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INEQUALITIES FOR A DISTRIBUTION WITH MONOTONE HAZARD RATE

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Summary

Let X be a positive random variable with the survival function $\bar{F}$ and the density f. Let X have the moments $\mu=\mathrm{E}(X)$ and $\mu_2=\mathrm{E}(X^2)$ and put $\varepsilon=|1-\mu_2/2\mu^2|$. Put $q(x)=f(x)/\bar{F}(x)$ and $q_1(x)=\bar{F}(x)\Big/\int_x^\infty \bar{F}(u)du$. It is proved that the following inequalities hold: $|\bar{F}(x)-e^{-x/\mu}|\leqq\varepsilon/(1-\varepsilon e)$, for all $x>0$, if $q(x)$ is monotone and that $\int_0^\infty |\bar{F}(x)-e^{-x/\mu}|dx\leqq 2\varepsilon\mu$, if $q_1(x)$ is monotone. It is also shown that Brown's inequality $|\bar{F}(x)-e^{-x/\mu}|\leqq\varepsilon/(1+\varepsilon)$ which holds whenever $q_1(x)$ is increasing is not valid in general when q_1 is decreasing.

1. Introduction

A positive random variable X or its distribution is said to have decreasing (or increasing) mean residual life if

$$R(x)=\mathrm{E}(X-x\,|\,X>x)$$

exists and monotone decreasing (or increasing) for $0\leqq x<\omega(F)=\sup\{x|F(x)<1\}$. Here and in the following the term "decreasing" and "increasing" are used to mean "non-increasing", and "non-decreasing", respectively. If $\bar{F}(x)=1-F(x)$ is the survival function of F, then the ratio

$$q_1(x)=\frac{\bar{F}(x)}{\int_x^\infty \bar{F}(u)du}$$

is defined for $0<x<\omega(F)$ and is equal to the reciprocal of $R(x)$. If F has a pdf $f(x)$, then the ratio

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$$q(x)=\frac{f(x)}{\bar{F}(x)}$$

is called the hazard rate. The distribution F is said to have increasing hazard rate (IHR) (or decreasing hazard rate (DHR)) if the ratio $q(x)$ is increasing (or decreasing) in x. It is easy to show that IHR (or DHR) implies the decreasing (or increasing) mean residual life (see Lemma 2 below).

The class $\mathcal{M}$ of the distributions having IHR or DHR includes the Polya type 2 distributions (IHR) [9] and the scale mixtures of exponential distributions (DHR) [2]. The implication of the property of monotone hazard rate has been well investigated (see e.g., [2], [3], [6], and [7]). In particular, it is known that such distributions satisfy some moment inequalities, and the exponential distribution is characterized in the class $\mathcal{M}$ as the one which satisfies the condition $\mu_k=k!\mu^k$, or more specifically, the condition $\mu_2/2\mu^2=1$, where $\mu=\mathrm{E}(X)$ and $\mu_k=\mathrm{E}(X^k)$.

As the exponential distribution is the simplest and most important distribution in the class $\mathcal{M}$, it is natural and interesting to ask how close is a distribution F in the class $\mathcal{M}$ to the exponential distribution when the ratio $\mu_2/2\mu^2$ is nearly equal to one. The purpose of the present article is to give bounds on the distance between a distribution in the class $\mathcal{M}$ and the exponential distribution in term of $\varepsilon=|1-\mu_2/2\mu^2|$. Heyde and Leslie [4] showed that if F is a scale mixture of the exponential distribution, then

$$|\bar{F}(x)-e^{-x/\mu}|\leqq 3.74\varepsilon\,, \qquad x>0\,.$$

Hall [5] improved the right hand side to 2.77ε. Azlarov and Volodin [1] obtained a further improvement; They showed that if F has a monotone hazard rate, then,

$$(1)\qquad |\bar{F}(x)-e^{-x/\mu}|\leqq\sqrt{2}\,\varepsilon\,, \qquad x>0\,.$$

Quite recentry, Brown [4] obtained the sharp bound

$$(2)\qquad \sup|\bar{F}(x)-e^{-x/\mu}|\leqq\frac{\varepsilon}{1+\varepsilon}=1-\frac{2\mu^2}{\mu_2}\,,$$

for a distribution F with a increasing mean residual life.

