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MOMENTS OF COVERAGE OF A RANDOM ELLIPSOID

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Summary

Exact expressions are obtained for the moments of coverage of the random ellipsoid

$$T_r(\bar{\boldsymbol{x}}, S(X)) = \{\boldsymbol{y} \mid (\boldsymbol{y}-\bar{\boldsymbol{x}})' S^{-1}(X)(\boldsymbol{y}-\bar{\boldsymbol{x}}) \leqq r\}$$

where $X=(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n)$ is a sample from a $N_p(\boldsymbol{\mu}, \Sigma)$ distribution. This leads to approximations for the distribution of coverage and a solution to a problem in tolerance regions. An alternative expression is obtained for the distribution function of a quadratic form in normal variables.

1. Introduction

Suppose that $X=(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n) \in R^{p\times n}$ is a sample from the $N_p(\boldsymbol{\mu}, \Sigma)$ distribution where $\boldsymbol{\mu} \in R^p$ and $\Sigma \in R^{p\times p}$ positive definite are unknown. A measurable map $T: R^{p\times n} \to \mathscr{B}^p$ satisfying $P^n_{(\boldsymbol{\mu}, \Sigma)}(P_{(\boldsymbol{\mu}, \Sigma)}(T(X)) \geqq \beta) \geqq \gamma$ for every $(\boldsymbol{\mu}, \Sigma)$ is called a β-content tolerance region at confidence γ.

The random variable $P_{(\boldsymbol{\mu}, \Sigma)}(T(X))$ is called the coverage of T. The purpose of this paper is to study the distribution of the coverage when

$$T(X) = T_r(\bar{\boldsymbol{x}}, S(X)) = \{\boldsymbol{y} \mid (\boldsymbol{y}-\bar{\boldsymbol{x}})' S^{-1}(X)(\boldsymbol{y}-\bar{\boldsymbol{x}}) \leqq r\} \tag{1}$$

where $\bar{\boldsymbol{x}} = n^{-1} \sum_{i=1}^{n} \boldsymbol{x}_i$, $S(X) = (X - \bar{\boldsymbol{x}}1')(X - \bar{\boldsymbol{x}}1')'$ and $r \geqq 0$. We will denote this coverage by $C_r(\bar{\boldsymbol{x}}, S(X))$ hereafter. As is easily seen the distribution of $C_r(\bar{\boldsymbol{x}}, S(X))$ is independent of $(\boldsymbol{\mu}, \Sigma)$. Thus if we know the distribution of $C_r(\bar{\boldsymbol{x}}, S(X))$ for each r we can obtain a β-content tolerance region at confidence γ by choosing r so that β is the $1-\gamma$ quantile of the distribution of $C_r(\bar{\boldsymbol{x}}, S(X))$.

The problem of obtaining such an r has been considered elsewhere. In particular when $p=1$, Wald and Wolfowitz [14] show how to calcu-

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late r exactly using numerical integration. Odeh [6] tabulates r for various values of n, β and γ and further compares these exact results with the approximation suggested by Wald and Wolfowitz [14]. Wallis [15] considers this problem in the context of simple regression and Lieberman and Miller [5] generalize to multiple regression situations.

John [3] considers this problem for arbitrary p and obtains a large sample approximation. Siotani [12] develops large sample approximations for dimensions $p=2$ and 3. Guttman [2] obtains large sample approximations to the mean and variance of $C_r(\bar{x}, S(X))$ and then fits by the method of moments a Beta (p, q) distribution. In all these papers the adequacy of the approximations is not formally established although they seem intuitively reasonable.

In this paper we provide a method of approximating r which is applicable to any sample size. This approximation is based on the fact that the distribution of $C_r(\bar{x}, S(X))$ is concentrated on $(0, 1)$ and is thus determined by its moments. This leads to a finite series approximation for the distribution function of $C_r(\bar{x}, S(X))$ based on m moments and we can then solve iteratively for the value of r for which β is the $(1-\gamma)$th quantile of the distribution. We emphasize that the approximation is appropriate for any sample size.

