exists i \in I$ such that w, $v \in W_i$ and $R_i wv$.

In order to be able to apply Jónsson's theorem we need the following lemma.

2.3 LEMMA. *Let $\{A_i \mid i \in I\}$ be a family of modal algebras and for $i \in I$, $F_i = (W_i, R_i)$ a frame such that $A_i \in S(F_i^+)$. For any $A \in P_U(\{A_i \mid i \in I\})$ there is an $F = (W, R) \in P_U(\{F_i \mid i \in I\})$ and $A' \in S(F^+)$ such that $A \cong A'$. Furthermore, if for every $i \in I$ and $w \in W_i$, $\{w\} \in A_i$ then for every $w \in W$, $\{w\} \in A'$.*

Proof. Let U be an ultrafilter on I and $A = \prod_{i \in I} A_i / {\sim U}$ where ${\sim U}$ is the equivalence relation on $\prod_{i \in I} A_i$ induced by U. Let $A_1 = \prod_{i \in I} F_i^+ / {\sim U}$ and $F = \prod_{i \in I} F_i / {\sim U} = (W, R)$. First we show that $A_1 \cong A_1' \in S(F^+)$. For any $x \in \prod_{i \in I} A_i$ or $x \in \prod_{i \in I} W_i$ let $\bar{x}$ denote the equivalence class containing x. If $\overline{(x_i)_{i \in I}} \in A_1$ is an atom then, since "being an atom" is expressible by a first order sentence, $\{i \in I \mid x_i$ is an atom in $F_i^+\} \in U$, hence there exist $w_i \in W_i$, $i \in I$ such that $\overline{(x_i)_{i \in I}} = \overline{(\{w_i\})_{i \in I}}$. For a modal algebra B let $\operatorname{At} B$ denote the set of atoms of B. Define a map $\phi': \operatorname{At} A_1 \to \mathcal{P}(W)$ by assigning to an atom $x = \overline{(x_i)_{i \in I}} \in \operatorname{At} A_1$ the set $\{w\}$ where

$w=\overline{(w_i)_{i\in I}}$ such that $\overline{(x_i)_{i\in I}}=\overline{(\{w_i\})_{i\in I}}$. The map ϕ' is well-defined and establishes a 1–1 correspondence between the atoms of A_1 and one-element subsets of W. Extend ϕ' to a map $\phi: A_1\to\mathcal{P}(W)$ by putting $x\to\bigcup\{\phi(a)\mid a\leq x,\ a\in \mathrm{At}\,A_1\}$. Since every F_i^+, $i\in I$, is atomic and atomicity is expressible by a first order sentence, A_1 is atomic; hence ϕ is an injection and further, clearly a Boolean homomorphism. It remains to show that $\phi: A_1\to F^+$ is in fact an $\underset{\sim}{M}$-homomorphism, i.e., that for $x\in A_1$ $\phi(x^0)=\phi(x)^0$ where $\phi(x)^0=\{w\in W\mid \forall v\in W[Rwv\Rightarrow v\in\phi(x)]\}$. If $x=\overline{(x_i)_{i\in I}}$ then

$$\phi(x^0)=\bigcup\{\phi(a)\mid a\leq x^0,\ a\in\mathrm{At}\,A_1\}$$
$$=\{w\mid w=\overline{(w_i)_{i\in I}} \text{ for some } w_i,\ i\in I \text{ with } w_i\in x_i^0,\ i\in I\},$$