In this article we shall give further results on this problem. In particular, the following inequalities will be proved for a distribution with monotone hazard rate or monotone mean residual life.

$$(3)\qquad \sup|\bar{F}(x)-e^{-x/\mu}|\leqq(1-\varepsilon e)^{-1}\,, \qquad \text{if } q(x) \text{ is increasing}$$

$$(4)\qquad \int_0^\infty|\bar{F}(x)-e^{-x/\mu}|dx\leqq 2\varepsilon\mu\,, \qquad \text{if } q_1(x) \text{ is monotone and also}$$

(5) $e^{-qx} \leqq \bar{F}(x) \leqq q\mu e^{-x/\mu}$, for all $x>0$, if $q_1(x) \uparrow q$ $(<\infty)$.

It will be shown, by an example, that the inequality (2) does not valid in the case of IHR. In fact, the bound from above can not be less than $\varepsilon+\varepsilon^3/6$.

2. Preliminary lemmas

By the very definition of the hazard rate $q(x)$, the survival function $\bar{F}(x)$ can be written as

$$\bar{F}(x)=\exp\left\{-\int_0^x q(u)du\right\}, \qquad x>0 . \tag{6}$$

It is easy, therefore, to verify that $\bar{F}(x)$ approaches to zero as quickly as an exponential function if $q(x)$ is increasing and that $\omega(F)=\infty$ if $q(x)$ is decreasing. In particular, $F(x)$ has moments of all orders if $q(x)$ is increasing.

Now, suppose that the distribution F has the moments $\mu=\mathrm{E}(X)$ and $\mu_k=\mathrm{E}(X^k)$, $k=1, 2, \cdots, K$ (a necessary and a sufficient conditions for this is given in Barlow, Marshall and Proschan [2], Theorem 6.2), and let $\{S_k(x)\}$, $k=0, 1, \cdots, K$, be the sequence of decreasing functions defined by

$$S_0(x)=\bar{F}(x) \quad \text{and} \quad S_k(x)=\int_x^\infty S_{k-1}(u)du$$
$$=\int_0^\infty \bar{F}(x+y)\frac{y^{k-1}}{(k-1)!}dy .$$

Let $S_{-1}(x)=f(x)$ be the pdf of F if it exists. It is easy to see that

$$S_k(x)=\mu_k/k! , \quad \text{and} \quad S'(x)=-S_{k-1}(x) , \qquad \text{for } k=0, 1, 2, \cdots, K . \tag{7}$$

Also the ratio $S_k(x)/S_k(0)$ is a survival function of a distribution and

$$q_k(x)=S_{k-1}(x)/S_k(x)$$

is its hazard rate. The following lemma, which is of some independent interest, is essentially given in [2] (see Theorem 4.2 and Lemma 6.5).

LEMMA 1. *Suppose $q_j(x)$ is monotone increasing (or decreasing) for some j $(0\leqq j<K)$. Then q's are also increasing (or decreasing) both in x and in k.*

We can use the lemma to prove the following well-known characterization theorem for the exponential distribution in the class of distributions with monotone mean residual life. The proof is straightforward and we omit the details. For more general forms of inequalities under less restrictive conditions on F, see Marshall and Proschan [8], Theorems 4.1 and 4.2.

LEMMA 2. *If $q_0(x)$ is increasing (F is IHR), then*

$$f(0)=q_0(0)\leqq\mu^{-1}\leqq q_0(\infty)\,. \tag{8}$$

If $q_1(x)$ is increasing (F has decreasing mean residual life), then

$$(k+1)\mu_k^2\geqq k\mu_{k+1}\mu_{k-1}\,. \tag{9}$$

In particular, we have

$$1\geqq\frac{\mu_2}{2\mu^2}\geqq\frac{\mu_3}{3!\,\mu^3}\geqq\cdots\geqq\frac{\mu_K}{K!\,\mu^K}\,. \tag{10}$$

All the inequality signs are reversed if $q_0(x)$ or $q_1(x)$ is decreasing. If any one of the inequality signs becomes an equality, then the distribution F must be exponential: $\bar{F}(x)=e^{-x/\mu}$.

