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BLOCK PLAN FOR A FRACTIONAL 2^m FACTORIAL DESIGN DERIVED FROM A 2^r FACTORIAL DESIGN

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Summary

For a given fractional 2^m factorial (2^m-FF) design T, the constitution of a block plan to divide T into k ($2^{r-1}<k\leqq 2^r$) blocks with r block factors each at two levels is proposed and investigated. The well-known three norms of the confounding matrix are used as measures for determining a "good" block plan. Some theorems concerning the constitution of a block plan are derived for a 2^m-FF design of odd or even resolution. Two norms which may be preferred over the other norm are slightly modified. For each value of N assemblies with $11\leqq N\leqq 26$, optimum block plans for $k=2$ blocks with block sizes $[N/2]$ and $N-[N/2]$ minimizing the two norms are presented for A-optimal balanced 2^4-FF designs of resolution V given by Srivastava and Chopra (*Technometrics*, 13, 257–269).

1. Introduction

Consider a 2^m factorial experiment with m factors. An assembly (or treatment combination) is represented by an m-rowed vector $(j_1, j_2, \cdots, j_m)$, where j_t (level of tth factor) is equal to 0 or 1. As unknown effects, θ_0, θ_t, and in general, $\theta_{t_1\cdots t_k}$ denote the general mean, main effect of tth factor, and k-factor interaction of $t_1,\cdots,t_k$th factors, respectively. For a fixed integer l ($1\leqq l\leqq m$), let $\boldsymbol{\theta}$ be the $\nu\times 1$ vector composed of effects up to l-factor interactions, where $\nu=\sum_{i=0}^{l}\binom{m}{i}$, i.e.,

$$\boldsymbol{\theta}'=(\theta_0;\, \theta_1, \theta_2,\cdots, \theta_m;\, \theta_{12},\cdots, \theta_{m-1\,m};\, \theta_{12\cdots l},\cdots, \theta_{m-l+1\cdots m})\ .$$

Assume throughout this paper that $(l+1)$-factor and higher order interactions are negligible and that the m factors are different from block factors. Let T be a fractional 2^m factorial (2^m-FF) design which is a

Key words: Confounding matrix, norm, balanced array.

suitable set of N assemblies. Note that the assemblies in T are not always distinct. Using a design T, we consider the estimation of a $\nu_0\times 1$ vector of linear parametric functions $\boldsymbol{\theta}_0=C\boldsymbol{\theta}$ for some $\nu_0\times\nu$ matrix C. For an $N\times 1$ observation vector $\boldsymbol{y}_T$ of T (whose observations are assumed to be independent random variables with common variance σ^2), consider the model

$$\mathrm{E}(\boldsymbol{y}_T)=E\theta \tag{1.1}$$

where $\mathrm{E}(\cdot)$ stands for an expected value and E is the $N\times\nu$ design matrix with elements ± 1 (see, e.g., Yamamoto, Shirakura and Kuwada [13]). Suppose that there exists a $\nu_0\times\nu$ matrix K satisfying $KM=C$ (i.e., rank $M=$rank $[M: C']$), which is equivalent to the estimability of $\boldsymbol{\theta}_0$, where $M=E'E$ is called the information matrix of T. The best linear unbiased estimate of $\boldsymbol{\theta}_0$ can then be given by

$$\hat{\boldsymbol{\theta}}_0=KE'\boldsymbol{y}_T\ . \tag{1.2}$$

When $\boldsymbol{\theta}_0=\boldsymbol{\theta}$, i.e., $C=I$ (identity matrix of appropriate order) and $\nu_0=\nu$, T corresponds to a 2^m-FF design of resolution $2l+1$. In this case, note that the nonsingularity of M is assumed. On the other hand, when $\boldsymbol{\theta}_0=(\theta_1,\cdots,\theta_m;\cdots;\theta_{12\cdots l-1},\cdots,\theta_{m-l+2\cdots m})'$, i.e., $C=[0:I:O]$ (0 and O are respectively zero vector and zero matrix of appropriate orders) and $\nu_0=\nu-1-\binom{m}{l}$, T corresponds to a design of resolution $2l$ (see Box and Hunter [1]).

In order to get $\hat{\boldsymbol{\theta}}_0$ in (1.2), it is required to make the plots of N assemblies under conditions as homogeneous as possible for the m factors. After planning a design T for $\hat{\boldsymbol{\theta}}_0$, however, it may occur that N observations for T can not be yielded simultaneously by physical, chemical and/or economical reasons, etc. For example, consider an experiment of a certain reaction for a mixture of m raw materials each at two levels. Then, after accomodating the N mixtures in a given T, it may occur that each reaction of them can not be observed under a homogeneous condition. Therefore we consider an arrangement of T in some blocks. The less is the number of assemblies in which we have to experiment simultaneously, the larger is the possibility that we obtain a homogeneous condition. Of course, the number of blocks (say k) should be small compared to N. The problem is to constitute the k blocks such that the estimate $\hat{\boldsymbol{\theta}}_0$ is not sensitive to the block division. The present paper discusses this problem under special situations. For the k blocks, we use a 2^r design with r factors each at two levels which consists of k distinct assemblies ($2^{r-1}<k\leqq 2^r$). As block factors, for example, consider experimenter, day and place. Our situation is then the case where the N observations in a given T have to be yielded

by two experimenters, in two days and/or at two places. As measures of the insensitivity to block effects, we use well-known three norms of a confounding matrix of T. In Section 2, we introduce the three norms of a confounding matrix for r block factors and present a procedure for the constitution of k $(2^{r-1}<k\leqq 2^r)$ blocks with some block sizes for a design T. For a 2^m-FF design of resolution $2l$ or $2l+1$, we give the constitution of the k blocks such that the norms have appropriate values and that the confounding matrix has a certain desirable pattern. In particular, it is shown that for a 2^m-FF design of even resolution, there exist the blocks for which the three norms are zero (i.e., its confounding matrix is zero). Section 3 deals with the constitution of $k=2$ blocks for a balanced fractional 2^m factorial (2^m-BFF) design of resolution $2l+1$. Balanced designs (including orthogonal designs) have various desirable properties and wide applications. A 2^m-BFF design of resolution V ($l=2$) was first discussed by Srivastava [9] and a design of resolution $2l+1$ has been generalized by Yamamoto, Shirakura and Kuwada [13], [14]. Some properties of the preferable two norms are presented for 2^m-BFF designs of resolution $2l+1$. Also, the norms are slightly modified. For A-optimal 2^4-BFF designs of resolution V given by Srivastava and Chopra [10], the constitutions of 2 blocks with block sizes $[N/2]$ and $N-[N/2]$ minimizing the modified norms are presented for the values of N satisfying $11\leqq N\leqq 26$.