2. The moments of coverage

We note that $T_r(\bar{x}, S(X))$ is equivariant under the action of the positive affine group given by $G=\{[\boldsymbol{a}, B] \mid \boldsymbol{a} \in R^p,\ B \in R^{p\times p},\ \det B>0\}$, where $[\boldsymbol{a}_1, B_1][\boldsymbol{a}_2, B_2]=[\boldsymbol{a}_1+B_1\boldsymbol{a}_2, B_1B_2]$ with action $[\boldsymbol{a}, B]X=\boldsymbol{a}\mathbf{1}'+BX$ on $R^{p\times n}$ inducing action $[\boldsymbol{a}, B](\bar{x}, S(X))=(\boldsymbol{a}+B\bar{x}, BS(X)B')$ on the minimal sufficient statistic $(\bar{x}, S(X))$. If $S \in R^{p\times p}$ is positive definite then denote by S_T the unique lower triangular factorization of S as $S=S_TS_T'$. Thus

$$(2)\qquad \begin{aligned} C_r(\bar{x}, S(X)) &= P_{(\mu, \Sigma)}(T_r(\bar{x}, S(X)) \\ &= P_{(0,I)}([\boldsymbol{\mu}, \Sigma_T]^{-1}[\bar{x}, S(X)_T]T_r(0, I)) = P_{(0,I)}(T_r(z, S)) \end{aligned}$$

where $z=\Sigma_T^{-1}(\bar{x}-\boldsymbol{\mu})\sim N_p(0, n^{-1}I)$ is statistically independent of

$$(3)\qquad S=\Sigma_T^{-1}S(X)_TS(X)_T'(\Sigma_T^{-1})'\sim W_p(I, n-1)\ .$$

This establishes our earlier comment that the distribution of C_r is independent of $(\boldsymbol{\mu}, \Sigma)$.

Denoting the joint distribution of (z, S) by P and mth moment of C_r by μ_m we have

$$(4)\qquad \begin{aligned} \mu_m &= \mathrm{E}\,[C_r^m] \\ &= \int_{R^p\times R^{p\times p}} \{P_{(0,I)}(T_r(z, S))\}^m dP(z, S) \end{aligned}$$

$$=\int_{R^{p\times m}} P(\{y_1,\cdots,y_m\}\subset T_r(z,S))dP^m_{(0,I)}(Y)$$
$$=\int_{R^{p\times m}} P(\max_{1\leqq i\leqq m}(y_i-z)'S^{-1}(y_i-z)\leqq r)dP^m_{(0,I)}(Y)$$
$$=\int_{R^p\times R^{p\times p}} P^m_{(0,I)}(\max_{1\leqq i\leqq m}(y_i-z)'S^{-1}(y_i-z)\leqq r)dP(z,S)$$

where $Y=(y_1,\cdots,y_m)$ is a sample from the $N_p(0,I)$ distribution, the third equality follows from Robbins [7], the fourth equality follows from the fact that $y\in C_r(z,S)$ if and only if $(y-z)'S^{-1}(y-z)\leqq r$ and the last equality is just an application of Fubini's Theorem. We note that the third equality in (4) establishes that evaluating the mth moment of C_r is equivalent to calculating the probability that the random ellipsoid covers m points in R^p chosen independently from the $N_p(0,I)$ distribution.

Given (z,S) we have that $(y_i-z)'S^{-1}(y_i-z)$ for $i=1,\cdots,m$ is a sample of m from a distribution on R with distribution function $G_{(z,S)}$. As is well-known the largest order statistic from a sample of m from distribution function F has distribution function F^m. Thus from (4) we have that

$$\mu_m=\mathrm{E}\,[G^m_{(z,S)}(r)]\ . \tag{5}$$

The ease of evaluating this expectation depends greatly on the expression we use for $G_{(z,S)}$. We consider this problem in the next section and for convenience suppress (z,S) as part of the notation. Note that when $m=1$ then $\mu_1=P\left(F(p,n-p+1)\leqq\left(\frac{n-p+1}{p}\frac{n}{n+1}r\right)\right)$ from standard results on the distribution of quadratic forms.

3. Expression for G

Many authors have considered the distribution of $(y-z)'S^{-1}(y-z)$ where $y\sim N_p(0,I)$ and $z\in R^p$, $S\in R^{p\times p}$ positive definite are fixed. Various expressions have been obtained for the distribution function G; for example see the bibliography in Johnson and Kotz [4], Chap. 29. For our purposes, as we must raise G to the mth power and then take expectations with respect to z and S, none of these seem suitable. Accordingly a different expression is developed here. This expression is seen to be a generalization of that obtained in Robbins [8]. We note that our result is obtained by directly integrating the $N_p(0,I)$ density over the non-central ellipsoid $T_r(z,S)$. This is perhaps the most straightforward method of calculating this probability, proceeding along the lines of the results in Ruben [9], and is an alternative to the approach taken in Ruben [10].