and $w_i\in x_i^0$ iff $\forall v_i\in W_i[Rw_iv_i\Rightarrow v_i\in x_i]$. Now suppose that $w\in\phi(x^0)$ and $v\in W$ such that Rwv, $w=\overline{(w_i)_{i\in I}}$, $v=\overline{(v_i)_{i\in I}}$. Then $\{i\in I\mid R_iw_iv_i\}\in U$, and since we may assume that for all $i\in I$ $w_i\in x_i^0$, $\{i\in I\mid v_i\in x_i\}\in U$ hence $v\in\phi(x)$. Thus $\phi(x^0)\subseteq\phi(x)^0$. Conversely, if $w=\overline{(w_i)_{i\in I}}\notin\phi(x^0)$ then $\{i\in I\mid w_i\notin x_i^0\}\in U$, hence we may assume that for all $i\in I$ $w_i\notin x_i^0$. Choose $v_i\in W_i$ such that $R_iw_iv_i$ but $v_i\notin x_i$. Then $v=\overline{(v_i)_{i\in I}}\in W$ and Rwv but $v\notin\phi(x)$, hence $w\notin\phi(x)^0$. It follows that ϕ is an $\underset{\sim}{M}$-embedding, hence $A_1\cong A_1'=\phi[A_1]\in S(F^+)$.

In the general case we have $A_i\in S(F_i^+)$, hence $\prod_{i\in I}A_i\in S(\prod_{i\in I}F_i^+)$ and $A=\prod_{i\in I}A_i/{\sim}U\in S(A_1)$. Thus $A\cong A'=(\phi\mid A)[A]\in S(F^+)$. If for all $w_i\in W_i$, $\{w_i\}\in A_i$ then $\overline{(\{w_i\})_{i\in I}}\in\mathrm{At}\,A$ hence $\{w\}=\{\overline{(w_i)_{i\in I}}\}\in(\phi\mid A)[A]$, for every $w\in W$.

To illustrate the use of the lemma we give an example. Recall that an algebra A in a variety $\underset{\sim}{K}$ is said to be a splitting algebra in $\underset{\sim}{K}$ if there exists a variety $\underset{\sim}{K}_1\subseteq\underset{\sim}{K}$ such that for any variety $\underset{\sim}{K}'\subseteq\underset{\sim}{K}$ either $A\in\underset{\sim}{K}'$ or $\underset{\sim}{K}'\subseteq\underset{\sim}{K}_1$, but not both, i.e. the lattice $\Lambda(\underset{\sim}{K})$ of subvarieties of $\underset{\sim}{K}$ satisfies $\Lambda(\underset{\sim}{K})=[V(A))\cup(\underset{\sim}{K}_1]$ with $[V(A))\cap(\underset{\sim}{K}_1]=\emptyset$. It is not difficult to verify that if $\underset{\sim}{K}$ is generated by its finite members and congruence distributive then for any splitting algebra A there is a finite subdirectly irreducible algebra A_1 such that $V(A)=V(A_1)$, whence we may restrict ourselves in that case to finite subdirectly irreducible splitting algebras. In [2] it was shown that any finite subdirectly irreducible interior algebra, i.e. a modal algebra satisfying the equation $x^0\cdot x=x^0$ and $x^{00}=x^0$ is a splitting algebra in the variety of interior algebras and a similar statement holds for the variety $\underset{\sim}{K}_m\subseteq\underset{\sim}{M}$ defined by the equation $x^{0^m}=x^{0^{m+1}}$, $m=1,2,3,\dots$ (cf. [7]).

Let L_n be the frame (W_n,R_n) where $W_n=\{1,\dots,n\}$ and R_nij iff $i=j$ or $j=i+1$ and let $C_n=(W_n,R_n')$ where $R_n'ij$ iff $i=j$ or $j=i+1 \bmod n$. In [1] it was proven that the C_n^+, $n=1,2,\dots$ are simple and that $S(C_n^+)=\{\underset{\sim}{2},C_n^+\}$ if n is a prime number, so $V(C_n^+)$ covers $V(\underset{\sim}{2})$ if n is prime. Here $\underset{\sim}{2}$ denotes as usual, the two-element modal algebra $\{0, 1\}$, with $0^0 = 0$, $1^0 = 1$. The set of natural numbers $1, 2, 3, \ldots$ is denoted by $\mathbf{N}$.

2.4. THEOREM. *Let* $\underline{K} = V(\{L_n^+ \mid n \in \mathbf{N}\})$. *Then* $\underline{K} = V(\{C_n^+ \mid n \in M\})$ *for any infinite set* $M \subseteq \mathbf{N}$.

