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Uniform congruence schemes

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1. Introduction

Mal'cev's Lemma (see [8]) gives a description of principal congruence relations in universal algebras. The general scheme contains many parameters; we define below a Congruence Scheme to formalize this and we shall say that an equational class **K** has a *Uniform Congruence Scheme* if in the whole class Mal'cev's Lemma applies with the same scheme.

We shall examine the consequences of the assumption that an equational class has a Uniform Congruence Scheme. The most important one is that congruence relations of a direct product can be described by the congruence relations of the direct factors.

We shall also relate Uniform Congruence Schemes and the Congruence Extension Property.

These lead to a close relationship between Uniform Congruence Schemes and the concepts of filtrality and ideal congruences of R. Magari (see [14]). This relationship will be explored more fully in §7.

2. Congruence schemes

Let us restate Mal'cev's Lemma:

2.1. MAL'CEV'S LEMMA ([8] and [16]). *Let $\mathfrak{A}$ be a (finitary) algebra and $a, b, c, d \in A$. Let $\Theta(a, b)$ denote the smallest congruence relation under which $a \equiv b$. Then $c \equiv d(\Theta(a, b))$ iff there exists an integer $n \geq 1$, a sequence $c = e_0, \ldots, e_n = d$ of elements of A, and a sequence $p_0, \ldots, p_{n-1}$ of unary algebraic functions such that $\{p_i(a), p_i(b)\} = \{e_i, e_{i+1}\}$, for $i = 0, 1, \ldots, n-1$.*

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Our basic definitions are motivated by 2.1:

2.2. DEFINITION. A *Congruence Scheme S* for a given type τ is given by two integers n and m, $n \geq 2$, $m \geq 1$, by n m-ary polynomials $p_0, \ldots, p_{n-1}$, and by a function $f: \{0, 1, \ldots, n-1\} \to \{0, 1\}$.

2.3. DEFINITION. Let S be a Congruence Scheme as given in 2.2; let $\mathfrak{A}$ be an algebra whose type includes τ, and let $a_0, a_1, b_0, b_1 \in A$. We say that $\langle a_0, a_1, b_0, b_1 \rangle$ is in *S-relation in* $\mathfrak{A}$ (or $S(a_0, a_1, b_0, b_1)$ holds in $\mathfrak{A}$) iff there exist $c_1, \ldots, c_m \in A$ satisfying

$$b_0 = p_0(a_{f(0)}, c_1, \ldots, c_m),$$

$$p_i(a_{1-f(i)}, c_1, \ldots, c_m) = p_{i+1}(a_{f(i+1)}, c_1, \ldots, c_m) \quad \text{for} \quad 0 \leq i \leq n-2,$$

$$p_{n-1}(a_{1-f(n-1)}, c_1, \ldots, c_m) = b_{n-1}.$$

2.4. COROLLARY. *Let φ be a homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ and let S be given as in* 2.2. *If $S(a_0, a_1, b_0, b_1)$ holds in $\mathfrak{A}$ then $S(a_0\varphi, a_1\varphi, b_0\varphi, b_1\varphi)$ holds in $\mathfrak{B}$.*

Proof. Indeed, it does, using $c_1\varphi, \ldots, c_n\varphi$.

2.5. COROLLARY. *Let $\mathfrak{A}$ be an algebra and let $a, b, c, d \in A$. Then $c \equiv d(\Theta(a, b))$ iff there exists a Congruence Scheme S such that $S(a, b, c, d)$ holds in $\mathfrak{A}$.*

Proof. This is a restatement of 2.1.

2.6. EXAMPLE. Let **D** denote the class of distributive lattices. Consider the following 5-ary polynomials in the variables z, x_0, x_1, y_0, y_1:

$$p_0 = (y_0 \vee y_1) \wedge (y_0 \vee (x_0 \wedge z)),$$

$$p_1 = (y_0 \vee y_1) \wedge (y_0 \vee x_0 \vee z),$$

$$p_2 = y_1 \vee (y_0 \wedge (x_0 \vee z)),$$

$$p_3 = y_1 \vee (y_0 \wedge x_0 \wedge z),$$

and let S be the Congruence Scheme with $n = 4$, $m = 5$, $p_0, \ldots, p_3$ as given, and $f(0) = f(2) = 1$, $f(1) = f(3) = 0$. Then (see [9] and also [11]) for any $L \in \mathbf{D}$ and $a, b, c, d \in L$, $c \equiv d(\Theta(a, b))$ iff $S(a, b, c, d)$ in L.

This gives us the motivation for the next definition.

2.7. DEFINITION. Let **K** be a class of algebras of the same type. Then **K** is said to have a *Uniform Congruence Scheme* (UCS, for short) iff there exists a

Congruence Scheme S of the same type satisfying the following condition:

For any $\mathfrak{A} \in K$ and $a, b, c, d \in A$, $c \equiv d(\Theta(a, b))$ iff $S(a, b, c, d)$ holds in $\mathfrak{A}$.

Thus 2.6 shows that the class of **D** has a Uniform Congruence Scheme. It is not difficult to see that if **K** is an equational class having permutable congruence relations and **K** has a UCS, then **K** has a UCS S with $n = 1$ and, conversely, if **K** has a UCS S with $n = 1$, then **K** has permutable congruences.