For the case where one knows in advance that block factors are explicitly needed for the estimation of $\boldsymbol{\theta}_0$, a design should be planned such that the effect $\boldsymbol{\theta}_0$ is unconfounded with block effects. For the case of $\boldsymbol{\theta}_0=\boldsymbol{\theta}$, however, it is in general difficult to plan such a design and even if we do it, a larger number of assemblies are required. If the effects of block factors may not be so large as effects in $\boldsymbol{\theta}$ and not be so neglected as higher order interactions, the methods given in this paper would be also useful.

2. Constitution of a confounding block plan

Let T be a 2^m-FF design with N assemblies whose αth assembly is given by $(j_1^{(\alpha)}, j_2^{(\alpha)},\cdots, j_m^{(\alpha)})$ for $\alpha=1,\cdots,N$. For $2^r<N$, consider another 2^r design D with r factors and N assemblies whose αth assembly is given by $(d_1^{(\alpha)}, d_2^{(\alpha)},\cdots, d_r^{(\alpha)})$ for $\alpha=1,\cdots,N$. Suppose k $(2^{r-1}<k\leqq 2^r)$ distinct assemblies are exactly included in D. Further let $[T;D]$ be a set of assemblies obtained by juxtaposing T and D in a way such that its αth assembly is $(j_1^{(\alpha)},\cdots, j_m^{(\alpha)}; d_1^{(\alpha)},\cdots, d_r^{(\alpha)})$ for $\alpha=1,\cdots,N$. The set $[T;D]$ can be considered as a 2^{m+r}-FF design with $m+r$ factors. However, the r factors play here the role of block factors and D gives a block plan for T. That is, k blocks $B_{(d_1\cdots d_r)}$ for T are constituted in a

way such that the αth assembly of $[T; D]$ is $(j_1^{(\alpha)}, \cdots, j_m^{(\alpha)}; d_1^{(\alpha)}, \cdots, d_r^{(\alpha)})$ if and only if the αth assembly of T belongs to $B_{(d_1^{(\alpha)}\cdots d_r^{(\alpha)})}$. Then the number of times an assembly $(d_1, \cdots, d_r)$ occurs in D, $k(d_1\cdots d_r)$, is called a block size for $B(d_1\cdots d_r)$. Consider now the $N\times 1$ observation vector $\boldsymbol{y}_{[T;D]}$ of $[T; D]$. According to model (1.1), we then have the following model:

$$\text{(2.1)} \qquad \mathrm{E}\,(\boldsymbol{y}_{[T;D]}) = E\boldsymbol{\theta} + X(D)\boldsymbol{\eta}\ ,$$

where $\boldsymbol{\eta}' = (\eta_1, \cdots, \eta_r)$, ($\eta_t$ is the main effects of tth block factor), and $X(D)$ is the $N\times r$ design matrix for $\boldsymbol{\eta}$ of D with elements ± 1. Assume that there are no interactions between the m factors and r block factors, and between the r block factors. Note that the level of jth factor for the αth assembly in D is 0 and 1 if and only if the (α, j) element of $X(D)$ is -1 and 1, respectively. We still insist on using $\hat{\boldsymbol{\theta}}_0$ in (1.2) for the estimation of $\boldsymbol{\theta}_0$, because T is a design which is in advance planned to obtain $\hat{\boldsymbol{\theta}}_0$. Under model (2.1), the expected value of $\hat{\boldsymbol{\theta}}_0$, ($\boldsymbol{y}_T$ in $\hat{\boldsymbol{\theta}}_0$ is replaced with $\boldsymbol{y}_{[T;D]}$), becomes

$$\text{(2.2)} \qquad \mathrm{E}\,(\hat{\boldsymbol{\theta}}_0) = \boldsymbol{\theta}_0 + A(D)\boldsymbol{\eta}\ ,$$

where $A(D) = KE'X(D)$ is said to be a confounding matrix of D. As measures how $\mathrm{E}\,(\hat{\boldsymbol{\theta}}_0)$ in (2.2) can be close to $\boldsymbol{\theta}_0$, the following three norms of $A(D)$ may be considered:

$$\text{(2.3)} \qquad \begin{aligned} &(\mathrm{i}) \quad \|A(D)\|_1 = \max_{1\le i\le \nu_0} \left\{\sum_{j=1}^{r} |a_{ij}|\right\}\ , \\ &(\mathrm{ii}) \quad \|A(D)\|_2 = \{\mathrm{tr}\,(A(D)'A(D))\}^{1/2} = \left\{\sum_{i=1}^{\nu_0}\sum_{j=1}^{r} a_{ij}^2\right\}^{1/2}\ , \\ &(\mathrm{iii}) \quad \|A(D)\|_3 = \max_{1\le j\le r} \left\{\sum_{i=1}^{\nu_0} |a_{ij}|\right\}\ , \end{aligned}$$

where a_{ij} are the (i, j) elements of $A(D)$.

Under (2.1), the mean square error of $\hat{\boldsymbol{\theta}}_0$ for $\boldsymbol{\theta}_0$ is given by

$$\mathrm{MSE}\,(\hat{\boldsymbol{\theta}}_0) = KMK'\sigma^2 + A(D)\boldsymbol{\eta}\boldsymbol{\eta}'A(D)'\ .$$

It is seen that for a given design T, $\mathrm{MSE}\,(\hat{\boldsymbol{\theta}}_0)$ is dependent on $A(D)$ only through $A(D)\boldsymbol{\eta}\boldsymbol{\eta}'A(D)'$, because σ^2 and $\boldsymbol{\eta}$ are constant. Therefore it is reasonable to be considered that minimizing each norm in (2.3) may also reflect decreasing a magnitude of $\mathrm{MSE}\,(\hat{\boldsymbol{\theta}}_0)$. In particular, this is more obvious for the norm $\|A(D)\|_2$ for the case $r=1$ which will be treated later, since

$$\text{(2.4)} \qquad \mathrm{tr}\,(\mathrm{MSE}\,(\hat{\boldsymbol{\theta}}_0)) = \sigma^2\,\mathrm{tr}\,(KMK') + \eta_1^2\|A(D)\|_2^2\ .$$

Let $\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ denote the collection of all possible D with N assemblies in which the assembly $(d_1,\cdots,d_r)$ occurs $k(d_1\cdots d_r)$ times. For a given T, D in $\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ is said to be a confounding block (CB) plan when D is selected in a consideration of the norm $\|A(D)\|_i$, $(i=1,2,3)$. A plan D is said to be an optimum confounding block (OCB) plan (with block sizes $k(d_1\cdots d_r)$) for T with respect to the norm $\|A(D)\|_i$ if $\|A(D)\|_i$ is a minimum in the class $\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ for each $i=1,2,3$. Also the set $\boldsymbol{B}_{T,\{k(d_1\cdots d_r)\}}$ of the corresponding blocks $B_{(d_1\cdots d_r)}$ is said to be an optimum block for T. This idea is due to Hedayat, Raktoe and Federer [4]. However they have proposed the norm $\|A\|=(\mathrm{tr}\,(A'A))^{1/2}$ of alias matrix A as a measure in selecting a design T.