Proof. To prove that $\underline{K} \subseteq V(\{C_n^+ \mid n \in M\})$, let $A \in P_U(\{C_n^+ \mid n \in M\})$ be infinite. By 2.3 we may assume that A is an algebra of subsets of some infinite ultraproduct (W, R) of $\{C_n \mid n \in M\}$. Using the known facts about preservation of first order sentences we see easily that $W = \bigcup_{\alpha \in \Gamma} Z_\alpha$, where the Z_α, $\alpha \in \Gamma$ are disjoint copies of the integers and that R is defined by Rij iff $\exists \alpha \in \Gamma$ $i, j \in Z_\alpha$ and $i = j$ or $i + 1 = j$. Let $a \in W$ be arbitrary and F_a the smallest generated subframe of (W, R) containing a. Then $F_a = (W_a, R \mid W_a)$ where $W_a = \{a, a+1, a+2, \ldots\}$ and by 2.2 $A \mid F_a \in H(A)$. Since A contains $\{w\}$ for $w \in W$ by 2.3, $\{a\}$, $\{a+1\}, \ldots, \{a+n-1\}$ and $W_a \setminus \{a, a+1, \ldots, a+n-1\}$ belong to $A \mid F_a$ and clearly form the atoms of a subalgebra of $A \mid F_a$ isomorphic to L_n^+. Hence, for any $n \in \mathbf{N}$ $L_n^+ \in SHP_U(\{C_n^+ \mid n \in M\})$, thus $\underline{K} \subseteq V(\{C_n^+ \mid n \in M\})$.

To prove the converse, we will show that for any $k \in \mathbf{N}$ $C_k^+ \in V(\{L_n^+ \mid n \in \mathbf{N}\})$. Let $A \in P_U(\{L_n^+ \mid n \in \mathbf{N}\})$, say $A = \prod_{n=1}^\infty L_n^+/{\sim U}$, where U is a non-principal ultrafilter on $\mathbf{N}$, and $\sim U$ the congruence relation induced by U. We may assume that A is an algebra of subsets of (W, R) where $W = \bigcup_{\alpha \in \Gamma} X_\alpha$ such that $X_{\alpha_0} \cong \mathbf{N}$, $X_{\alpha_1} \cong \{n \in Z \mid n \leqslant 0\}$ for certain $\alpha_0, \alpha_1 \in \Gamma$ and $X_\alpha \cong Z$ for $\alpha \in \Gamma$, $\alpha \neq \alpha_0, \alpha_1$, and $X_\alpha \cap X_\beta = \emptyset$ if $\alpha, \beta \in \Gamma$, $\alpha \neq \beta$. The relation R is determined by Rij iff $\exists \alpha \in \Gamma$ $i, j \in X_\alpha$ and $i = j$ or $i + 1 = j$. Let $a \in X_\alpha$ for some $\alpha \in \Gamma$, $\alpha \neq \alpha_1$ and $F_a = (W_a, R \mid W_a)$ be the smallest generated subframe containing a. We claim that $x_0 = \{a + nk \mid n = 0, 1, 2, \ldots\} \in A \mid F_a$. For if $\{a_n\} \in L_n^+$ are such that $a = \overline{(a_n)_{n \in \mathbf{N}}}$ then if for $n \in \mathbf{N}$ $x_n = \{a_n + jk \bmod n \mid 0 \leqslant j \leqslant [n/k]\} \in L_n^+$ and $x = \overline{(x_n)_{n \in \mathbf{N}}}$ one easily verifies that $x_0 = x \cap W_a$. Similarly, $x_i = \{a + i + nk \mid n = 0, 1, 2, \ldots\} \in A \mid F_a$ for $i = 1, \ldots, k-1$ and clearly $x_0, x_1, \ldots, x_{k-1}$ are the atoms of a subalgebra of $A \mid F_a$ isomorphic to C_k^+. It follows that $V(\{C_n^+ \mid n \in M\}) \subseteq \underline{K}$.