It is easily seen that the class **G** of all groups and the class **A** of all abelian groups have no UCS-s. Further examples shall be given in the next section.

3. Equational definability and factor determined congruences

The following description of $\Theta(a, b)$ in distributive lattices (see [9]) is much simpler than the one given in 2.6:

$$c \equiv d(\Theta(a, b)) \quad \text{iff} \quad a \wedge b \wedge c = a \wedge b \wedge d \quad \text{and} \quad a \vee b \vee c = a \vee b \vee d.$$

This motivates the following definition:

3.1. DEFINITION. Let **K** be a class of algebras of the same type. **K** has *Equationally Definable Principal Congruences* (EDPC, for short) iff there is a set of equations $\{p_i = q_i \mid i \in I\}$ such that for any $\mathfrak{A} \in K$ and $a, b, c, d \in A$, $c \equiv d(\Theta(a, b))$ is equivalent to the existence of $e_0, e_1, \ldots \in A$ such that $p_i(a, b, c, d, e_0, e_1, \ldots) = q_i(a, b, c, d, e_0, e_1, \ldots)$ for all $i \in I$.

3.2. EXAMPLE. The class of distributive lattices with pseudocomplementation. It was shown in [13] by H. Lakser that if $\mathfrak{B} = \langle L; \vee, \wedge, {}^* \rangle$ is a distributive lattice with pseudocomplementation and $a, b, c, d \in L$, then $c \equiv d(\Theta(a, b))$ iff $c \wedge a = d \wedge a$ and $(c \vee b) \wedge (a^* \wedge b)^* = (d \vee b) \wedge (a^* \wedge b)^*$.

Now we are ready to state our first result.

3.3. THEOREM. *Let* **K** *be an equational class. Then* **K** *has a Universal Congruence Scheme iff* **K** *has Equationally Definable Principal Congruences.*

Before we prove this theorem we consider some more concepts.

3.4. DEFINITION. A class $\mathbf{K}$ of algebras of the same type has *Factor Determined Principal Congruences* (FDPC, for short) *on Direct Products* iff whenever $\mathfrak{A}_i \in \mathbf{K}$ for $i \in I$, $a_i, b_i, c_i, d_i \in A_i$ and $c_i \equiv d_i(\Theta(a_i, b_i))$, then in the direct product $\mathfrak{A} = \Pi(\mathfrak{A}_i \mid i \in I)$ there holds $c \equiv d(\Theta(a, b))$, where $a = \langle a_i \mid i \in I \rangle$, $b = \langle b_i \mid i \in I \rangle$, $c = \langle c_i \mid i \in I \rangle$, and $d = \langle d_i \mid i \in I \rangle$.

3.5. THEOREM. *Let* $\mathbf{K}$ *be an equational class. Then* $\mathbf{K}$ *has a Universal Congruence Scheme iff* $\mathbf{K}$ *has Factor Determined Principal Congruences on Direct Products.*

We shall consider one more property of an equational class $\mathbf{K}$:

(F) There exists an algebra $\mathfrak{A}$ in $\mathbf{K}$ and elements $a, b, c, d \in A$ such that $c \equiv d(\Theta(a, b))$ and for any algebra $\mathfrak{B} \in \mathbf{K}$ and elements $a', b', c', d' \in B$ if $c' \equiv d'(\Theta(a', b'))$, then there is a homomorphism φ of $\mathfrak{A}$ into $\mathfrak{B}$ satisfying $a\varphi = a'$, $b\varphi = b'$, $c\varphi = c'$, and $d\varphi = d'$.

Proof of 3.3 *and* 3.5. We prove that the three conditions in 3.3 and 3.5 (UCS, EDPC, FDPC (on Direct Products)) are equivalent to each other and to (F) for an equational class $\mathbf{K}$.

UCS *implies* EDPC. This is trivial.

EDPC *implies* FDPC on Direct Products. This is also trivial.