2^m-FF designs of resolution $2l$ or $2l+1$ are particularly important for our practical uses. Therefore we shall discuss some properties of the norms in (2.3) for these designs.

THEOREM 2.1. *Let T be a 2^m-FF design of resolution $2l$. Suppose an $N\times r$ submatrix of E whose r columns correspond to r l-factor interactions has k $(2^{r-1}<k\leqq 2^r)$ distinct rows. Then there exists an OCB plan $D\in\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ for T such that $\|A(D)\|_i=0$, $(i=1,2,3)$, i.e., $A(D)$ is a zero matrix, where $k(d_1\cdots d_r)$ is the number of row vectors $(2d_1-1,\cdots,2d_r-1)$ occurring in the submatrix.*

PROOF. Denote the submatrix by Y. Assume that the r l-factor interactions are at the s_1th, s_2th, $\cdots$, s_rth positions of $\boldsymbol{\theta}$ and let Z be the $\nu\times r$ matrix in which the ith element of s_ith row is 1 and 0 elsewhere. Then $Y=EZ$ holds. Consider D satisfying $X(D)=Y$. Then, from the assumption, $D\in\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$. Also, $A(D)=KE'X(D)=KMZ=CZ=O$. This completes the proof.

THEOREM 2.2. *Let T be a 2^m-FF design of resolution $2l+1$. Consider the same submatrix and $\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ as in Theorem 2.1. Then, there exists a CB plan $D\in\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ for T such that $\|A(D)\|_i=1$, $(i=1,3)$, and $\|A(D)\|_2=\sqrt{r}$. Moreover, in the plan D, every effect up to $(l-1)$-factor interactions is unconfounded with block effects $\boldsymbol{\eta}$.*

PROOF. Consider the same matrices Y and Z, and the plan $D\in\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ as in Proof of Theorem 2.1. Then $A(D)=Z$, since $C=I$. From the structure of Z, $\|Z\|_i=1$, $(i=1,3)$, $\|Z\|_2=\sqrt{r}$, and furthermore from (2.2), $\mathrm{E}(\hat{\boldsymbol{\theta}}_0)=\boldsymbol{\theta}+Z\boldsymbol{\eta}$, which completes the proof.

In what follows, (optimum) CB plans with two blocks (i.e., $r=1$) are constituted for designs. Note that $A(D)$ and $X(D)$ become column vectors. In view of Theorem 2.1, we can easily obtain an OCB plan with two blocks for some 2^m-FF design of even resolution. The following theorem implies that for a 2^m-FF design of odd resolution, there exists a CB plan better than one in Theorem 2.2 w.r.t. the norm $\|A(D)\|_1$:

THEOREM 2.3. *Let $2 \leq l \leq m-2$. Then there exists a CB plan D in $\mathcal{D}^N_{\{k(d_1)\}}$, ($k(d_1)$ are certain block sizes), for a 2^m-FF design of resolution $2l+1$ such that $\|A(D)\|_1 = 1/2$ holds. Moreover, in the plan D, any effects up to $(l-1)$-factor interactions are unconfounded with block effect η.*

PROOF. Since $2 \leq l \leq m-2$, there exist two distinct main effects θ_t and $\theta_{t'}$ such that $t, t' \notin \{t_1, \cdots, t_{l-1}, t'_1, \cdots, t'_{l-1}\}$ for some two distinct $(l-1)$-factor interactions $\theta_{t_1 \cdots t_{l-1}}$ and $\theta_{t'_1 \cdots t'_{l-1}}$. Denote a column vector of E corresponding to an effect θ_z in $\boldsymbol{\theta}$ by e_z. Consider the $N \times 1$ vector $\boldsymbol{x}$ given as follows:

$$\boldsymbol{x} = \frac{1}{2}\{e_t * (e_{t_1 \cdots t_{l-1}} + e_{t'_1 \cdots t'_{l-1}}) + e_{t'} * (e_{t_1 \cdots t_{l-1}} - e_{t'_1 \cdots t'_{l-1}})\} .$$

where $*$ denotes the product operation defined by $(a_1, a_2, \cdots, a_n)' * (b_1, b_2, \cdots, b_n)' = (a_1 b_1, a_2 b_2, \cdots, a_n b_n)'$. Then it is easy to verify that $\boldsymbol{x}$ is a vector with elements 1 or -1. From the property of design matrix E, $\boldsymbol{x}$ can also be writen by

$$\boldsymbol{x} = \frac{1}{2}(e_{t t_1 \cdots t_{l-1}} + e_{t t'_1 \cdots t'_{l-1}} + e_{t' t_1 \cdots t_{l-1}} - e_{t' t'_1 \cdots t'_{l-1}}) .$$

This means that $\boldsymbol{x}$ can be expressed as a linear combination of columns of E corresponding to the four l-factor interactions $\theta_{t t_1 \cdots t_{l-1}}$, $\theta_{t t'_1 \cdots t'_{l-1}}$, $\theta_{t' t_1 \cdots t_{l-1}}$ and $\theta_{t' t'_1 \cdots t'_{l-1}}$. Suppose now $\boldsymbol{z}$ is a $\nu \times 1$ vector with elements 0, 1 or -1 obtained from $\boldsymbol{\theta}$ by replacing the above first three l-factor interactions with 1, the last one with -1, and the other effects with 0. Further consider D given by $X(D) = \boldsymbol{x}$. Then we have $X(D) = E\boldsymbol{z}/2$. Therefore, $A(D) = M^{-1}E'X(D) = \boldsymbol{z}/2$, since $K = M^{-1}$. Hence $\|A(D)\|_1 = 1/2$. This completes the proof.

It is remarked that the block sizes $k(0)$ and $k(1)$ of the plan D in the above theorem are equal to the numbers of -1 and 1 in $\boldsymbol{x}$, respectively.