2.5 COROLLARY. *Neither* L_n^+, $n \in \mathbf{N}$, $n > 1$ *nor* C_n^+, $n \in \mathbf{N}$, $n > 1$ *are splitting algebras in* $\underline{M}$.

Proof. Suppose that C_n^+ is a splitting algebra for some $n \in \mathbf{N}$, $n > 1$. Let $\underline{M} : C_n^+$ denote the corresponding splitting variety. Since $C_n^+ \notin V(L_m^+)$ for any $m \in \mathbf{N}$,—in fact, $V(L_m^+)_{SI} \subseteq HS(L_m^+) = \{L_n^+ \mid 0 \leqslant n \leqslant m\}$, as one easily verifies—$L_m^+ \in \underline{M} : C_n^+$ for all $m \in N$, hence $V(\{L_m^+ \mid m \in \mathbf{N}\}) \subseteq \underline{M} : C_n^+$. But this contradicts our result in 2.4 which implies that $C_m^+ \in V(\{L_n^+ \mid n \in \mathbf{N}\})$. In a similar way one shows that the L_n^+ are not splitting algebras in $\underline{M}$.

Note that in fact we have proved that the L_n^+ and C_n^+, $n \in \mathbf{N}$, $n > 1$ are not splitting in the variety of modal algebras defined by the equation $x^0 \cdot x = x^0$.

3. An interval $[\underline{K}_1, \underline{K}_2]$ in which $\underline{K}_1$ does not possess a cover

In our construction we use a modification of the recession frame (cf. [1]). Let B be the subalgebra of finite and cofinite subsets of F^+ where F is the frame $(\mathbf{N}, R)$ and

$$Rij \text{ iff } j = i-1 \quad \text{or} \quad j \geq i+1 \quad \text{or} \quad i = j \quad \text{and} \quad i \notin \{k^2 \mid k \in \mathbf{N},\ k > 1\}.$$

Let B^n be the subalgebra of B generated by the element $\{1, \ldots, n\} \in B$, so $B^n = [\{1, \ldots, n\}] \subseteq B$. If $\underline{K}_1 = \bigcap_{n=1}^{\infty} V(B^n)$ and $\underline{K}_2 = V(B)$, then the interval $[\underline{K}_1, \underline{K}_2]$ satisfies the announced property. The one element modal algebra is denoted by $\underline{1}$, and if $x \in A \in \underline{M}$ then $x^{0^0} = x$ and $x^{0^n} = (x^{0^{n-1}})^0$ for $n \in \mathbf{N}$. The following facts about the algebra B will be used repeatedly.

(1) x finite $\Rightarrow x^0 = 0$

For x cofinite, choose i minimum such that $[i, \infty) \subseteq x$.

(2) $i = 1 \Rightarrow x = x^0 = 1$

(3) $i \geq 2$, $i-2 \in x$, $i-1$ a square $\Rightarrow x^0 = \{i-1, i+1, i+2, \ldots\}$

(4) $i \geq 2$, $i-2 \notin x$ or $i-1$ not a square $\Rightarrow$
$x^0 = \{i+1, i+2, \ldots\}$.

(5) $i \geq 2 \Rightarrow x^{00} = [i+2, \infty)$ & $x^0 \cdot x = [i+1, \infty)$

3.1 LEMMA. (i) $H(B^n) = \{\underline{1}, \underline{2}, B^n\}$

(ii) $S(B^n) \subseteq I(\{B^m \mid m \geq n\} \cup \{\underline{2}\})$.

(iii) *If* $B^n \cong B^m$ *then* $n = m$.

Proof. (i) Let $h: B^n \to A$ be an onto homomorphism which is not 1–1; say $x \in B^n$, $x \neq 1$, $h(x) = 1$. Then $h(x^{0^n}) = 1$ for $n = 1, 2, \ldots$ and one easily verifies that for each cofinite set y in B^n there is an $m \in \mathbf{N}$ such that $y \geq x^{0^m}$, so $h(y) = 1$ for every cofinite set y. Thus $A \cong \underline{2}$ or $A \cong \underline{1}$ and clearly both possibilities occur.