Using the spectral decomposition for S we have that $S=QDQ'$ where $Q\in R^{p\times p}$ is orthogonal and $D=\operatorname{diag}(d_1^2,\cdots,d_p^2)$ where $d_1\geqq d_2\geqq\cdots\geqq d_p>0$. Now $Q'\boldsymbol{y}\sim N_p(\mathbf{0},I)$ and thus we can consider the distribution of $(\boldsymbol{y}-\boldsymbol{b})'D^{-1}(\boldsymbol{y}-\boldsymbol{b})$ where $\boldsymbol{b}=Q'\boldsymbol{z}$ and $\boldsymbol{y}\sim N_p(\mathbf{0},I)$. The ellipsoidal region $T_r(\boldsymbol{b},D)$ can be divided into 2^p subregions bounded by the boundary of $T_r(\boldsymbol{b},D)$ and the hyperplanes containing the principal axes. A particular subregion is completely characterized by $\Lambda=\operatorname{diag}(\lambda_1,\cdots,\lambda_p)$ where $\lambda_i=\operatorname{sgn}(y_i-b_i)$. Within a particular subregion make the change of variable $\boldsymbol{y}\to t, v_1,\cdots,v_{p-1}$ where $t=(\boldsymbol{y}-\boldsymbol{b})'D^{-1}(\boldsymbol{y}-\boldsymbol{b})$ and $v_i^{1/2}=\lambda_i t^{-1/2}d_i^{-1}(y_i-b_i)$. Note that $v_p=1-v_1-\cdots-v_{p-1}$. The Jacobian is given by

$$J(\boldsymbol{y}\to t, v_1,\cdots,v_{p-1})=2^{-p}d_1\cdots d_p t^{p/2-1}\prod_{i=1}^{p} v_i^{-1/2}\,. \tag{6}$$

Thus we have that

$$\begin{aligned} G(r)&=P_{(\mathbf{0},I)}((\boldsymbol{y}-\boldsymbol{b})'D^{-1}(\boldsymbol{y}-\boldsymbol{b})\leqq r)\\ &=2^{-p}(2\pi)^{-p/2}\exp\left\{-\frac{1}{2}\boldsymbol{b}'\boldsymbol{b}\right\}d_1\cdots d_p\sum_{\Lambda}\int_0^r\int_{S_*}f_\Lambda(t,\boldsymbol{v})d\boldsymbol{v}dt \end{aligned} \tag{7}$$

where $S_*=\{\boldsymbol{v}\,|\,0\leqq v_i\leqq 1,\ v_p=1-v_1-\cdots-v_{p-1}\}$ and

$$\begin{aligned} f_\Lambda(t,\boldsymbol{v})&=t^{p/2-1}\prod_{i=1}^{p}v_j^{-1/2}\exp\left\{-\frac{t}{2}(d_1^2v_1+\cdots+d_p^2v_p)\right\}\\ &\quad\times\prod_{i=1}^{p}\exp\{-\lambda_i t^{1/2}d_i b_i v_i^{1/2}\}\\ &=\exp\left\{-\frac{1}{2}(d_1^2v_1+\cdots+d_p^2v_p)\right\}\\ &\quad\times\sum_{k=0}^{\infty}\left\{\sum_{k.=k}\prod_{j=1}^{p}\frac{(-\lambda_j)^{k_j}d_j^{k_j}b_j^{k_j}}{k_j!}v_j^{(k_j+1)/2-1}\right\}t^{(p+k)/2-1} \end{aligned} \tag{8}$$

where $k.=k_1+\cdots+k_p$ for $k_i\in N_0$ and we sum over all $(k_1,\cdots,k_p)$ satisfying these constraints. Taking the Cauchy product of the p series is justified by Theorem 3.5, Rudin [11] as they are individually absolutely convergent. Then by the Dominated Convergence Theorem, Rudin [11], we have that $\sum_{\Lambda}\int_0^r\int_{S_*}f_\Lambda(t,\boldsymbol{v})d\boldsymbol{v}dt$ equals

$$\begin{aligned} &\sum_{k=0}^{\infty}\left\{\sum_{k.=k}\delta\left(\frac{\boldsymbol{k}+\mathbf{1}}{2}\right)\sum_{\Lambda}\prod_{j=1}^{p}\frac{(-\lambda_j)^{k_j}d_j^{k_j}b_j^{k_j}}{k_j!}\right.\\ &\quad\left.\times\int_0^r m\left(-\frac{t}{2}d_1^2,\cdots,-\frac{t}{2}d_p^2\,\middle|\,\frac{\boldsymbol{k}+\mathbf{1}}{2}\right)t^{(p+k)/2-1}dt\right\} \end{aligned} \tag{9}$$

where $\boldsymbol{n}=(n_1,\cdots,n_p)'$, $\delta(\boldsymbol{n})=(\Gamma(n_1)\cdots\Gamma(n_p))/\Gamma(n_1+\cdots+n_p)$ and $m(\cdot\,|\boldsymbol{n})$ is the moment generating function of the Dirichlet $(\boldsymbol{n})$ distribution.