Example 1. Consider

$$T = \{\Omega(5, 0), \Omega(5, 2), \Omega(5, 4)\} ,$$

where $\Omega(m, h)$, $(0 \leq h \leq m)$, is the set of $\binom{m}{h}$ distinct assemblies with weight h (number of 1's). Then it is easy to see that T is a 2^5-FF design of resolution V ($l=2$) with $N=16$ assemblies such that $M=16I$. This design is called an orthogonal design which is the most popular design. T can be expressed by the 16×5 matrix

$$T' = \left[\begin{array}{c|cccccccccc|ccccc} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right],$$

whose αth row denotes the αth assembly. Suppose two main effects are θ_2 and θ_3, and the other main effects are θ_4, θ_5 (i.e., $t=2$, $t'=3$, $t_1=4$ and $t_1'=5$) in Proof of Theorem 2.3. Therefore $\boldsymbol{x}=(\boldsymbol{e}_{24}+\boldsymbol{e}_{25}+\boldsymbol{e}_{34}-\boldsymbol{e}_{35})/2=(1; -1, 1, -1, 1, -1, -1, 1, 1, -1, -1; 1, -1, 1, -1, 1)'$. Thus we have the plan $D=\{\boldsymbol{x}+(1, 1, \cdots, 1)'\}/2 \in \boldsymbol{D}^{16}_{\{k(0)=8, k(1)=8\}}$ with $\|A(D)\|_1=1/2$. The corresponding blocks $B_{(0)}$ and $B_{(1)}$ are given by

$$B_{(0)}=\{(1, 1, 0, 0, 0), (1, 0, 0, 1, 0), (0, 1, 1, 0, 0), (0, 1, 0, 1, 0), (0, 0, 1, 0, 1), (0, 0, 0, 1, 1), (1, 1, 1, 0, 1), (1, 0, 1, 1, 1)\},$$

$$B_{(1)}=T-B_{(0)}.$$

In fact, the above plan D turns out to be an OCB plan over $\boldsymbol{D}^{16}_{\{k(0)=8, k(1)=8\}}$ w.r.t. $\|A(D)\|_1$.

As compared with the norms $\|A(D)\|_2$ and $\|A(D)\|_3$, the decrease of $\|A(D)\|_1$ directly acts to reduce the bias for each estimate in $\hat{\boldsymbol{\theta}}_0$. In the sense, $\|A(D)\|_1$ is preferred over $\|A(D)\|_2$ and $\|A(D)\|_3$. Also, for the present case $r=1$, $\|A(D)\|_2$ is mathematically preferred over $\|A(D)\|_1$ and $\|A(D)\|_3$ in view of (2.4). For reference, note that $\|A(D)\|_2=1$ and $\|A(D)\|_3=2$ for the plan D in Theorem 2.3. We are henceforth interested in an OCB plan with two blocks w.r.t. $\|A(D)\|_i$, $(i=1, 2)$.

3. OCB plans for balanced designs

CB plans in $\boldsymbol{D}^{N}_{\{k(d_1)\}}$, $(N=k(0)+k(1))$, are developed for 2^m-BFF designs of resolution $2l+1$. We first define a balanced design. A 2^m-FF design of resolution $2l+1$ is called balanced if the covariance matrix $\mathrm{Var}(\hat{\boldsymbol{\theta}})=M^{-1}\sigma^2$ of $\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\theta}}_0$ in (1.2) is invariant under any permutation of m factors, i.e., if for two estimates $\hat{\theta}_{t_1\cdots t_u}$ and $\hat{\theta}_{t_1'\cdots t_v'}$ in $\hat{\boldsymbol{\theta}}$,

$$\text{(3.1)} \qquad \begin{aligned} &\mathrm{Var}(\hat{\theta}_{t_1\cdots t_u})=\mathrm{Var}(\hat{\theta}_{\tau(t_1\cdots t_u)}), \\ &\mathrm{Cov}(\hat{\theta}_{t_1\cdots t_u}, \hat{\theta}_{t_1'\cdots t_v'})=\mathrm{Cov}(\hat{\theta}_{\tau(t_1\cdots t_u)}, \hat{\theta}_{\tau(t_1'\cdots t_v')}) \end{aligned}$$

where $\mathrm{Var}(\cdot)$ and $\mathrm{Cov}(\cdot, \cdot)$ stand for variance and covariance of estimates, respectively, and τ is an element of the symmetric group $C=\left\{\tau;\ \tau=\begin{pmatrix} 1 & 2 & \cdots & m \\ \tau(1) & \tau(2) & \cdots & \tau(m)\end{pmatrix}\right\}$ of order m. In practice, for a given m, u'-factor interactions $(m/2<u'\leqq m)$ are often assumed to be negligible.

Therefore, we here give the restriction $2l \leqq m$. It is known from Srivastava [9] and Yamamoto, Shirakura and Kuwada [13] that a 2^m-BFF design of resolution $2l+1$ with N assemblies is equivalent to a balanced array of size N, m constraints, strength $2l$ and index set $\mathcal{U} = \{\mu_0, \mu_1, \cdots, \mu_{2l}\}$, (simply, B-array $[N, m, 2l, \mathcal{U}]$), provided M is non-singular. For the save of space, see the above papers for the definition of a balanced array. In particular, a B-array $[N, m, 2l, \mathcal{U}]$ is called an orthogonal array with index λ if $\lambda = \mu_0 = \mu_1 = \cdots = \mu_{2l}$, which is equivalent to an orthogonal 2^m-FF design of resolution $2l+1$ with $M = 2^{2l}\lambda I$ (see Example 1).

Now, for a $\nu \times 1$ vector $\boldsymbol{c}$, attach the same subindices as the effects of $\boldsymbol{\theta}$ to the elements of $\boldsymbol{c}$, i.e.,

$$\boldsymbol{c}' = (c_0; c_1, \cdots, c_m; c_{12}, \cdots, c_{m-1m}; \cdots; c_{12\cdots l}, \cdots, c_{m-l+1\cdots m}) .$$

Further consider a $\nu \times 1$ vector $\tau(\boldsymbol{c})$ for $\tau \in \mathcal{C}$ defined by

$$\tau(\boldsymbol{c})' = (c_0; c_{\tau(1)}, \cdots, c_{\tau(m)}; c_{\tau(12)}, \cdots, c_{\tau(m-1m)}; \cdots; c_{\tau(12\cdots l)}, \cdots, c_{\tau(m-l+1\cdots m)}) .$$

Then, we have the following lemma which can easily be proved:

LEMMA 3.1. *For any* $\tau \in \mathcal{C}$,

$$\boldsymbol{c}_1'\tau(\boldsymbol{c}_2) = \tau^{-1}(\boldsymbol{c}_1)'\boldsymbol{c}_2$$

holds where $\boldsymbol{c}_1$ *and* $\boldsymbol{c}_2$ *are* $\nu \times 1$ *vectors.*

THEOREM 3.2. *Let T be a 2^m-BFF design of resolution $2l+1$ with N assemblies and D be a plan in $\boldsymbol{D}^N_{\{k(d_1)\}}$. Then, for any* $\tau \in \mathcal{C}$,

$$\|A(D)\|_i = \|A^\tau(D)\|_i , \qquad i = 1, 2$$

hold where $A^\tau(D) = M^{-1}\tau(E'X(D))$.