FDPC on Direct Products *implies* (F). Let $\mathbf{K}_f$ be a set of algebras of $\mathbf{K}$ containing, up to isomorphism, all the finitely generated algebras of $\mathbf{K}$. Consider all sequences: $\langle \mathfrak{A}_i, a_i, b_i, c_i, d_i \rangle$, $i \in I$, where $\mathfrak{A}_i \in \mathbf{K}_f$, $a_i, b_i, c_i, d_i \in A_i$, and $c_i \equiv d_i(\Theta(a_i, b_i))$. Set $\mathfrak{A} = \Pi(\mathfrak{A}_i \mid i \in I)$, $a = \langle a_i \mid i \in I \rangle$, $b = \langle b_i \mid i \in I \rangle$, $c = \langle c_i \mid i \in I \rangle$, and $d = \langle d_i \mid i \in I \rangle$. Then by FDPC on Direct Products, we have $c \equiv d(\Theta(a, b))$. We claim that $\mathfrak{A}$ and a, b, c, d satisfy (F) in $\mathbf{K}$. The first clause is trivial; to verify the second clause let $\mathfrak{B} \in \mathbf{K}$ and $a', b', c', d' \in B$, $c' \equiv d'(\Theta(a', b'))$. It is clear from 2.5 that there exists a finitely generated subalgebra $\mathfrak{B}_1$ of $\mathfrak{B}$ such that $a', b', c', d' \in B_1$ and $c' \equiv d'(\Theta(a', b'))$ in $\mathfrak{B}_1$. By the definition of $\mathbf{K}_f$, there is a $\mathfrak{C} \in \mathbf{K}_f$ and an isomorphism $\psi : \mathfrak{C} \to \mathfrak{B}_1$. Since $\mathfrak{C} \in \mathbf{K}_f$ and $c'\psi^{-1} \equiv d'\psi^{-1}(\Theta(a'\psi^{-1}, b'\psi^{-1}))$ in $\mathfrak{C}$, the sequence $\langle \mathfrak{C}, a'\psi^{-1}, b'\psi^{-1}, c'\psi^{-1}, d'\psi^{-1} \rangle$ is of the form $\langle \mathfrak{A}_i, a_i, b_i, c_i, d_i \rangle$. Thus if π_i is the i-th projection, $\pi_i\psi$ is the homomorphism satisfying the second clause of (F).

(F) *implies* UCS. Let the Congruence Scheme S satisfy $S(a, b, c, d)$ in $\mathfrak{A}$, where $\mathfrak{A}$ and a, b, c, d are given in (F). By 2.4 and (F), $S(a', b', c', d')$ holds whenever $a', b', c', d' \in B$ and $\mathfrak{B} \in \mathbf{K}$ and $c' \equiv d'(\Theta(a', b'))$.

This completes the proofs of Theorems 3.3 and 3.5.

3.6. COROLLARY. *Let* **K** *be an equational class of algebras having* EDPC *and let* $p_i = q_i$, $i \in I$ *be a set of equations defining the principal congruences in* **K**. *Then there exists a finite subset* I_1 *of* I *such that* $p_i = q_i$, $i \in I_1$ *defines the principal congruences in* **K**.

Proof. Consider the algebra $\mathfrak{A}$ and $a, b, c, d \in A$ which exists in **K** by (F). Let us expand the type of **K** by adding nullary operations denoted by $a, b, c, d, e_1, e_2, \ldots$ so the equations $p_i = q_i$, $i \in I$ turn into identities. Let $Id(\mathbf{K})$ denote the identities holding in **K** and set $\Sigma = ID(\mathbf{K}) \cup \{p_i = q_i \mid i \in I\}$.

Now consider the algebra $\mathfrak{A}$ and $a, b, c, d \in A$ as given by (F). By EDPC, there exist $e_1, e_2, \ldots \in A$ such that $p_i(a, b, c, d, e_1, e_2, \ldots) = q_i(a, b, c, d, e_1, e_2, \ldots)$ for all $i \in I$. Hence the same holds in $\mathfrak{A}_1$, the subalgebra generated by $a, b, c, d, e_1, e_2, \ldots$. With the obvious interpretation of constants, $\mathfrak{A}_1$ satisfies Σ.

Let S be a Congruence Scheme satisfying $S(a, b, c, d)$ in $\mathfrak{A}_1$ with the auxiliary element $c_1, \ldots, c_m$. Since $c_1, \ldots, c_m \in A_1$, they are all polynomials of constants, hence the equations of 2.3 turn into a finite set of identities Ω. Σ implies Ω, hence there is a finite subset Σ_1 of Σ that implies Ω by the Compactness Theorem for Equational Logic. Thus $Id(\mathbf{K})$ and $\{p_i = q_i \mid i \in I\} \cap \Sigma_1 = \{p_i = q_i \mid i \in I_1\}$ imply Ω. But Ω is sufficient to prove that S is a UCS which now easily yields that $p_i = q_i$, $i \in I_1$ is a finite equational definition of principal congruences.

The following result provides some nontrivial examples of equational classes having a UCS.

3.7. THEOREM. *Let* **K** *be a congruence permutable and congruence distributive variety generated by finitely many finite algebras* $\mathfrak{A}_1, \ldots, \mathfrak{A}_n$. *Furthermore, let every subalgebra of each* $\mathfrak{A}_i$ *be a subdirect product of simple algebras. Then* **K** *has a UCS.*