We have that $\sum_{\Lambda}(-\lambda_1)^{k_1}\cdots(-\lambda_p)^{k_p}=2^p$ when the k_j are all even and is 0 otherwise. Now

$$(10)\qquad m(\boldsymbol{t}|\boldsymbol{n})=\sum_{j_1=0}^{\infty}\cdots\sum_{j_p=0}^{\infty}\mu_{\boldsymbol{j}}\frac{t_1^{j_1}}{j_1!}\cdots\frac{t_p^{j_p}}{j_p!}=\sum_{l=0}^{\infty}\left\{\sum_{l.=l}\mu_{\boldsymbol{l}}\frac{t_1^{l_1}}{l_1!}\cdots\frac{t_p^{l_p}}{l_p!}\right\}$$

where $\mu_{\boldsymbol{l}}=\delta(\boldsymbol{n}+\boldsymbol{l})/\delta(\boldsymbol{n})$ is the $\boldsymbol{l}$th moment of the Dirichlet $(\boldsymbol{n})$. Thus we have that (9) equals

$$(11)\qquad 2^p\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k.=k}\sum_{l.=l}\delta(\boldsymbol{k}+\boldsymbol{l}+1/2)(-1/2)^l\prod_{m=1}^{p}\frac{d_m^{2(k_m+l_m)}b_m^{2k_m}}{(2k_m!)l_m!}\,\frac{r^{p/2+k+l}}{(p/2+k+l)}$$

and putting $j_m=k_m+l_m$, $j=k+l$, (11) becomes

$$(12)\qquad 2^p r^{p/2}\sum_{j=0}^{\infty}\left\{\sum_{j.=j}\delta(\boldsymbol{j}+1/2)\prod_{m=1}^{p}d_m^{2j_m}\sum_{i=0}^{j_m}\frac{(-2b_m^2)^i}{(2i)!(j_m-i)!}\right\}\left(\frac{p}{2}+j\right)^{-1}\left(-\frac{r}{2}\right)^j .$$

Therefore combining (7) and (12) we have that

$$(13)\qquad G(r)=\pi^{-p/2}\exp\left(-\frac{1}{2}\boldsymbol{b}'\boldsymbol{b}\right)\left(\frac{r}{2}\right)^{p/2}\sum_{j=0}^{\infty}\left\{\sum_{j.=j}\left(\frac{p}{2}+j\right)^{-1}\delta(\boldsymbol{j}+1/2)\right.$$
$$\left.\times\prod_{k=1}^{p}d_k^{2j_k+1}\sum_{i=0}^{j_k}\frac{(-2b_k^2)^i}{(2i)!(j_k-i)!}\right\}\left(-\frac{r}{2}\right)^j .$$

When $b_1=b_2=\cdots=b_p=0$ this expression agrees with that given in Robbins [8].

We obtain an expression for $G^m(r)$ by raising (13) to the mth power and expressing this as a series in r. This gives

$$(14)\qquad G^m(r)=\pi^{-mp/2}\exp\left(-\frac{m}{2}\boldsymbol{b}'\boldsymbol{b}\right)\left(\frac{r}{2}\right)^{mp/2}$$
$$\times\sum_{j=0}^{\infty}\left\{\sum_{j.=j}\prod_{k=1}^{m}\left(\frac{p}{2}+j_k\right)^{-1}\sum_{j_{k.}=j_k}\delta(\boldsymbol{j}_k+1/2)\right.$$
$$\left.\times\prod_{l=1}^{p}d_l^{2j_{kl}+1}\sum_{i=0}^{j_{kl}}\frac{(-2b_l^2)^i}{(2i)!(j_{kl}-i)!}\right\}\left(-\frac{r}{2}\right)^j$$

where $j.=j_1+\cdots+j_m$, $j_{i.}=j_{i1}+\cdots+j_{ip}$ for $i=1,\cdots,m$ and the summation convention is as before.