PROOF. For $\tau_1, \tau_2 \in \mathcal{C}$, suppose $V(\tau_1(\boldsymbol{\theta}), \tau_2(\boldsymbol{\theta}))\sigma^2$ is the covariance matrix for $\boldsymbol{\theta}$ in which the row and column orders correspond to $\tau_1(\boldsymbol{\theta})$ and $\tau_2(\boldsymbol{\theta})$, respectively. In particular, note that $V(\boldsymbol{\theta}, \boldsymbol{\theta}) = M^{-1}$ since $\mathrm{Var}(\hat{\boldsymbol{\theta}}) = M^{-1}\sigma^2$. It follows from Lemma 3.1 that

$$A^\tau(D) = M^{-1}\tau(E'X(D)) = V(\boldsymbol{\theta}, \boldsymbol{\theta})\tau(E'X(D)) = V(\boldsymbol{\theta}, \tau^{-1}(\boldsymbol{\theta}))(E'X(D)) .$$

Therefore, we have

$$\tau^{-1}(A^\tau(D)) = V(\tau^{-1}(\boldsymbol{\theta}), \tau^{-1}(\boldsymbol{\theta}))(E'X(D)) .$$

From (3.1), $V(\boldsymbol{\theta}, \boldsymbol{\theta}) = V(\tau^{-1}(\boldsymbol{\theta}), \tau^{-1}(\boldsymbol{\theta}))$ holds. Hence we have

$$\|A^\tau(D)\|_i = \|\tau^{-1}(A^\tau(D))\|_i = \|M^{-1}E'X(D)\|_i = \|A(D)\|_i .$$

A design T is called a simple array with parameters $\lambda_0, \lambda_1, \cdots, \lambda_m$, (simply, S-array $[m; \lambda_0, \lambda_1, \cdots, \lambda_m]$), if the assemblies in T can be obtained by λ_h repetitions of the set $\Omega(m, h)$ for each $h=0, 1, \cdots, m$, where $\Omega(m, h)$ are explained in Example 1 in Section 2. It is easily seen that an S-array $[m; \lambda_0, \cdots, \lambda_m]$ is a B-array $[N, m, 2l, \mathcal{U}]$, where

$$N=\sum_{h=0}^{m} \lambda_h \binom{m}{h} ,$$

$$\mu_i=\sum_{h=0}^{m} \lambda_h \binom{m-2l}{h-i} , \qquad i=0, 1, \cdots, 2l .$$

As will be seen from Chopra [2], Chopra and Srivastava [3], Shirakura [6], [7], Srivastava and Chopra [10], [11], etc., most of balanced arrays are of simple types for practical values of m and N. Therefore, it is desirable to study CB plans for 2^m-BFF designs derivable from simple arrays.

THEOREM 3.3. *Let T be an S-array $[m, \lambda_0, \cdots, \lambda_m]$ with $N=\sum_{h=0}^{m} \lambda_h \cdot \binom{m}{h}$. Then, for any $D \in \boldsymbol{D}^N_{\{k(d_1)\}}$ and $\tau \in C$, there exists D_0 in $\boldsymbol{D}^N_{\{k(d_1)\}}$ such that*

$$\tau(E'X(D))=E'X(D_0) .$$

PROOF. The design matrix E is represented by

$$E=(e_0; e_1, \cdots, e_m; e_{12}, \cdots, e_{m-1m}; \cdots; e_{12\cdots l}, \cdots, e_{m-l+1\cdots m}) ,$$

where e_z is defined in Proof of Theorem 2.3. Suppose $\tau(E)$ is the $N\times \nu$ matrix obtained from E by exchanging e_z with $e_{\tau(z)}$. (In particular, $e_{\tau(0)}=e_0$.) Further define the $N\times m$ matrices

$$E_1=(e_1, \cdots, e_m) \qquad \text{and} \qquad \tau(E_1)=(e_{\tau(1)}, \cdots, e_{\tau(m)}) .$$

Since T is a simple array, both E_1 and $\tau(E_1)$ then include just λ_h matrices of size $\binom{m}{h}\times m$ with elements ± 1 composed of $\binom{m}{h}$ distinct m-rowed vectors in which the numbers of 1's are h for each $h=0, 1, \cdots, m$. Therefore, there exists a permutation matrix P_τ of order N such that

$$\tau(E_1)=P_\tau E_1 . \tag{3.2}$$

On the other hand, we have

$$e_{\tau(t_1 t_2\cdots t_u)}=e_{\tau(t_1)} * e_{\tau(t_2)} * \cdots * e_{\tau(t_u)} ,$$

where $*$ denotes the product operator in Theorem 2.3. By (3.2), $e_{\tau(t_i)} =P_\tau e_{t_i}$ hold for $i=1, \cdots, u$. Thus it can be shown that

$$e_{\tau(t_1t_2\cdots t_u)}=(P_\tau e_{t_1})*\cdots*(P_\tau e_{t_u})=P_\tau(e_{t_1}*\cdots*e_{t_u})=P_\tau e_{t_1\cdots t_u}\ .$$

Also, $e_0=P_\tau e_0$, since the elements of e_0 are all one. Therefore, the matrix P_τ satisfies $\tau(E)=P_\tau E$. This means that

$$\tau(E'X(D))=\tau(E)'X(D)=E'P_\tau'X(D)\ .$$

From the one-to-one correspondence of $X(D)$ and D, there exists D_0 in $\boldsymbol{D}^N_{\{k(d_1)\}}$ satisfying $X(D_0)=P_\tau'X(D)$. This completes the proof.