(ii) Let $A \in S(B^n)$, $A \neq \underline{2}$, and let $m = \min\{k \mid [k, \infty) \in A,\ k > 1\}$; then m is well-defined and $m>n$, and for all $k\geq 0$ $\{m+k\}\in A$ since $\{m+k\}=[m,\infty)^{0^k}\cdot([m,\infty)^{0^{k+1}})'$. If x is an atom of A such that $x\leq[1,m-1]$ then either $[1,n]\leq x$ or $x\leq[n+1,m-1]$. There are at most two atoms of A contained in $[1,m-1]$ for if there were three or more then at least one of them, say x_1, would satisfy $x_1\leq[1,m-3]$ and then $x_1'^0x_1'$ has the form $[p,\infty)$, $1<p\leq m-1$, contradictory to our choice of m. If x_1 is the atom of A contained in $[1,m-1]$ such that $m-1\notin x_1$, x_2 the other atom of A such that $x_2\leq[1,m-1]$ then the map $x_1\mapsto[1,m-2]$, $x_2\to\{m-1\}$, $\{m+k\}\mapsto\{m+k\}$ for $k=0,1,2,\dots$ is easily seen to induce an isomorphism from A to B^{m-2}. And if there is only one atom of A contained in $[1,m-1]$ then $A\cong B^{m-1}$.

(iii) In B^n the element $x_n=\mathbf{N}\backslash\{1,\dots,n\}$ is the unique element x satisfying: $\underline{1}$ covers x and for all y, if $y\neq 1$ then $y\cdot y^0\leq x^0$. It follows that if $h:B^n\to B^m$ is an isomorphism then $h(x_n)=x_m$ and hence $h(\{n+k\})=h(x_n^{0^{k-1}}\cdot(x_n^{0^k})')=x_m^{0^{k-1}}\cdot(x_m^{0^k})'=\{m+k\}$, $k=1,2,3,\dots$. It follows that $n+k$ is irreflexive iff $m+k$ is irreflexive which can only be if $n=m$ (note that for any $i>1$, i is a square iff $\{i\}\leq\{i\}'^0$).

Observe that in order to represent the algebra B^n, $n\in\mathbf{N}$, $n>1$ we do not need the full frame F. In fact, B^n is isomorphic to the algebra of finite and cofinite subsets of $F_n=(\mathbf{N}_n,R_n)$, where $\mathbf{N}_n=\{m\in\mathbf{N}\mid m\geq n\}$ and

$$R_nij \text{ iff } i,j\in\mathbf{N}_n \quad\text{and}\quad j=i-1$$

$$\text{or } j\geq i+1 \text{ or } i=j \text{ and } i\notin\{k^2\mid k\in\mathbf{N}, k^2>n\},$$

also to be denoted by B^n.

In order to guarantee that the interval $[\underline{K}_1,\underline{K}_2]$ is proper we prove

3.2 THEOREM. *For every* $n\in\mathbf{N}$ $V(B^{n+1})\subsetneq V(B^n)$.

Proof. Suppose that $B^n\in V(B^{n+1})_{SI}$ for some $n\in\mathbf{N}$. Then there exists an ultrapower A_1 of B^{n+1} and a homomorphism $h:A_1\to A_2$ which is onto such that $B^n\in S(A_2)$, in virtue of Jónsson's result and the fact that for any class $\underline{K}$ of modal algebras $HS(\underline{K})=SH(\underline{K})$. Indeed, congruence relations on a modal algebra A correspond to filters F on the Boolean part of A such that $x\in F\Rightarrow x^0\in F$. Hence if A is a subalgebra of B, F is a filter on A, and G is the filter on B generated by F, then $G\cap A=F$. From these facts it is routine to check that $HS(\underline{K})=SH(\underline{K})$.