Proof. By the well-known theorem of B. Jónsson [12], the subdirectly irreducible algebras in **K** are all in $HS(\mathfrak{A}_1, \ldots, \mathfrak{A}_n)$. Now if $\mathfrak{A} \subseteq \mathfrak{A}_i$, then the congruence lattice of $\mathfrak{A}$ is Boolean. Hence all subdirectly irreducible algebras in **K** are simple. Since **K** is congruence permutable, all finite algebras in **K** are direct products of simple algebras. Let $\langle \mathfrak{B}_i; a_i, b_i, c_i, d_i \rangle$, $i \in I$ be all sequences where $\mathfrak{B}_i \in \mathbf{K}$ is simple, $a_i, b_i, c_i, d_i \in B_i$, and $c_i \equiv d_i(\Theta(a_i, b_i))$. Note that $c_i \equiv d_i(\Theta(a_i, b_i))$ iff either $a_i = b_i$ and $c_i = d_i$ or $a_i \neq b_i$, and that $|I|$ is finite. Let $\mathfrak{B} = \Pi(\mathfrak{B}_i \mid i \in I)$, $a = \langle a_i \mid i \in I \rangle$, $b = \langle b_i \mid i \in I \rangle$, $c = \langle c_i \mid i \in I \rangle$, $d = \langle d_i \mid i \in I \rangle$. We claim that $c \equiv d(\Theta(a, b))$. To see this, first note that each congruence of $\mathfrak{B}$ is the kernel of a projection onto the product of a subset of $\{\mathfrak{B}_i \mid i \in I\}$. But then $\Theta(a, b)$ corresponds to the subset $E(a, b) = \{i \mid a_i = b_i\}$. On the other hand, $a_i = b_i$ implies $c_i = d_i$ so $E(a, b) \subseteq E(c, d)$; hence $c \equiv d(\Theta(a, b))$. Next we claim that $\mathfrak{B}$ and a, b, c, d satisfy (F). For this, let $\mathfrak{C} \in \mathbf{K}$, $a', b', c', d' \in C$, and $c' \equiv d'(\Theta(a', b'))$. Without loss of generality we may assume that $\mathfrak{C}$ is finitely generated. Hence $\mathfrak{C} \cong \Pi(\mathfrak{C}_j \mid j \in J)$ where each $\mathfrak{C}_i$ is simple. For $j, k \in J$ define $j \sim k$ iff $a'_j = a'_k$, $b'_j = b'_k$, $c'_j = c'_k$, $d'_j = d'_k$. Let $C' = \{e \in C \mid e_j = e_k \text{ if } j \in k\}$; then $\mathfrak{C}'$ is a subalgebra of $\mathfrak{C}$. In fact $\mathfrak{C}' \cong \Pi(\mathfrak{C}_j \mid j \in J')$ for some $J' \subseteq J$. But it is also clear that $\mathfrak{C}' \cong \Pi(\mathfrak{B}_i \mid i \in I')$ for some $I' \subseteq I$. Hence there is a homomorphism $\varphi : \mathfrak{B} \to \mathfrak{C}$ such that $\varphi(a) = a'$, $\varphi(b) = b'$, $\varphi(c) = c'$, $\varphi(d) = d'$. Thus, as in the proof of 3.4, **K** has UCS.

3.8. EXAMPLE. Varieties generated by finitely many finite simple Kirkman algebras (see [19]). In [19] the third author introduced the concept of a near-Boolean algebra (that is, an algebra of type $\langle 2, 3, 1, 0, 0\rangle$ satisfying all 2-variable identities true in Boolean algebras). Thus the variety of all near-Boolean algebras is congruence permutable and congruence distributive. Among the near-Boolean algebras are the simple Kirkman algebras; for each Steiner triple system of order $n \geq 7$ there is a simple Kirkman algebra of order $2n+2$. Moreover, each subalgebra of a simple Kirkman algebra is again a simple Kirkman algebra or is a Boolean algebra. Hence by 3.7 a variety generated by finitely many finite simple Kirkman algebras has a UCS.

4. Congruence extension property

We start with a definition (see [10]) and a result of A. Day.

4.1. DEFINITION. Let **K** be a class of algebras of the same type. **K** is said to have the *Congruence Extension Property* (CEP, for short) iff for any algebra $\mathfrak{A} \in \mathbf{K}$, subalgebra $\mathfrak{B}$ of $\mathfrak{A}$, and congruence relation Θ of $\mathfrak{B}$, there exists a congruence relation Φ of $\mathfrak{A}$ whose restriction to $\mathfrak{B}$ is Θ. In other words, every congruence relation of $\mathfrak{B}$ can be extended to $\mathfrak{A}$.

4.3. EXAMPLES. The class of distributive lattices has CEP (see [8]). The **K** *has CEP iff for any* $\mathfrak{A} \in \mathbf{K}$ *and* $a, b, c, d \in A$, $c \equiv d(\Theta(a, b))$ *in* $\mathfrak{A}$ *iff* $c \equiv d(\Theta(a, b))$ *holds in the subalgebra of* $\mathfrak{A}$ *generated by* a, b, c, *and* d.

4.3. EXAMPLES. The class of distributive lattices has CEP (see [8]). The class of distributive lattices with pseudocomplementation has CEP (see [10]).

4.2 gives us guidance as to how to modify the definitions of §§2 and 3 to accommodate CEP.

4.4. DEFINITIONS. A *Restricted Congruence Scheme* S is a Congruence Scheme (as in 2.2) with $m = 5$; the relation $S(a, b, c, d)$ is defined as in 2.3 with the restriction that $c_1 = a$, $c_2 = b$, $c_3 = c$, $c_4 = d$. URCS stands for Uniform Restricted Congruence Scheme. If in 3.1 the p_i and q_i are 4-ary, then **K** is said to have REDPC. If 3.4 is required to hold for any subalgebra of $\mathfrak{A}$, we say that **K** has FDPC for Subdirect Products. Finally, (RF) stands for (F) with the additional hypothesis that $\mathfrak{A}$ is generated by a, b, c, and d.