LEMMA 3. *If the distribution F has decreasing mean residual life, then*

$$S_k(x)\leqq S_k(0)e^{-x/\mu}\,,\qquad k=1,2,\cdots, \tag{11}$$

and

$$S_k(x)\geqq\mu S_{k-1}(0)e^{-x/\mu}-(\mu S_{k-1}(0)-S_k(0))\,,\qquad k=2,3,\cdots. \tag{12}$$

If F has increasing mean residual life, then the inequality signs are reversed.

PROOF. Suppose that $q_1(x)$ is increasing. Then we have for $k\geqq1$

$$R(x)\equiv\mu S_{k-1}(x)-S_k(x)\geqq0\,.$$

Solving the differential equation $\mu S_k'(x)+S_k(x)=-R(x)$, we obtain

$$S_k(x)=S_k(0)e^{-x/\mu}-\mu^{-1}e^{-x/\mu}\int_0^x e^{u/\mu}R(u)du\,.$$

The inequality (11) is clear from this and $R(x)\geqq0$. To prove (12), let $k\geqq2$, and note that $R'(x)=-\mu S_{k-2}(x)+S_{k-1}(x)\leqq0$ for all x, which implies $0\leqq R(x)\leqq R(0)=\mu S_{k-1}(0)-S_k(0)$ for all x. It follows that

$$\begin{aligned}S_k(x)&\geqq S_k(0)e^{-x/\mu}-\mu^{-1}R(0)e^{-x/\mu}\int_0^x e^{u/\mu}du\\&=uS_{k-1}(0)e^{-x/\mu}-R(0)\,.\end{aligned}$$

A similar argument can apply when $q(x)$ is decreasing.

The inequality (11) with $k=1$ was obtained by Marshall and Proschan [8], Theorem 4.6, under less restrictive condition that F is new better than used in expectation, which is equivalent to $q_1(x)\geqq q_1(0)$ for all $x\geqq 0$.

3. The main results

In this section we shall prove some stability theorems of the characterizations of the exponential distribution stated in Lemma 2. The following theorem gives slightly better bounds than (1) when ε is sufficiently small.

THEOREM 1. *If F has increasing hazard rate and if $\varepsilon\equiv 1-\mu_2/2\mu^2 < e^{-1}$, then*

$$-\varepsilon\leqq\bar{F}(x)-e^{-x/\mu}\leqq(1-\varepsilon e)^{-1}\varepsilon\,. \tag{13}$$

PHOOF. Suppose $q_0(x)$ is increasing. The first inequality of (13) easily follows from Lemmas 1–3:

$$\bar{F}(x)\geqq\frac{S_0(0)}{S_2(0)}S_2(x)\geqq\frac{1}{\mu^2}(\mu^2e^{-x/\mu}-\mu^2\varepsilon)=e^{-x/\mu}-\varepsilon\,.$$

The lemmas can also be used to show that for $x\leqq\mu$

$$\bar{F}(x)\leqq\frac{S_1^2(x)}{S_2(x)}\leqq\frac{\mu^2e^{-2x/\mu}}{\mu^2e^{-x/\mu}-\mu^2\varepsilon}\leqq e^{-x/\mu}+(1-\varepsilon e)^{-1}\varepsilon\,.$$

Let x_1 ($\geqq\mu$) be the positive number defined by

$$x_1=\sup\{u\,|\,\bar{F}(x)\leqq e^{-x/\mu}+(1-\varepsilon e)^{-1}\varepsilon\,,\ \text{for } x\leqq u\}\,.$$

We have only to show that $x_1\geqq\omega(F)$. Assume the contrary and put $g(x)\equiv\bar{F}(x)-e^{-x/\mu}$. Then $g(x_1)=(1-\varepsilon e)^{-1}\varepsilon$, and we can find a sequence of positive numbers δ_n such that $\delta_n\to 0$ and $g(x_1+\delta_n)\geqq g(x_1)$. This means $(\bar{F}(x_1+\delta_n)-\bar{F}(x_1))/\delta_n\geqq e^{-x_1/\mu}(e^{-\delta_n/\mu}-1)/\delta_n$. Letting $n\to\infty$, we obtain $f(x_1+0)\leqq\mu^{-1}e^{-x_1/\mu}$. Therefore,

$$q_0(x_1)=\frac{f(x_1)}{\bar{F}(x_1)}\leqq\frac{\mu^{-1}e^{-x_1/\mu}}{e^{-x_1/\mu}+(1-\varepsilon e)^{-1}\varepsilon} \leqq(1-\varepsilon e)/\mu\,. \tag{14}$$