By 2.3 A_1 is an algebra of subsets of some ultrapower of F_{n+1}, which can be represented as (X,R), where $X=\bigcup_{\alpha\in\Gamma}X_\alpha$, Γ being a densely linearly ordered set with first element α_0 and without last element, $X_{\alpha_0}=\mathbf{N}_{n+1}$ and the X_α, $\alpha\neq\alpha$ being disjoint copies of the integers. The relation R will be of the following form:

$$Rij \text{ iff } \begin{cases} i, j \in X_{\alpha_0} \text{ and } j = i-1 \text{ or } j \geq i+1 \text{ or } i = j \text{ and } i \neq k^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad k \in \mathbf{N},\ k^2 > n \\ i, j \in X_\alpha \text{ for some } \alpha \in \Gamma,\ \alpha > \alpha_0 \text{ and } j = i-1 \text{ or } i+1 \leq j \text{ or} \\ \qquad\qquad\qquad\qquad i = j, \text{ for all } i \in X_\alpha \text{ except at most one} \\ i \in X_\alpha,\ j \in X_\beta,\ \alpha < \beta. \end{cases}$$

So the frame starts with a copy of F_{n+1} and consists further of copies of Z with the relation of the recession frame, except that one element may be irreflexive. Now suppose that h is not 1–1. Then, we claim, $A_2 \cong A_1 \mid F'$ for some generated subframe $F' = (X', R \mid X')$, where $X' = \bigcup_{\alpha \in \Gamma'} X_\alpha$, Γ' being a proper final segment of Γ. Indeed, let $X' = \bigcap \{x \in A_1 \mid h(x) = 1\}$. Since $h(x) = 1$ implies $h(x^0) = 1$ and h is not 1–1 X' satisfies the requirements. Since $B^n \in S(A_2)$ we may assume that $X' \neq \emptyset$. It suffices now to verify that for $x \in A_1$ if $x \cap X' = X'$ then $h(x) = 1$. Note that $x^0 \supsetneq X'$, for if $x^0 = X'$ then $x^0 = x^{00}$, contradicting the fact that the algebra B^{n+1} satisfies the sentence

$$\forall x[x \neq 1 \wedge x^0 \neq 0 \Rightarrow x^{00} < x^0]$$

hence so does A_1.

Let $y \in A_1$ be such that $h(y) = 1$ and $x^{00} \not\leq y$. Then $X' \subseteq y$, and $X' \subsetneq y^0 \leq x^{00} \leq x$, hence $h(x) = 1$, as was to be shown. Let $i: B^n \to A_2$ be an $\underline{M}$-embedding and let $i((\mathbf{N}_n \setminus \{n\})^{00}) = u \in A_2$. Then there is precisely one $\alpha \in \Gamma'$ such that $u \cap X_\alpha = [m, \infty)$ for some $m \in X_\alpha$. Using the fact that $\mathbf{N}_n \setminus \{n\}$ generates B^n we see that $i(\{n+k\}) = \{m+k-3\}$ for $k \geq 3$. However, there is then a $k \in \mathbf{N}$, $k \geq 3$ such that $\{n+k\}$ is irreflexive, hence $\{n+k\} \leq \{n+k\}'^0$, while $i(\{n+k\})$ is reflexive, so $i(\{n+k\}) \not\leq i(\{n+k\})'^0$, a contradiction. Hence our assumption that h is not 1–1 cannot be true, so there is an $\underline{M}$-embedding $i: B^n \to A_1$. If $i((\mathbf{N}_n \setminus \{n\})^{00}) = u$ and $u \cap X_\alpha = [m, \infty)$ for some $m \in X_\alpha$, $\alpha > \alpha_0$ then we arrive at the same contradiction as before, hence $u \cap X_\alpha = [m, \infty)$ for some $m \in X_{\alpha_0}$. But then, since $\mathbf{N} \setminus \{n\}$ generates B^n, B^n is isomorphic to a subalgebra of the algebra $\{x \in A_1 \mid x \cap X_{\alpha_0}$ is finite or $x' \cap X_{\alpha_0}$ is finite$\}$ which is isomorphic to B^{n+1}. By 3.1 (ii) and (iii) this is impossible.