4.5. THEOREM. *Let* **K** *be an equational class of algebras. Then the following conditions are equivalent:*

(i) **K** *has a* UCS *and* **K** *has* CEP;
(ii) **K** *has a* URCS;
(iii) **K** *has* REDPC;
(iv) **K** *has* FDPC *for Subdirect Products*;
(v) **K** *satisfies* (RF).

Proof. The proof of this result is very similar to the proofs presented in §3; one only has to observe that if $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ and $a, b, c, d \in B$, then CEP implies that $c \equiv d(\Theta(a, b))$ in $\mathfrak{A}$ exactly if $c \equiv d(\Theta(a, b))$ in $\mathfrak{B}$.

4.6. EXAMPLE. A simple Kirkman algebra contains an 8-element Boolean algebra which, of course, is not simple. Thus a variety generated by finitely many finite simple Kirkman algebras has a UCS but not CEP and so has no URCS.

5. Ideal congruences and filtrality

In the previous section we have seen how principal congruences in a (sub) direct product are determined by the factors. Now we extend this to arbitrary congruences. The underlying idea is due to R. Magari [14].

5.1. DEFINITIONS. Let $\mathfrak{A} = \Pi(\mathfrak{A}_j \mid j \in J)$ and let I be an ideal of the join-semilattice $\Pi(\text{Comp}\,\mathfrak{A}_j \mid j \in J)$, where $\text{Comp}\,\mathfrak{A}_j$ is the join-semilattice of compact congruence relations of $\mathfrak{A}_j$. We define a congruence relation Θ_I on $\mathfrak{A}$ by the rule

$a \equiv b(\Theta_I)$ iff there is a $\Theta = \langle \Theta_j \mid j \in J \rangle \in I$ satisfying $a_j \equiv b_j(\Theta_j)$ for all $j \in J$, where $a = \langle a_j \mid j \in J \rangle$ and $b = \langle b_j \mid j \in J \rangle$.

K is said to have *Ideal Congruences for Direct Products* iff for all $\mathfrak{A} \in$ **K** and for all direct product representations of $\mathfrak{A}$, all congruences of $\mathfrak{A}$ are of the form Θ_I. **K** is said to have *Ideal Congruences for Subdirect Products* iff the same condition holds for all subdirect representations.

This definition is slightly different from, but equivalent to, that of R. Magari [14].

5.2. THEOREM. *An equational class* **K** *has a Uniform Congruence Scheme iff* **K** *has Ideal Congruences for Direct Products.*

Proof. Let **K** have Ideal Congruences for Direct Products and let $\mathfrak{A} = \Pi(\mathfrak{A}_j \mid j \in J)$, $a = \langle a_j \mid j \in J\rangle, \ldots, d = \langle d_j \mid j \in J\rangle \in A$. Then $\Theta(a, b) = \Theta_I$ for some ideal I of $\Pi(\text{Comp}\,\mathfrak{A}_j \mid i \in J)$. Thus there is a $\Theta = \langle \Theta_j \mid j \in J\rangle \in I$ such that $a_j \equiv b_j(\Theta_j)$, for all $j \in J$ and so $\Theta(a_j, b_j) \leq \Theta_j$. We conclude that $c_j \equiv d_j(\Theta(a_j, b_j))$, for all $j \in J$, implies that $c_j \equiv d_j(\Theta_j)$ and so $c \equiv d(\Theta_I)$, that is $c \equiv d(\Theta(a, b))$. Thus **K** has Factor Determined Principal Congruences on Direct Products, hence by Theorem 3.5, **K** has a Universal Congruence Scheme.

Conversely, let **K** have a Universal Congruence Scheme or, equivalently by 3.5, Factor Determined Principal Congruences on Direct Products. Let $\mathfrak{A} = \Pi(\mathfrak{A}_j \mid j \in J) \in \mathbf{K}$ and $a = \langle a_j \mid j \in J\rangle$, $b = \langle b_j \mid j \in J\rangle \in A$. Then $\Theta(a, b) = \Theta_I$, where $I = (\Theta]$ and $\Theta = \langle \Theta(a_j, b_j) \mid j \in J\rangle$. Now let $\varphi = \langle \varphi_j \mid j \in J\rangle \in \Pi(\text{Comp}\,\mathfrak{A}_j \mid j \in J)$, let $I = (\varphi]$, and take the elements $a_0 = \langle a_{0j} \mid j \in J\rangle$ and $a_1 = \langle a_{1j} \mid j \in J\rangle \in A$. We wish to show that $\Theta_I \vee \Theta(a_0, a_1) = \Theta_{I'}$, where $I' = (\langle \varphi_j \vee \Theta(a_{0j}, a_{1j}) \rangle \mid j \in J\rangle]$. Clearly, $\Theta_I \vee \Theta(a_0, a_1) \leq \Theta_{I'}$ so let $b_0 \equiv b_1(\Theta_{I'})$. We form $\mathfrak{A}' = \mathfrak{A}/\Theta_I$ and for $x \in A$ let x' denote the image of x in $\mathfrak{A}'$. Then $\mathfrak{A}/\Theta_I \cong \Pi(\mathfrak{A}_j/\varphi_j \mid j \in J)$. It is obvious that $\Theta_{I'}/\Theta_I = \Theta(a'_0, a'_1)$. Thus $b'_0 \equiv b'_1(\Theta(a'_0, a'_1))$ and so there exists a Congruence Scheme S (as given in 2.2 and 2.3) satisfying $S(a'_0, a'_1, b'_0, b'_1)$ in $\mathfrak{A}'$; that is, there exist $c'_1, \ldots, c'_m \in A'$ satisfying the equations in 2.3. Thus in $\mathfrak{A}$ we have