On the other hand, as $q_0(x)$ is increasing, we have $q_0(x)\leqq q_0(x_1)$ or $f(x)\leqq q_0(x_1)\bar{F}(x)$ for all $x\leqq\mu\leqq x_1$. Integrating,

$$\int_0^\mu dx\int_0^x f(u)du\leqq q_0(x_1)\int_0^\mu dx\int_0^x\bar{F}(u)du\,,\qquad\text{or}$$

$S_1(\mu) \leqq q_0(x_1)\{S_2(\mu)+\varepsilon\mu^2\}$. But by Lemma 2, we have, unless F is exponential, $S_1(\mu) > S_2(\mu)/\mu$. Therefore, applying Lemma 2 again

$$q_0(x_1) > \frac{S_2(\mu)/\mu}{S_2(\mu)+\varepsilon\mu^2} \geqq (1-\varepsilon e)/\mu \, .$$

This contradicts (14).

It is hard to expect that the inequalities (13) give the sharp bounds, but, as we shall see later, the inequality (2), which holds for a distribution with DHR is not valid in the case of IHR. See Example 2 below.

We can prove the following somewhat stronger result.

THEOREM 2. *If F has decreasing mean residual life, then for* $0 \leqq \delta < \mu^{-1}$

$$J(\delta) = \int_0^\infty |\bar{F}(x) - e^{-x/\mu}| e^{\delta x} dx \leqq 2\mu\left(1 - \frac{\mu_2}{2\mu^2}\right) + \frac{2-\delta\mu}{1-\delta\mu}\delta\mu^2 \, . \tag{15}$$

In particular,

$$J(0) = \int_0^\infty |\bar{F}(x) - e^{-x/\mu}| dx \leqq \Delta_0 \equiv 2\mu\left|1 - \frac{\mu_2}{2\mu^2}\right| . \tag{16}$$

Inequality (16) *also holds true when F has increasing mean residual life. If F has decreasing hazard rate, then we have*

$$\int_0^\infty |\bar{F}(x) - e^{-x/\mu}| dx \leqq 2\mu\left(1 - \frac{2\mu^2}{\mu^2}\right) \leqq J(0) \, . \tag{17}$$

PROOF. Let A be the set of positive numbers x such that $\bar{F}(x) \leqq e^{-x/\mu}$. Then

$$\begin{aligned} J_0 \equiv J(0) &= \int_0^\infty |\bar{F}(x) - e^{-x/\mu}| dx = 2\int_A (e^{-x/\mu} - \bar{F}(x)) dx \\ &\leqq 2\int_A (e^{-x/\mu} - \mu^{-1}S_1(x)) dx \leqq 2\int_0^\infty (e^{-x/\mu} - \mu^{-1}S_1(x)) dx \\ &= 2(\mu - \mu^{-1}S_2(0)) = 2\mu(1 - \mu_2/2\mu^2) \, . \end{aligned}$$

A similar reasoning can apply when F has increasing mean residual life. To prove (17), let F have decreasing hazard rate. Then there exists an $a \geqq \mu$ such that the inequality $\bar{F}(x) \leqq e^{-x/\mu}$ or $\bar{F}(x) \geqq e^{-x/\mu}$ holds according as $x \leqq a$ or $x \geqq a$. Therefore,

$$\int_0^\infty |\bar{F}(x) - e^{-x/\mu}| dx = 2\int_a^\infty (\bar{F}(x) - e^{-x/\mu}) dx = 2(S_1(a) - \mu e^{-a/\mu}) \, .$$

But, according to theorem (iii) of Brown [4], $S_1(x)$ is bounded from above by $\mu\left(e^{-x/\mu}+\frac{\varepsilon}{1+\varepsilon}\right)$, where $\varepsilon=-1+\mu_2/2\mu^2$, which implies (17). To prove (15), let again F have decreasing mean residual life. Then for $k\geqq 1$, we have as before,

$$J_k\equiv\int_0^\infty|\bar F(x)-e^{-x/\mu}|x^k dx\leqq k!\{\mu^{k+1}+S_{k+1}(0)-2\mu^{-1}S_{k+2}(0)\}\,.$$