It will follow from the next theorem that $\underline{K}_1$ does not possess a cover in $[\underline{K}_1, \underline{K}_2]$.

3.3 THEOREM. *Let $\underline{K}$ be a variety such that $\underline{K}_1 \subsetneq \underline{K} \subseteq \underline{K}_2$. Then $\underline{K} = V(B^m)$ for some $m \in \mathbf{N}$.*

Proof. Let $A \in \underline{K}$ be a finitely generated subdirectly irreducible algebra such that $A \notin \underline{K}_1$. We will show that $V(A) = V(B^m)$ for some $m \in \mathbf{N}$. Let A_1 be some ultrapower of B, say $A_1 = \prod_{i \in I} B/\sim U$ where $\sim U$ is the congruence relation on $\prod_{i \in I} B$ induced by the ultrafilter U on I, $h: A_1 \twoheadrightarrow A_2$ a surjective homomorphism and $A \in S(A_2)$. We claim that if h is not 1–1 then $A_2 \in \bigcap_{n=1}^{\infty} V(B^n) = \underline{K}_1$, hence also $A \in \underline{K}_1$. Let $B_1^n = \prod_{i \in I} B^n/\sim U$ and define $g: B_1^n \twoheadrightarrow A_2$ by $g = h \mid B_1^n$. This makes sense since $B^n \in S(B)$, hence $B_1^n \in S(A_1)$. The map g is an $\underline{M}$-homomorphism and it remains to show that g is onto. Let $a \in A_2$ and $x = \overline{(x_i)_{i \in I}} \in A_1$ such that $h(x) = a$. Let $y_i = [n+1, \infty) \in B$, and $y = \overline{(y_i)_{i \in I}}$, then $h(y) = 1$. Indeed, there is a $z \in A_1$, $z \neq 1$ such that $h(z) = 1$. If $z = \overline{(z_i)_{i \in I}}$ then $\{i \in I \mid z_i^{0^n} \leq y_i\} \in U$, so $z^{0^n} \leq y$ hence $h(y) = 1$. But $h(y \cdot x) = h(y) \cdot h(x) = 1 \cdot h(x) = h(x) = a$, and $y \cdot x = \overline{(y_i \cdot x_i)_{i \in I}}$ where $y_i \cdot x_i \in B^n$ thus $y \cdot x \in B_1^n$. So $g(y \cdot x) = h(y \cdot x) = a$, hence g is onto. Therefore h cannot be 1–1, so we may assume that $A \in S(A_1)$. Recall that we may think of A_1 as an algebra of subsets of a frame (X, R) as described in the proof of 3.2. Since in B

$$\forall x[(x^0 \neq 0 \vee x'^0 \neq 0) \wedge (x^0 \neq 0, 1 \Rightarrow (x^{00} \neq 0 \wedge x^{00} < x^0)],$$