Theorems 3.2 and 3.3 are useful in determining an OCB plan D for a design T in which $\|A(D)\|_i$, $(i=1, 2)$, has a minimum over $\boldsymbol{D}^N_{\{k(d_1)\}}$. Suppose $c=(c_0; c_1, \cdots, c_m; \cdots, c_{m-l+1\cdots m})'=E'X(D)$ and consider $\boldsymbol{D}^{*N}_{\{k(d_1)\}}=\{D\in\boldsymbol{D}^N_{\{k(d_1)\}};\ c_1\leqq c_2\leqq\cdots\leqq c_m\}$. Now let D^* be an OCB plan over $\boldsymbol{D}^N_{\{k(d_1)\}}$. Further consider a permutation $\tau\in C$ such that $(c_1^*, \cdots, c_m^*)$ transforms $(c^*_{\tau(1)}, \cdots, c^*_{\tau(m)})$, where $c^*=E'X(D^*)=(c_0^*; c_1^*, \cdots, c_m^*; \cdots, c^*_{m-l+1\cdots m})'$ and $c^*_{\tau(1)}\leqq c^*_{\tau(2)}\leqq\cdots\leqq c^*_{\tau(m)}$. Then, by Theorems 3.2 and 3.3, there exists $D_0^*\in\boldsymbol{D}^{*N}_{\{k(d_1)\}}$ satisfying $\|A(D^*)\|_i=\|A^\tau(D^*)\|_i=\|A(D_0^*)\|_i$. This means that D_0^* is also an OCB plan over $\boldsymbol{D}^N_{\{k(d_1)\}}$. That is, an OCB plan over $\boldsymbol{D}^N_{\{k(d_1)\}}$ can be selected in $\boldsymbol{D}^{*N}_{\{k(d_1)\}}$.

Consider the subclass of $\boldsymbol{D}^N_{\{k(d_1)\}}$,

$$\boldsymbol{S}^N=\{D\in\boldsymbol{D}^N_{\{k(d_1)\}};\ k(1)=[N/2]\}\ .$$

where $[x]$ denotes the greatest integer less than or equal to x. Also we take no interest in the confounding of the general mean and block effect. Therefore, the following slightly modified norms may be introduced:

(i) $\|A^*(D)\|_1=\max_{1\leqq i\leqq \nu-1}|a_i^*|$,

(ii) $\|A^*(D)\|_2=\{\mathrm{tr}\,(A^*(D)'A^*(D))\}^{1/2}$,

where $A^*(D)$ is the $(\nu-1)\times 1$ vector obtained by removing the first element of $A(D)$ and a_i^*'s are the elements of $A^*(D)$. Our interest now lies in a CB plan such that $\|A^*(D)\|_i$ is a minimum over $\boldsymbol{S}^N$. The vector $A^*(D)$ can be written by $A^*(D)=M^*E'X(D)$, where M^* is the $(\nu-1)\times\nu$ matrix obtained by removing the first row of M^{-1}. Arguments similar to Proofs of Theorems 3.2 and 3.4 give the following two theorems:

THEOREM 3.4. *Consider the design T and plan D of Theorem* 3.2. *Then, for any $\tau\in C$,*

$$\|A^*(D)\|_i=\|A^{*\tau}(D)\|_i\ ,\qquad i=1, 2\ ,$$

where $A^{*\tau}(D)=M^*\tau(E'X(D))$.

THEOREM 3.5. *Consider the simple array T of Theorem* 3.3. *Then, for any $D\in\boldsymbol{S}^N$ and $\tau\in C$, there exists $D_0\in\boldsymbol{S}^N$ such that*

$$\tau(E'X(D))=E'X(D_0) .$$

From the above theorems, we observe similarly that an OCB plan (w.r.t. $\|A^*(D)\|_i$) over S^N can also be selected in the subclass

$$S^{*N}=\{D \in S^N;\ D \in \boldsymbol{D}^{*N}_{\{k(d_1)\}}\} .$$

On the other hand, the calculation of $\|A(D)\|_i$ or $\|A^*(D)\|_i$ requires the inverse of $\nu\times\nu$ matrix M. However, Yamamoto, Shirakura and Kuwada [13] have shown that for a 2^m-BFF design of resolution $2l+1$, M^{-1} has at most $\binom{l+3}{3}$ distinct elements. Furthermore, Shirakura and Kuwada [14] have given an explicit expression of these elements.

In Table, optimum blocks $\boldsymbol{B}_T$ $(=\boldsymbol{B}_{T,\{k(d_1)\}})$ corresponding to OCB plans w.r.t. $\|A^*(D)\|_1$ and/or $\|A^*(D)\|_2$ over S^N are listed for A-optimal 2^4-BFF designs T of resolution V (minimizing tr M^{-1}) for the values of N with $11\leqq N\leqq 26$. The A-optimal balanced designs have been already given by Srivastava and Chopra [10]. Note that a B-array $[N,\ m=4,\ 4,\ \mathcal{U}=\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}]$ is equivalent to an S-array $[m=4;\ \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4]$, where $N=\lambda_0+4\lambda_1+6\lambda_2+4\lambda_3+\lambda_4$ and $\mu_i=\lambda_i$, $(i=0, 1, 2, 3, 4)$. Therefore, 2^4-BFF designs of resolution V can be represented by the sets $\Omega(m=4, h)$, $(h=0, 1, 2, 3, 4)$, of assemblies. Briefly, we write $T=\{\lambda_0\Omega(0), \lambda_1\Omega(1), \lambda_2\Omega(2), \lambda_3\Omega(3), \lambda_4\Omega(4)\}$, where $\Omega(h)=\Omega(4, h)$ for $h=0,\cdots, 4$. In Table, note that the other set $B_{(1)}$ can be obtained by $B_{(1)}=T-B_{(0)}$ for each T. Fortunately, it turns out that relatively many optimum blocks $\boldsymbol{B}_T$ are simply expressed by $\Omega(h)$ and that the optimum blocks w.r.t. $\|A^*(D)\|_{i=1,2}$ are identical except for $N=12$, 14, 15 and 21. The values of $\|A^*(D)\|_{i=1,2}$ are also given in the table for reference. From the above theorems, it may be remarked that an OCB plan is not unique for a given design. Furthermore, it is easily seen that if D is an OCB plan for T, then it is also so for $\bar{T}$, where $\bar{T}$ is the design obtained by an interchange of 0 and 1 in T. By the interchange, the most of properties for T are preserved (e.g., balanced property or A-, D-, E-optimalities). The following example is helpful in referring to Table.

Example 2.