Therefore,

$$\begin{aligned}J(\delta)&=\sum_{k=0}^\infty\frac{\delta^k}{k!}J_k\leqq J_0+\sum_{k=1}^\infty\delta^k\{\mu^{k+1}+S_{k+1}(0)-2\mu^{-1}S_{k+2}(0)\}\\&=J_0+\delta\mu^2(1-\delta\mu)^{-1}+\delta S_2(0)+\left(1-\frac{2}{\delta\mu}\right)\sum_{k=2}^\infty\delta^k S_{k+1}(0)\\&\leqq J_0+\delta\mu^2(1-\delta\mu)^{-1}+\delta\mu^2\,,\end{aligned}$$

as was to be proved. Note that $1-2/\delta\mu<-1$.

Finally we prove the following theorem, which gives better estimates for $|\bar F(x)-e^{-x/\mu}|$ when x is large.

THEOREM 3. *If F has decreasing mean residual life and if $q\equiv\lim_{x\to\infty}q_1(x)<\infty$, then*

(18) $$\max(e^{-qx},\ \{1-(q-\mu^{-1})x\}e^{-x/\mu})\leqq\bar F(x)\leqq q\mu e^{-x/\mu}\,.$$

If F has increasing mean residual life and if $q\equiv\lim_{x\to\infty}q_1(x)>0$, then the inequalities are reversed.

PROOF. By l'Hopital's rule and Lemma 1

$$\mu^{-1}=q_1(0)\leqq q_1(x)=\frac{S_0(x)}{S_1(x)}\leqq\lim q_1(x)=q\,.$$

The second inequality of (18) follows from this and Lemma 3. For $\mu^{-1}\leqq c\leqq q$, put $R_c(x)\equiv S_0(x)-cS_1(x)$ $(\leqq(q-c)S_1(x))$. Then

$$\begin{aligned}\bar F(x)&\geqq\mu^{-1}S_1(x)=\mu^{-1}\left\{\mu e^{-cx}-e^{-cx}\int_0^x e^{cu}R_c(u)du\right\}\\&\geqq e^{-cx}-\mu^{-1}(q-c)e^{-cx}\int_0^x e^{cu}S_1(u)du\\&\geqq e^{-cx}-(q-c)e^{-cx}\int_0^x e^{(c-\mu^{-1})u}du\,.\end{aligned}$$

Taking $c=q$ and then $c=\mu^{-1}$, we obtain

$$\bar F(x)\geqq e^{-qx}\,,\qquad\text{and}$$

$$\bar F(x)\geqq e^{-x/\mu}-(q-\mu^{-1})e^{-x/\mu}x\,.$$

4. Examples

If a non-exponential unimodal distribution F with $\mu=1$ and with the p.d.f. $f(x)$ continuous on $(0, \omega(F))$ has increasing hazard rate, then the errors $\bar{K}=\sup(\bar{F}(x)-e^{-x})$ and $\underline{K}=\sup(e^{-x}-\bar{F}(x))$ are attained, respectively, by the minimum and the maximum zeros b and c of the function $f(x)-e^{-x}$. On the other hand if a is the unique zero of the function $\bar{F}(x)-e^{-x}$, then $J=\int_0^\infty|\bar{F}(x)-e^{-x}|dx=2(e^{-a}-S_1(a))$. Similar remark can apply to a distribution with decreasing hazard rate. Here are some examples.

1) Uniform distribution: Let $X_1, X_2, \cdots, X_n$ be a random sample from the uniform distribution on the unit interval $(0, 1)$. The distribution F_n of the random variable $X=(n+1)\cdot\min\{X_1, X_2, \cdots, X_n\}$ has the survival function

$$\bar{F}_n(x)=\begin{cases}\left(1-\dfrac{x}{n+1}\right)^n, & 0<x<n+1\\ 0, & x\geqq n+1\end{cases}$$

and moments $\mu=1$ and $\mu_2=2(n+1)/(n+2)$. For each n, the hazard rate $q_0(x)=n/(n+2-x)$ is defined and strictly increasing on the finite interval $(0, n+1)$.