this sentence also holds in A_1, so there is an $\alpha \in \Gamma$, $n \in X_\alpha$, such that $\{m \mid m \in X_\alpha, m \geq n \text{ or } m \in X_\beta, \beta > \alpha\}$, denoted by $[n)$, belongs to A, because, as one easily verifies, every element x^{00} such that $x^{00} \neq 0$ is of this form. Now assume that for no $n \in X_{\alpha_0}$, $n > 1$, $[n) \in A$. Then there exists a homomorphism $h: A_1 \twoheadrightarrow A_2$ which is onto and not 1–1 such that $A \in S(A_2)$, and we are back in the previous case. For let $F' = (\bigcup_{\substack{\alpha \in \Gamma \\ \alpha > \alpha_0}} X_\alpha, R \mid \bigcup_{\substack{\alpha \in \Gamma \\ \alpha > \alpha_0}} X_\alpha)$ then the map $h: A_1 \twoheadrightarrow A_1 \mid F'$ defined by $x \to x \cap \bigcup_{\substack{\alpha \in \Gamma \\ \alpha > \alpha_0}} X_\alpha$ is a homomorphism which is onto but not 1–1, and $h \mid A$ is 1–1, since if $x \in A$ such that $x \neq 1$ then $x^{00} = [n)$ for some $n \in \bigcup_{\alpha > \alpha_0} X$, hence $h(x^{00}) \neq 1$ thus $h(x) \neq 1$. So there is an $n \in N$, $n > 1$ such that $n \in X_{\alpha_0}$ and $[n) \in A$. Let n_0 be the smallest such number. Now let $C = \{y \cap X_{\alpha_0} \mid y \in A, y \cap X_{\alpha_0} \text{ is finite or cofinite}\}$; then C is isomorphic to a subalgebra of A. Identifying X_{α_0} with $\mathbf{N}$, we may consider C to be a subalgebra of B. Let $j: C \twoheadrightarrow B^m \subseteq B$ denote the isomorphism described in the proof of 3.1 (ii). Observe that $m = n_0 - 2$ if $[1, n_0 - 1]$ contains two atoms of A (and hence of C), $m = n_0 - 1$ otherwise, i.e. if $[1, n_0 - 1]$ is an atom of A (and hence of C). It follows that $B^m \in V(A)$. To prove that $A \in V(B^m)$ as well, note that if $y \in A$, $y = \overline{(y_i)_{i \in I}}$, then we may assume that $y_i \in C$, $i \in I$. For in virtue of the last statement of 2.3, for any $k \in \mathbf{N}$, $y \in A$, $k \in y$ if and only if $\{i \in I \mid k \in y_i\} \in U$. Since $[1, n_0 - 1]$ is finite, it follows therefore that $\{i \in I \mid y_i \cap [1, n_0 - 1] = y \cap [1, n_0 - 1]\} \in U$, so we may assume that for all $i \in I$, $y_i \cap [1, n_0 - 1] \in C$. Since for any $z \in B$, $z \in C$ if and only if $z \cap [1, n_0 - 1] \in C$ we conclude that we may assume that $y_i \in C$, for all $i \in I$. Using the fact that $C \in S(B)$ we see that $A \in S(\prod_{i \in I} C/\sim U) \subseteq IS(\prod_{i \in I} B^m/\sim U)$, whence $A \in V(B^m)$. Now let

$m_0 = \min\{m \mid B^m \in \underline{K}\}$. Then $V(B^{m_0}) \subseteq \underline{K}$, and conversely, for any $A \in \underline{K}_{SI}$ either $A \in \underline{K}_1 \subseteq V(B^{m_0})$ or $V(A) = V(B^m)$ for some $m \geqslant m_0$, in which case $A \in V(B^{m_0})$ as well. Thus $\underline{K} = V(B^{m_0})$.

3.4 COROLLARY. *The lattice* $[\underline{K}_1, \underline{K}_2]$ *is isomorphic to the chain* $1 + \omega^*$.

The use of frames which are not reflexive in our example is not essential. One can also choose $\underline{K}_1$ and $\underline{K}_2$ in such a way that they satisfy the law $x^0 \leqslant x$. Consider for that purpose the frame $F = (W, R)$ with

$$W = \mathbf{N} \cup \{k^{2+} \mid k \in \mathbf{N},\ k \geqslant 2\}$$

and

$$Rnm \text{ iff } \begin{cases} n, m \in \mathbf{N} \quad \text{and} \quad m \geqslant n-1 \\ \{n, m\} \subseteq \{k^2, k^{2+}\} \quad \text{for some} \quad k \in \mathbf{N},\ k \geqslant 2. \end{cases}$$

Let D be the algebra of finite and cofinite subsets of W and $D^n = [\{[1, n]\}] \subseteq D$. If $\underline{K}_1 = \bigcap_{n=1}^{\infty} V(D^n)$ and $\underline{K}_2 = V(D)$ then also $[\underline{K}_1, \underline{K}_2] \cong 1 + \omega^*$.

However, the question, if the lattice of varieties of interior algebras or even the lattice of varieties of Heyting algebras is strongly atomic or not, has not been answered.

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