When $d_1=\cdots=d_p=1$ we have that $(\boldsymbol{y}-\boldsymbol{b})'(\boldsymbol{y}-\boldsymbol{b})\sim$Chi-square $(p,\boldsymbol{b}'\boldsymbol{b})$. In this case the distribution function of the largest order statistic in a sample of m from this distribution is given by

$$(15)\qquad G^m(r)=\exp\left(-\frac{m}{2}\boldsymbol{b}'\boldsymbol{b}\right)\left\{\sum_{j=0}^{\infty}\frac{(\boldsymbol{b}'\boldsymbol{b}/2)^j}{j!}\Gamma^{-1}\left(\frac{p}{2}+j\right)\right.$$
$$\left.\times\int_0^r\left(\frac{x}{2}\right)^{p/2+j-1}e^{-x/2}\frac{1}{2}dx\right\}^m$$
$$=\exp\left(-\frac{m}{2}\boldsymbol{b}'\boldsymbol{b}\right)\left(\frac{r}{2}\right)^{mp/2}\sum_{j=0}^{\infty}\left\{\sum_{j.=j}\sum_{k=1}^{m}\sum_{i=0}^{j_k}\frac{(-\boldsymbol{b}'\boldsymbol{b}/2)^i}{i!(j_k-i)!}\right.$$

$$\times\Gamma^{-1}\left(\frac{p}{2}+i\right)\left(\frac{p}{2}+j_k\right)^{-1}\Bigg\}\left(-\frac{r}{2}\right)^j .$$

Then setting $d_1=\cdots=d_p=1$ in (14) and equating coefficients between (14) and (15) we obtain the nonintuitive identity

$$(16)\qquad \sum_{j.=j}\prod_{k=1}^{m}\sum_{j_{k.}=j_k}\delta(j_k+1/2)\prod_{l=1}^{p}\sum_{i=0}^{j_{kl}}\frac{(-2b_l^2)^i}{(2i)!(j_{kl}-1)!}$$
$$=\pi^{mp/2}\sum_{j.=j}\prod_{k=1}^{m}\sum_{i=0}^{j_{kl}}\frac{(-\boldsymbol{b}'\boldsymbol{b}/2)^i}{i!(j_k-i)!}\Gamma^{-1}\left(\frac{p}{2}+i\right) .$$

From this we deduce that the absolute value of the jth term in the series in (14) is bounded above by

$$(17)\qquad \pi^{mp/2}d_1^{2j+mp}\left(\frac{r}{2}\right)^j\Gamma^{-1}\left(\frac{p}{2}\right)\left(\frac{p}{2}\right)^{-m/2}\sum_{j.=j}\prod_{k=1}^{m}\frac{|1-\boldsymbol{b}'\boldsymbol{b}/2|^{j_k}}{j_k!}$$
$$=\pi^{mp/2}d_1^{2j+mp}\left(\frac{r}{2}\right)^j\Gamma^{-m}\left(\frac{p}{2}\right)\left(\frac{p}{2}\right)^{-m}|1-\boldsymbol{b}'\boldsymbol{b}/2|^j\frac{m^j}{j!} .$$

Then putting $x=md_1^2(r/2)|1-\boldsymbol{b}'\boldsymbol{b}/2|$ and taking j terms in (14) we have that the absolute error in this estimate of $G^m(r)$ is bounded above by

$$(18)\qquad \exp\left(-\frac{m}{2}\boldsymbol{b}'\boldsymbol{b}\right)\left(\frac{r}{2}\right)^{mp/2}d_1^{mp}\Gamma^{-m}\left(\frac{p}{2}\right)\left(\frac{p}{2}\right)^{-m}\sum_{k=j+1}^{\infty}\frac{x^j}{j!}$$
$$=\exp\left(-\frac{m}{2}\boldsymbol{b}'\boldsymbol{b}\right)\left(\frac{r}{2}\right)^{mp/2}d_1^{mp}\Gamma^{-m}\left(\frac{p}{2}\right)\left(\frac{p}{2}\right)^{-m}\frac{1}{j!}\int_0^x e^t(x-t)^j dt$$
$$\leqq\exp\left\{-\frac{m}{2}(\boldsymbol{b}'\boldsymbol{b}-d_1^2r|1-\boldsymbol{b}'\boldsymbol{b}/2|)\right\}\left(\frac{r}{2}\right)^{mp/2+j+1}$$
$$\times d_1^{2j+mp+1}\Gamma^{-m}\left(\frac{p}{2}\right)\left(\frac{p}{2}\right)^{-m}m^{j+1}|1-\boldsymbol{b}'\boldsymbol{b}/2|^{j+1}/(j+1)!$$

where the first equality is justified by Taylors' Integral Remainder Theorem.

4. Evaluation of μ_m

We now use the expression in (14) to evaluate μ_m. To do this we need to calculate the expectation of (14) with respect to the distribution for $(\boldsymbol{b}, D)$ where $\boldsymbol{b}\sim N_p(0, n^{-1}I)$ statistically independent of D which has the distribution of the characteristic roots from a $W_p(I, n-1)$ distribution. By the Dominated Convergence Theorem this expectation can be evaluated by calculating the expectation of each term in (14) and summing.