Our bounds for $\bar{K}_n=\sup(\bar{F}_n(x)-e^{-x})$ and $\underline{K}_n=\sup(e^{-x}-\bar{F}(x))$ are $1/(n+2-e)$ and $1/(n+2)$, respectively. A numerical computation shows $\bar{K}_{10}=0.02189$, $\bar{K}_{100}=0.00229$, $\bar{K}_{1000}=0.00023$, $\underline{K}_{10}=0.00864$, $\underline{K}_{100}=0.00080$ and $\underline{K}_{1000}=0.00008$. In this example, b and c tend to $2\pm\sqrt{2}$ as $n\to\infty$. If $(1-a/(n+1))^n=e^{-a}$, then $J=\int_0^\infty|\bar{F}(x)-e^{-x}|dx=2\{e^{-a}-(1-a/(n+1))^n\}=2e^{-a}\cdot a/(n+1)$. It is easy to show that a lies in the interval $(1, 2)$ and tends to 2 as $n\to\infty$, so that $4e^{-2}/(n+1)\leqq J\leqq 2e^{-1}/(n+1)$ and $\lim n\cdot J=4e^{-2}$. Our bound on J is $2/(n+2)$.

2) Shifted exponential: The distribution with the survival function

$$\bar{F}(x)=\begin{cases}1 & \text{if } x\leqq b\equiv 1-\sqrt{1-2\varepsilon}\\ e^{-(x-b)/\sqrt{1-2\varepsilon}} & \text{if } x>b\end{cases}$$

has moments $\mu=1$ and $\mu_2=2(1-\varepsilon)$. Hazard rate is increasing and $\bar{K}(\varepsilon)\equiv\sup(\bar{F}(x)-e^{-x})=1-e^{-b}$ $(\geqq\varepsilon+\varepsilon^3/6)$ and $\underline{K}(\varepsilon)\equiv\sup(e^{-x}-\bar{F}(x))=be^{-b}$ $(\geqq\varepsilon-\varepsilon^2)$ are attained at $x=b$, at which the density is discontinuous and at $x=c\equiv(b-\log\sqrt{1-2\varepsilon})/\sqrt{1-2\varepsilon}$, respectively. Note that $\bar{K}(\varepsilon)$ is greater than Brown's bound $\varepsilon/(1+\varepsilon)$ which is applicable to a distribution with increasing mean residual life. As $a=1$ is the unique zero of

$\bar{F}(x)-e^{-x}$, we have

$$J\equiv\int_0^\infty|\bar{F}(x)-e^{-x}|dx=2(e^{-1}-\sqrt{1-2\varepsilon}\,e^{-(1-b)/\sqrt{1-2\varepsilon}})$$
$$=2be^{-1}\ (\geqq 2\varepsilon e^{-1})\,.$$

Our upper bound on J is 2ε.

3) Gamma distribution: The gamma distribution with the p.d.f.

$$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}\,,\qquad x>0$$

has moments $\mu=\alpha/\lambda$ and $\mu_2=\alpha(\alpha+1)/\lambda^2$. The hazard rate $q_0(x)=f(x)/\bar{F}(x)$ is increasing or decreasing according as $\alpha>1$ or $\alpha<1$. Theorem 3 gives

$$-(\alpha-1)\frac{\lambda}{\alpha}xe^{-\lambda x/\alpha}\leqq\bar{F}(x)-e^{-\lambda x/\alpha}\leqq(\alpha-1)e^{-\lambda x/\alpha}\,,$$

for $\alpha>1$. The inequalities should be reversed when $\alpha<1$. Theorem 2 gives

$$J(\alpha)\equiv\int_0^\infty|\bar{F}(x)-e^{-\lambda x/\alpha}|dx\leqq\Delta(\alpha)\equiv\frac{|\alpha-1|}{\lambda}\,.$$

Numerical computations show that when $\lambda=1$, $J(0.95)=0.0164$, $J(0.96)=0.0131$, $J(0.97)=0.0099$, $J(0.98)=0.0066$, $J(0.99)=0.0033$, $J(1.01)=0.0033$, $J(1.02)=0.0067$, $J(1.03)=0.0101$, $J(1.04)=0.0135$ and $J(1.05)=0.0169$.

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