$$b_0 \equiv p_0(a_{f(0)}, c_1, \ldots, c_m)(\Theta_I),$$

$$p_i(a_{f(i)}, c_1, \ldots, c_m) \equiv p_i(a_{1-f(i)}, c_1, \ldots, c_m)(\Theta(a_0, a_1)), \qquad \text{for } 0 \leq i \leq n-2,$$

$$p_i(a_{1-f(i)}, c_1, \ldots, c_m) \equiv p_{i+1}(a_{f(i+1)}, c_1, \ldots, c_m)(\Theta_I), \qquad \text{for } 0 \leq i \leq n-2,$$

$$p_{n-1}(a_{1-f(n-1)}, c_1, \ldots, c_m) \equiv b_1(\Theta_I).$$

Thus, $b_0 \equiv b_1(\Theta_I \vee \Theta(a_0, a_1))$, proving the equality.

This implies that every compact congruence relation is of the form Θ_I with a principal ideal I. Since every congruence relation is a set union of compact ones it follows readily that every congruence relation is of the form Θ_I. This completes the proof of the theorem.

The analogue of 5.2 for subdirect products is as follows.

5.3. DEFINITION. An equational class **K** has *Ideal Congruences for Subdirect Products* iff whenever $\mathfrak{A} \in \mathbf{K}$ has a subdirect representation as a subalgebra of a direct product $\Pi(\mathfrak{A}_j \mid j \in J)$, then every congruence relation of $\mathfrak{A}$ is the restriction of a suitable Θ_I of $\Pi(\mathfrak{A}_j \mid j \in J)$.

5.4. THEOREM. *An equational class* **K** *has a Uniform Restricted Congruence Scheme iff* **K** *has Ideal Congruences for Subdirect Products.*

Proof. If **K** has URCS, then **K** has CEP by 4.5, and **K** has Ideal Congruences for Direct Products by 5.2, thus **K** has Ideal Congruences for Subdirect Products. Conversely, suppose **K** has Ideal Congruences for Subdirect Products. Let $\langle \mathfrak{A}_j, a_j, b_j, c_j, d_j \rangle$, $j \in J$, be all algebras in **K**, up to isomorphism, satisfying $c_j \equiv d_j(\Theta(a_j, b_j))$ and $A_j = [a_j, b_j, c_j, d_j]$. Then we form $\Pi(\mathfrak{A}_j \mid j \in J)$, $a = \langle a_j \mid j \in J \rangle, \ldots, d = \langle d_j \mid j \in J \rangle$ and $A = [a, b, c, d]$, $\mathfrak{A}$ a subalgebra of $\Pi(\mathfrak{A}_j \mid j \in J)$. Thus there exists an ideal I such that $\Theta(a, b)$ is the restriction of Θ_I to $\mathfrak{A}$. Since $a \equiv b(\Theta_I)$, there is a $\langle \Theta_j \mid j \in J \rangle \in I$ satisfying $a_j \equiv b_j(\Theta_j)$ for all $j \in J$. Therefore, $c_j \equiv d_j(\Theta_j)$ for all $j \in J$ and, by the definition of Θ_i, $c \equiv d(\Theta_I)$. Thus $c \equiv d(\Theta(a, b))$. We have verified that $\mathfrak{A}$ and a, b, c, d satisfy (RF) and so by 4.5, **K** has a URCS, completing the proof of the theorem.

5.2 is especially interesting in special classes:

5.5. DEFINITION. Let **K** be an equational class. **K** is *semisimple* iff all subdirectly irreducible algebras in **K** are simple. Let **K** be a semisimple equational class, $\mathfrak{A} \in \mathbf{K}$, and let $\mathfrak{A}$ be represented as a subdirect product of the simple algebras $\mathfrak{A}_j, j \in J$, $A \subseteq \Pi(A_j \mid j \in J)$. For $a = \langle a_j \mid j \in J \rangle$, $b = \langle b_j \mid j \in J \rangle \in A$, set $E(a, b) = \{j \mid a_j = b_j\} \subseteq J$, the *equalizer* of a and b. For a filter F over I (that is, a dual ideal of the lattice of all subsets of I) we define a relation Θ_F on $A : a \equiv b(\Theta_F)$ iff $E(a, b) \in F$, and call Θ_F a *filtral congruence* on $\mathfrak{A}$. We call $\mathfrak{A}$ *filtral* iff every congruence on $\mathfrak{A}$ is filtral (for *any* subdirect decomposition into subdirectly irreducible algebras). A semisimple equational class **K** is *filtral* iff all $\mathfrak{A} \in \mathbf{K}$ are filtral.