First we calculate the expectation with respect to the distribution for $\boldsymbol{b}$. We have that $\mathrm{E}[\exp\{-(m/2)\boldsymbol{b}'\boldsymbol{b}\}b_l^{2i}]=n^{p/2}(m+n)^{-i-p/2}(2i)!/(2^i i!)$

and thus

$$\text{(19)}\qquad \mathrm{E}\left[\exp\left\{-\frac{m}{2}\boldsymbol{b}'\boldsymbol{b}\right\}\sum_{i=0}^{j_{kl}}\frac{(-2b_l^2)^i}{(2i)!(j_{kl}-i)!}\right] =\left(\frac{n}{m+n}\right)^{p/2}\sum_{i=0}^{j_{kl}}\frac{(-1)^i(m+n)^{-i}}{i!(j_{kl}-i)!} =\left(\frac{n}{m+n}\right)^{p/2}\left(1-\frac{1}{m+n}\right)^{j_{kl}}\Big/ j_{kl}!\,.$$

Thus we can write the expectation of (14) with respect to $\boldsymbol{b}$ as

$$\text{(20)}\qquad \pi^{-mp/2}\left(\frac{r}{2}\right)^{mp/2}\left(\frac{n}{m+n}\right)^{p/2}\sum_{j=0}^{\infty}\left\{\sum_{j.=j}\prod_{k=1}^{m}\left(\frac{p}{2}+j_k\right)^{-1}\sum_{j_{k.}=j_k}\delta(\boldsymbol{j}_k+1/2)\times\prod_{l=1}^{p}\frac{d_l^{2j_{kl}+1}}{j_{kl}!}\left(1-\frac{1}{m+n}\right)^{j_{kl}}\right\}\left(-\frac{r}{2}\right)^j$$
$$=\pi^{-mp/2}\left(\frac{r}{2}\right)^{mp/2}\left(\frac{n}{m+n}\right)^{p/2}\sum_{j=0}^{\infty}\left\{\sum_{j.=j}k_{j_1\cdots j_p}(m)d_1^{2j_1+m}\cdots d_p^{2j_p+m}\right\}\times\left[-\left(1-\frac{1}{m+n}\right)\frac{r}{2}\right]^j$$

where

$$\text{(21)}\qquad k_{j_1\cdots j_p}(m)=\sum_{j_{.1}=j_1}\cdots\sum_{j_{.p}=j_p}\prod_{k=1}^{m}\Gamma^{-1}\left(\frac{p}{2}+j_{k1}+\cdots+j_{kp}+1\right)\times\frac{\Gamma(j_{k1}+1/2)\cdots\Gamma(j_{kp}+1/2)}{\Gamma(j_{k1}+1)\cdots\Gamma(j_{kp}+1)}\,.$$

Note that we have the following recursion to aid in the computation of k

$$\text{(22)}\qquad k_{j_1\cdots j_p}(m)=\sum_{j_{m1}=0}^{j_1}\cdots\sum_{j_{mp}=0}^{j_p}k_{j_1-j_{m1}\cdots j_p-j_{mp}}(m-1)\times\Gamma^{-1}\left(\frac{p}{2}+j_{m1}+\cdots+j_{mp}+1\right)\frac{\Gamma(j_{m1}+1/2)\cdots\Gamma(j_{mp}+1/2)}{\Gamma(j_{m1}+1)\cdots\Gamma(j_{mp}+1)}\,.$$

The density of $(d_1,\cdots,d_p)$ is given by

$$\text{(23)}\qquad \frac{A_{n-1}^{(p)}\cdots A_p^{(p)}}{2^p(2\pi)^{p(n-1)/2}}d_1^{n-1-p}\cdots d_p^{n-1-p}\prod_{i<j}(d_i^2-d_j^2)\exp\left\{-\frac{1}{2}(d_1^2+\cdots+d_p^2)\right\}$$

where $A_n=2\pi^{n/2}/\Gamma(n/2)$, $A_n^{(p)}=A_nA_{n-1}\cdots A_{n-p+1}$ and $0<d_p<\cdots<d_1<\infty$. We then want to evaluate expectations of the form $\mathrm{E}\,[d_1^{2j_1+m}\cdots d_p^{2j_p+m}]$ with respect to (23). We denote this expectation by $\lambda_{j_1\cdots j_p}(m)$.