(i) Consider a B-array $[N=12,\ m=4,\ 4,\ \mathcal{U}=\{1, 0, 1, 1, 1\}]$ given by

$$T=\{\Omega(0), \Omega(2), \Omega(3), \Omega(4)\} ,$$

which is a 2^4-BFF design of resolution V with 12 assemblies. Then, T can be rewritten by the 12×4 matrix

Table. Optimum blocks for 2^4-BFF designs of resolution V

N	μ_0	μ_1	μ_2	μ_3	μ_4	T	$B_{(0)}$	$\|\|A^*(D)\|\|_1$	$\|\|A^*(D)\|\|_2$
11	1	0	1	1	0	$\{\Omega(0), \Omega(2), \Omega(3)\}$	$\{\Omega(0), (1,0,1,0), (0,1,0,1), (0,0,1,1), (1,0,1,1), (0,1,1,1)\}$	0.3331	0.8333
12	1	0	1	1	1	$\{\Omega(0), \Omega(2), \Omega(3), \Omega(4)\}$	†$\{\Omega(2)\}$	0.25	0.7181
							††$\{(1,0,1,0), (1,0,0,1), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,1)\}$	0.5	0.7071
13	2	0	1	1	1	$\{2\Omega(0), \Omega(2), \Omega(3), \Omega(4)\}$	$\{\Omega(2), \Omega(4)\}$	0.1429	0.3499
14	0	1	1	1	0	$\{\Omega(1), \Omega(2), \Omega(3)\}$	†$\{(0,1,0,0), (0,0,1,0), (0,0,0,1), (1,1,0,0), (1,0,1,0), (1,0,0,1), (0,1,1,1)\}$	0.25	0.5
							††$\{(0,0,1,0), (0,0,0,1), (1,1,0,0), (0,1,1,0), (0,1,0,1), (1,0,1,1), (0,1,1,1)\}$	0.375	0.4330
15	1	1	1	1	0	$\{\Omega(0), \Omega(1), \Omega(2), \Omega(3)\}$	†$\{\Omega(1), \Omega(3)\}$	0.2	0.6325
							††$\{\Omega(0), \Omega(2), (0,1,1,1)\}$	0.25	0.5
16	1	1	1	1	1	$\{\Omega(0), \Omega(1), \Omega(2), \Omega(3), \Omega(4)\}$	$\{\Omega(1), \Omega(3)\}$	0	0
17	2	1	1	1	1	$\{2\Omega(0), \Omega(1), \Omega(2), \Omega(3), \Omega(4)\}$	$\{\Omega(0), \Omega(1), \Omega(3)\}$	0.0370	0.1171
18	2	1	1	1	2	$\{2\Omega(0), \Omega(1), \Omega(2), \Omega(3), 2\Omega(4)\}$	$\{\Omega(1), \Omega(3), \Omega(4)\}$	0.0833	0.1667
19	3	1	1	1	2	$\{3\Omega(0), \Omega(1), \Omega(2), \Omega(3), 2\Omega(4)\}$	$\{\Omega(0), \Omega(1), \Omega(3), \Omega(4)\}$	0.0437	0.1169
20	1	2	1	1	1	$\{\Omega(0), 2\Omega(1), \Omega(2), \Omega(3), \Omega(4)\}$	$\{(1,0,0,0), (0,0,1,0), (0,0,0,1), (0,1,0,0), (0,0,1,0), (0,0,0,1), \Omega(3)\}$	0.1429	0.2427
21	1	2	1	1	2	$\{\Omega(0), 2\Omega(1), \Omega(2), \Omega(3), 2\Omega(4)\}$	†$\{\Omega(1), \Omega(2), \Omega(4)\}$	0.1398	0.3424
							††$\{(0,0,1,0), (0,0,0,1), \Omega(1), \Omega(3), \Omega(4)\}$	0.1801	0.2724
22	1	1	2	1	1	$\{\Omega(0), \Omega(1), 2\Omega(2), \Omega(3), \Omega(4)\}$	$\{\Omega(1), \Omega(2), \Omega(4)\}$	0.125	0.25
23	2	1	2	1	1	$\{2\Omega(0), \Omega(1), 2\Omega(2), \Omega(3), \Omega(4)\}$	$\{2\Omega(0), \Omega(2), \Omega(3)\}$	0.0690	0.1873
24	2	1	2	1	2	$\{2\Omega(0), \Omega(1), 2\Omega(2), \Omega(3), 2\Omega(4)\}$	$\{2\Omega(0), \Omega(2), \Omega(3)\}$	0	0
24	1	2	1	2	1	$\{\Omega(0), 2\Omega(1), \Omega(2), 2\Omega(3), \Omega(4)\}$	$\{(1,0,0,0), (0,1,1,0), (0,1,0,1), (0,0,1,1), (1,1,1,0), (1,1,0,1), (1,0,1,1), \Omega(1), \Omega(4)\}$	0	0
25	1	2	1	2	2	$\{\Omega(0), 2\Omega(1), \Omega(2), 2\Omega(3), 2\Omega(4)\}$	$\{\Omega(0), \Omega(1), 2\Omega(3)\}$	0.0395	0.1101
26	2	2	1	2	2	$\{2\Omega(0), 2\Omega(1), \Omega(2), 2\Omega(3), 2\Omega(4)\}$	$\{2\Omega(1), \Omega(3), \Omega(4)\}$	0.0625	0.125

†These are optimum blocks w.r.t. $\|A^*(D)\|_1$; ††these are optimum blocks w.r.t. $\|A^*(D)\|_2$; the others are optimum blocks w.r.t. the two norms.

$$T'=\left[\begin{array}{c|cccccc|cccc|c} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right],$$

whose αth row denotes the αth assembly in T. Let $D_1=(1;\ 0,0,0,0,0,0;\ 1,1,1,1;\ 1)$ and $D_2=(1;\ 1,0,0,0,0,0;\ 1,1,1,1;\ 0)$ be CB plans in S^N, whose αth elements denote the αth assemblies in D_1 and D_2, respectively. Then $\|A^*(D_1)\|_1=0.25$, $\|A^*(D_1)\|_2=0.7181$, $\|A^*(D_2)\|_1=0.5$ and $\|A^*(D_2)\|_2=0.7071$. This means that D_1 is a CB plan better than D_2 w.r.t. $\|A^*(D)\|_1$ and D_2 is one better than D_1 w.r.t. $\|A^*(D)\|_2$ for T. In fact, $\|A^*(D_1)\|_1$ and $\|A^*(D_2)\|_2$ have minimum values over S^N. Hence D_1 and D_2 are OCB plans w.r.t. $\|A^*(D)\|_1$ and $\|A^*(D)\|_2$, respectively. An optimum block corresponding to D_1 is

$$\boldsymbol{B}_T=\{B_{(0)}=\{\Omega(2)\},\ B_{(1)}=\{\Omega(0),\Omega(3),\Omega(4)\}\}\ .$$

An optimum block to D_2 is $\boldsymbol{B}_T=\{B_{(0)},B_{(1)}\}$, where

$$B_{(0)}=\{(1,0,1,0),(1,0,0,1),(0,1,1,0),\\ (0,1,0,1),(0,0,1,1),(1,1,1,1)\}\ ,$$

$$B_{(1)}=\{(0,0,0,0),(1,1,0,0),\Omega(3)\}\ .$$

This is the indication of Table for $N=12$.