5.6. COROLLARY. *Let* **K** *be a semisimple equational class. Then* **K** *is filtral iff* **K** *has Ideal Congruences for Subdirect Products.*

Proof. This is obvious; the ideal I we get and the filter F are connected by $X \in I$ iff $J - X \in F$.

The concept of a filtral class is due to R. Magari [14]; see also G. M. Bergman [3].

5.7. THEOREM. *Let* **K** *be a semisimple equational class. Then* **K** *is filtral iff* **K** *has a Uniform Restricted* Mal'cev *Scheme.*

Proof. By 5.6 and 5.4.

6. Congruence distributive equational classes with CEP

The connection between CEP and URCS given in 4.5 is even closer for varieties with distributive congruence lattices. Our first result is equivalent to a weaker form of Theorem 1 in G. Mazzanti [17]:

6.1. THEOREM. *Let* **K** *be a congruence-distributive equational class generated by a finite algebra* $\mathfrak{B}$. *Then* **K** *has the Congruence Extension Property iff* **K** *has a Universal Restricted Congruence Scheme.*

Proof. Let us assume that **K** has CEP. Construct $\mathfrak{A}$ and a, b, c, d as in the proof "FDPC on Direct Products implies (F)" in §3 except that the $\mathfrak{A}_i$ $(i \in I)$ are generated by a_i, b_i, c_i, d_i. Since **K** is locally finite, I can be chosen finite. But congruence distributivity implies that every congruence relation on $\Pi(\mathfrak{A}_i \mid i \in I)$ is of the form $\Pi(\Theta_i \mid i \in I)$, where Θ_i is a congruence relation of $\mathfrak{A}_i$ (this remark is attributed to A. Hales in [6]). This implies immediately that $\Theta(a, b)$ is Factor Determined on this direct product, hence $\mathfrak{A}$ and a, b, c, d satisfy (RF). Thus **K** has a URCS by 4.5. The converse is contained in 4.5.

Various stronger forms of 6.1 are easily found. For instance, it is not necessary to assume that **K** is generated by a finite algebra; it is sufficient that $F_{\mathbf{K}}(4)$ be finite. However, the hypothesis of congruence distributivity cannot be dropped.

In contrast to this, R. N. McKenzie [18] has recently shown that the only equational classes of lattices which have definable principal congruences are the classes of distributive lattices and one element lattices (a class of algebras **K** has definable principal congruences if there is a first order formula $\Phi(x, y, u, v)$ with free variables x, y, u, v such that for any $\mathfrak{A} \in \mathbf{K}$ and any $a, b, c, d \in \mathfrak{A}$, $c \equiv d(\Theta(a, b))$ iff $\Phi(a, b, c, d)$ holds in $\mathfrak{A}$). Not coincidentally, these are the only equational classes of lattices with CEP. Hence CEP seems to play a crucial role. Note also that the equational class generated by the two element group has CEP, does not have distributive congruences and does not have UCS. Thus it is the combination of CEP and distributive congruences which give us positive results. In a recent paper, B. A. Davey [4] has proved the following important result:

THEOREM 6.2 (B. A. Davey [4]). *Let* **K** *be a congruence distributive equational class and let* $Si(\mathbf{K})$ *be the class of subdirectly irreducible algebras in* **K**. *Assume that* $Si(\mathbf{K})$ *is axiomatic (i.e., definable by a set of first order sentences). Then* **K** *has CEP iff* $Si(\mathbf{K})$ *has CEP.*

The crucial step in the proof of this theorem is Jónsson's Lemma, the key to all structure theorems about congruence distributive varieties:

THEOREM 6.3 (B. Jónsson [12]). *Let $\mathfrak{A} \subseteq \Pi\{\mathfrak{A}_i \mid i \in I\}$ and assume that the congruence lattice of $\mathfrak{A}$ is distributive. Let Θ be a congruence on $\mathfrak{A}$ such that $\mathfrak{A}/\Theta$ is subdirectly irreducible. Then there is an ultrafilter F on I such that Θ_F restricted to $\mathfrak{A}$ is contained in Θ.*

THEOREM 6.4. *Let* **K** *be a congruence distributive equational class such that $Si(\mathbf{K})$ is axiomatic, has CEP and has definable principal congruences. Then* **K** *has a URCS.*