To evaluate the λ's we must integrate an expression of the form (23) where the powers of the d_i now differ. Towards this end we write $\prod_{1\le i<j\le p}(d_i^2-d_j^2)$ as a polynomial in $d_1^2,\cdots,d_p^2$; namely $\sum_{I_p}l_{i_1\cdots i_p}d_1^{2i_1}\cdots d_p^{2i_p}$,

where

$$(24)\qquad I_p=\left\{(i_1,\cdots,i_p)\,|\,i_1+\cdots+i_p=\frac{p(p-1)}{2},\ 0\leqq i_j\leqq p-1\right\}$$

as it is easily shown that each term must have degree $p(p-1)/2$. While we are not able to obtain closed form expressions for these coefficients they can be calculated recursively as $\prod_{1<i<j<p}(d_i^2-d_j^2)=\prod_{1<i<j<p-1}(d_i^2-d_j^2)\cdot\prod_{i=1}^{p-1}(d_i^2-d_p^2)$ and from this we deduce $l_{i_1\cdots i_p}=(-1)^{i_p}\sum_{I(i_1,\cdots,i_p)}l_{h_1\cdots h_{p-1}}$ where $I(i_1,\cdots,i_p)=\{(h_1,\cdots,h_{p-1})\,|\,h_1+\cdots+h_{p-1}=(p-1)(p-2)/2,\ 0\leqq h_i\leqq p-2,\ i_p$ of the indices satisfy $h_j=i_j$ and the remainder satisfy $h_j+1=i_j\}$.

Thus to evaluate λ we must evaluate integrals of the form

$$(25)\qquad l(i_1,\cdots,i_p)=\int_0^\infty\int_0^{d_1}\int_0^{d_2}\cdots\int_0^{d_{p-1}}d_1^{i_1}\cdots d_p^{i_p}\exp\left\{-\frac{1}{2}(d_1^2+\cdots+d_p^2)\right\}\times dd_p\cdots dd_1$$

where $i_1,\cdots,i_p$ are nonnegative integers all of the same parity. These integrals can be evaluated in closed form via partial integration or, conveniently for computation, recursively. For example when $p=2$ this leads to $\lambda_{j_1j_2}(m)=[(n-3+m+j_1+j_2)!+2(j_1-j_2)l(n-3+m+2j_2,n-3+m+2j_1)]/(n-3)!$ and $l(i,j)=-1/2((i+j-2)/2)!+(j-1)l(i,j-2)$ when $j>1$, $l(i,j)=1/2((i+j-2)/2)!+(i-1)l(i-2,j)$ when $i>1$ and $l(0,0)=\pi/4$, $l(1,1)=1/2$. Therefore from (20) we have established.

THEOREM 1.

$$\mu_m=\pi^{-mp/2}\left(\frac{r}{2}\right)^{mp/2}\left(\frac{n}{m+n}\right)^{p/2}\sum_{j=0}^\infty\left\{\sum_{j_1+\cdots+j_p=j}k_{j_1\cdots j_p}(m)\lambda_{j_1\cdots j_p}(m)\right\}\times\left[-\left(1-\frac{1}{m+n}\right)\frac{r}{2}\right]^j.$$

No error bound is at present available for truncation of this series.

5. Approximations

As mentioned earlier, specification of the moments of the distribution in this context in effect specifies the distribution. Of course, as it is impossible to evaluate all the moments, we must make use of the information given by a finite number to obtain an approximation. One method of obtaining an approximation is via Bernstein polynomials; see Feller [1], pp. 219-227. In fact we have the result that if F_r represents the distribution function of the coverage C_r then $F_{(m,r)}(t)=\sum_{j\leqq mt}\binom{m}{j}(-1)^{m-j}\Delta^{m-j}\mu_j(r)$ converges uniformly to F_r as $m\to\infty$ where

$\mu_j(r)$ is the jth moment of C_r and Δ^k is the kth difference operator.

Now note that $r_1 < r_2$ implies $T_{r_1}(\bar{x}, S(X)) \subsetneqq T_{r_2}(\bar{x}, S(X))$ which implies $C_{r_1}(\bar{x}, S(X)) < C_{r_2}(\bar{x}, S(X))$ which in turn implies $F_{r_2}(t) < F_{r_1}(t)$ for $t \in (0, 1)$. Therefore $F_r(t)$ is a strictly decreasing function of r. Also $F_0(t)=1$ and $F_\infty(t)=0$ and thus there is a unique r such that $F_r(\beta)=1-\gamma$.