(ii) Consider a B-array $[20,\ 4,\ 4,\ \{1,2,1,1,1\}]$ given by

$$T=\{\Omega(0),2\Omega(1),\Omega(2),\Omega(3),\Omega(4)\}\ ,$$

which is a 2^4-BFF design of resolution V with 20 assemblies. Similarly, T can be rewritten by the following

$$T'=\left[\begin{array}{c|cccc|cccc|cccccc|cccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

Let $D=(1;\ 0,1,0,0;\ 1,0,0,0;\ 1,1,1,1,1,1;\ 0,0,0,0;\ 1)\in S^N$. Then, $\|A^*(D)\|_1=0.1429$ and $\|A^*(D)\|_2=0.2474$, which have minimum values over S^N. Hence D is an OCB plan for T w.r.t. the two norms and the corresponding optimum block $\boldsymbol{B}_T=\{B_{(0)},B_{(1)}\}$, where

$$B_{(0)}=\{(1,0,0,0),(0,0,1,0),(0,0,0,1),\\ (0,1,0,0),(0,0,1,0),(0,0,0,1),\Omega(3)\}\ ,$$

$$B_{(1)}=\{\Omega(0),(0,1,0,0),(1,0,0,0),\Omega(2),\Omega(4)\}\ .$$

Remark. For $r\geqq 2$, in general, it is an enormous amount of work to find an OCB plan D in $\boldsymbol{D}^N_{\{k(d_1\cdots d_r)\}}$ w.r.t. each norm for a design T. In the case where block factors rise in turn for the complete experiment of a design T, we can consider a successive block division. For simplicity, consider the case of $r=2$ and $k=4$. This is the case where after the division of two blocks for one factor (experimenter), both of the two blocks must be divided into two blocks for a new factor (day). First we find an optimum block $\boldsymbol{B}_{T,\{k(0),k(1)\}}=\{B_{(0)}, B_{(1)}\}$ w.r.t. each norm for the block effect η_1. Next we find the set $\boldsymbol{B}_{T,\{k(0,0),k(0,1),k(1,0),k(1,1)\}}=\{B_{(0,0)}, B_{(0,1)}, B_{(1,0)}, B_{(1,1)}\}$ of blocks minimizing the norm for the effect η_2, where $B_{(0,s)}$ and $B_{(1,s)}$, $(s=0, 1)$, are two divisions of $B_{(0)}$ and $B_{(1)}$, respectively, and $k(q)=k(q, 0)+k(q, 1)$ for $q=0, 1$. If $N=16$ and $k(0, 0)=\cdots=k(1, 1)=4$, and if the assemblies in T are all distinct, then the total number of plans in $\boldsymbol{D}^N_{\{k(d_1 d_2)\}}$ is $16!/(4!)^4=63{,}063{,}000$, whereas the plan in the above procedure is chosen among $\binom{16}{8}+\binom{8}{4}\binom{8}{4}=17{,}770$ plans.

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REFERENCES

[1] Box, G. E. P. and Hunter, J. S. (1961). The 2^{k-p} fractional designs, Part I, II, *Technometrics*, 3, 311-351, 449-458.

[2] Chopra, D. V. (1975). Balanced optimal 2^8 fractional factorial designs of resolution V, $52\leqq N\leqq 59$, A survey of statistical designs and linear models (ed. J. N. Srivastava), 91-100, Amsterdam, North-Holland.

[3] Chopra, D. V. and Srivastava, J. N. (1973). Optimal balanced 2^7 fractional factorial designs of resolution V, with $N\leqq 42$, *Ann. Inst. Statist. Math.*, 25, 587-604.

[4] Hedayat, A., Raktoe, B. L. and Federer, W. T. (1974). On a measure of aliasing due to fitting an incomplete model, *Ann. Statist.*, 2, 650-660.

[5] Kiefer, J. (1959). Optimum experimental designs, *J. Roy. Statist. Soc.*, Ser. B, 21, 272-319.

[6] Shirakura, T. (1976). Optimal balanced fractional 2^m factorial designs of resolution VII, $6\leqq m\leqq 8$, *Ann. Statist.*, 4, 515-531.

[7] Shirakura, T. (1977). Contributions to balanced fractional 2^m factorial designs derived from balanced arrays of strength $2l$, *Hiroshima Math. J.*, 7, 217-285.

[8] Shirakura, T. and Kuwada, M. (1976). Covariance matrices of the estimates for balanced fractional 2^m factorial designs of resolution $2l+1$, *J. Japan Statist. Soc.*, 6, 27-31.

[9] Srivastava, J. N. (1970). Optimal balanced 2^m fractional factorial designs, *S. N. Roy Memorial Volume*, Univ. of North Carolina and Indian Statistical Institute, 689-706.

[10] Srivastava, J. N. and Chopra, D. V. (1971). Balanced optimal 2^m fractional factorial designs of resolution V, $m\leqq 6$, *Technometrics*, 13, 257-269.

[11] Srivastava, J. N. and Chopra, D. V. (1973). Balanced optimal 2^7 fractional factorial designs of resolution V, with 56 to 68 runs, *Utilitas Mathematica*, **5**, 263-279.

[12] Srivastava, J. N., Raktoe, B. L. and Pesotan, H. (1976). On invariance and randomization in fractional replication, *Ann. Statist.*, **4**, 423-430.

[13] Yamamoto, S., Shirakura, T. and Kuwada, M. (1975). Balanced arrays of strength $2l$ and balanced fractional 2^m factorial designs, *Ann. Inst. Statist. Math.*, **27**, 143-157.

[14] Yamamoto, S., Shirakura, T. and Kuwada, M. (1976). Characteristic polynomials of the information matrices of balanced fractional 2^m factorial designs of higher $(2l+1)$ resolution, *Essays in Prob. and Statist., Birthday Volume in Honor of Prof. J. Ogawa* (ed. S. Ikeda, et al.), 73-94, Tokyo: Shinko-Tsusho.