Proof. By 4.5 we need only show that **K** satisfies (RF). Note that by 6.2, **K** has CEP. Consider all sequences $\langle \mathfrak{A}_i, a_i, b_i, c_i, d_i \rangle$ for $i \in I$ where $\mathfrak{A}_i \in Si(\mathbf{K})$ is generated by $\{a_i, b_i, c_i, d_i\}$ and $c_i \equiv d_i(\Theta(a_i, b_i))$. Let $\mathfrak{A}$ be the subalgebra of $\mathfrak{A}' = \Pi\{\mathfrak{A}_i \mid i \in I\}$ generated by the corresponding a, b, c, d. First we wish to show that $c \equiv d(\Theta(a, b))$. By CEP it is sufficient to show that $c \equiv d(\Theta(a, b))$ in $\mathfrak{A}'$. Now $\Theta(a, b) = \bigwedge\{\Theta_j \mid j \in J\}$ where each $\mathfrak{A}'/\Theta_j$ is subdirectly irreducible. Thus we need only show that $c \equiv d(\Theta_j)$ for each $j \in J$. By 6.3 there is an ultrafilter F on I such that $\Theta_F \leq \Theta_j$. Hence $\Theta_F \vee \Theta(a, b) \leq \Theta_j$. But since $\mathfrak{A}'/\Theta_F$ is an ultraproduct of members of $Si(\mathbf{K})$, and $S_i(\mathbf{K})$ is axiomatic, we conclude that $\mathfrak{A}'/\Theta_F \in Si(\mathbf{K})$. Moreover, $Si(\mathbf{K})$ has definable principle congruences and F is an ultrafilter, hence $c \equiv d(\Theta_F \vee \Theta(a, b))$; thus $c \equiv d(\Theta_j)$ as claimed. Next let $a', b', c', d' \in B$, $\mathfrak{B} \in \mathbf{K}$ with $c' \equiv d'(\Theta(a', b')$. By CEP we may assume that $\mathfrak{B}$ is generated by $\{a', b', c', d'\}$. Now a modification of the argument in the proof of 3.7 shows that the mapping $a \to a'$, $b \to b'$, $c \to c'$, $d \to d'$ induces a homomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$. Thus **K** satisfies (RF) and so has a URCS.

COROLLARY 6.5. *Let* **K** *be a congruence distributive equational class and $Sim(\mathbf{K})$ the class of simple algebras in* **K**; *let* **K**′ *be the equational class generated by $Sim(\mathbf{K})$. Then* **K**′ *is filtral iff $Sim(\mathbf{K})$ is a universal class (that is, definable by a set of universally quantified first order sentences).*

Proof. In [3] G. Bergman shows that $Sim(\mathbf{K})$ is a universal class for any filtral variety **K**. Conversely if $Sim(\mathbf{K})$ is universal, then it obviously has CEP; moreover, principal congruences in $Sim(\mathbf{K})$ are easily seen to be definable.

EXAMPLE 6.6. UBP-s. In [7] the first author introduced the class of UBP-s: weakly associative lattices with the unique bound property (that is, for all a, b in the UBP $\mathfrak{A}$, $a \vee b$ is the only common upper bound of a and b and dually for $a \wedge b$). For example, let $\mathfrak{M}_n$ be the n element lattice with $n-2$ atoms. In its partial ordering replace $0<1$ with $1<0$. Then the algebra so obtained, $\mathfrak{M}_n^*$, is a UBP. It was proved in [7] that each UBP is simple and that the class of UBP-s is characterized as those weakly associative lattices that contain no three element chains. Thus 6.5 tells us that the equational class generated by all UBP-s is filtral.

Let us now return to Example 3.8. Here we have congruence distributive equational classes each having a UCS but without CEP. We now give a generalization of 3.7. To see that it is indeed a generalization, note that in [18] R. N. McKenzie proves that if **K** is a congruence permutable equational class generated by its finite subdirectly irreducible algebras which are finite in number and all simple, then **K** has definable principal congruences.

THEOREM 6.9. *If* **K** *is a congruence distributive equational class and has definable principal congruences, then* **K** *has a* UCS.

Proof. We will show that **K** satisfies (F). Construct $\mathfrak{A}$ and a, b, c, d as in the proof that FDPC on direct products implies (F). Now verify that $c \equiv d(\Theta(a, b))$ as in the proof of 6.5 and then proceed as in the proof that FDPC on direct products implies (F).

7. Comments and questions

1. The concepts of ideal congruences and filtral congruences have been developed extensively by R. Magari and his school; the most accessible reference is [14]. The concept of an ideal congruence is a natural generalization of that of a filtral congruence; the latter is a formalization of some ideas developed by A. Foster, often in collaboration with A. Pixley (see [8] for an extensive bibliography of Foster's papers).

2. Magari defines ideal congruences and filtral congruences for arbitrary classes rather than just for varieties. This means that condition (F) cannot be formulated, and condition (F) is the linchpin connecting ideal congruences and factor determined congruences on the one hand with uniform congruence schemes on the other.

3. In [15] Magari gives a characterization of when a class **K** has ideal congruences; this characterization involves the concept of "good n-families" of polynomials. Essentially, a UCS is a good 1-family of polynomials.

4. Two other references to definable principal congruences are [1] and [2]. In fact, (ii) $\Rightarrow$ (i) of 4.5 is proved in [1], and it is implicit in [2]. Also in [2] it was proved that a locally finite equational class with CEP has definable principal congruences. Thus 6.8 is a generalization of 6.1 (by way of 6.2).

5. Is every filtral variety congruence distributive? All examples in this paper of varieties with a UCS are also congruence distributive. It is easily seen that if $\mathcal{K}$

is filtral and $\mathcal{F}_{\mathcal{K}}(3)$ is finite, then the congruence lattice of $\mathcal{F}_{\mathcal{K}}(3)$ is boolean and so $\mathcal{K}$ is congruence distributive. (Added in proof: An affirmative answer was given by P. Köhler and D. Pigozzi.)

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