From Feller ([1], p. 224) we have that $(-1)^{m-k}\Delta^{m-k}\mu_k(r) = \mathrm{E}\,[C_r^k(1-C_r)^{m-k}]$. Now reasoning as in Robbins [7] and as in Section 2 we have that this expectation can be written as

$$\begin{aligned}(29)\qquad &\int_{R^{p\times m}} P(\boldsymbol{y}_i \in T_r(z, S)\ \ i=1,\cdots,k)\\ &\qquad \times P(\boldsymbol{y}_i \in T_r^c(z, S)\ \ i=k+1,\cdots,m)dP^m_{(0,I)}(Y)\\ &= \int_{R^{p\times m}} P((\boldsymbol{y}_i - z)'S^{-1}(\boldsymbol{y}_i - z) \leqq r;\ i=1,\cdots,k)\\ &\qquad \times P((\boldsymbol{y}_i - z)'S^{-1}(\boldsymbol{y}_i - z) > r;\ i=k+1,\cdots,m)dP^m_{(0,I)}(Y)\\ &= \int_{R^p \times R^{p\times p}} G^k_{(z,S)}(r)(1-G_{(z,S)}(r))^{m-k}dP(z, S)\,.\end{aligned}$$

Therefore

$$(30)\qquad F_{(m,r)}(t) = \mathrm{E}\left[\sum_{k=0}^{\lfloor mt\rfloor}\binom{m}{k}G^k_{(z,S)}(r)(1-G_{(z,S)}(r))^{m-k}\right]$$

and the sum in the expectation is the probability that in a sample of m from $G_{(z,S)}$ at most $\lfloor mt\rfloor$ are less than or equal to r. This is the complement of the event that at least $\lfloor mt\rfloor+1$ of the sample values are less than or equal to r and this is the distribution function of the $\lfloor mt\rfloor+1$st order statistic in a sample of m from G evaluated at r. Therefore $F_{(m,r)}(t)$ is a strictly decreasing continuous function of r. Further $F_{(m,0)}(t)=1$, $F_{(m,\infty)}(t)=0$ which implies the existence of a unique r_m such that $F_{(m,r_m)}(\beta)=1-\gamma$ for every m.

Now let $\delta>0$ and suppose there are infinitely many $r_m < r-\delta$, say subsequence $\{r_{m_i}\}$. Then $1-\gamma = F_{(m_i,r_{m_i})}(\beta) > F_{(m_i,r-\delta)}(\beta) \to F_{r-\delta}(\beta) > F_r(\beta) = 1-\gamma$ and accordingly no such subsequence exists. Similarly there cannot exist infinitely many $r_m > r+\delta$ and we have proved

THEOREM 2. $r_m \to r$.

Thus the r_m we compute from our approximations to $F_r(\beta)$ are true approximations. We note that in proving Theorem 2 we have established the interesting fact that $F_{(m,r)}(t)$ is the probability the random ellipsoid $T_r(\bar{x}, S(X))$ will contain at most $\lfloor mt\rfloor$ of $\boldsymbol{y}_1,\cdots,\boldsymbol{y}_m$ where these values are a sample from an independent $N_p(\boldsymbol{\mu}, \Sigma)$ distribution.

Further combining Theorem 1 and (30) we obtain the following series expression for $F_{(m,r)}(\beta)$ where we have put $m_0=\lfloor m\beta\rfloor$; namely

$$(31)\qquad 1-\sum_{i=(m_0+1)p}^{\infty}\ \sum_{\{tp+j=i,m_0+1\le t\le m\}}\ \sum_{k=m_0+1}\binom{m}{k}\binom{m-k}{t-k}\left(\frac{n}{t+n}\right)^{p/2}$$
$$\times(-1)^{t-k+i}\left(1-\frac{1}{t+n}\right)^{i}\pi^{-tp/2}\sum_{j_1+\cdots+j_p=i}k_{j_1\cdots j_p}(t)\lambda_{j_1\cdots j_p}(t)\left(\frac{r}{2}\right)^{i/2}.$$

This series is used for calculations. By the monotonicity results obtained above once we have obtained a value of r for which the series gives a sum less than $1-\gamma$ the already calculated coefficients of the appropriately truncated series can be used in the iterative calculations to obtain the value of r solving $F_{(m,r)}(\beta)=1-\gamma$.

For example when $p=2$, $n=122$, $\beta=.9$, $\gamma=.1$ and where we record $r_{m^*}=((n-2)/2)(n/(n+1))r_m$ we obtain the following sequence for (m, r_{m^*}): (1, .106), (2, .380), (3, .651), (4, 795), (5, .864), (6, .897), (7, .912). These values were obtained by directly programming the expression in (31). Extensive tabulation of approximate values of r still remains a considerable programming problem but the above results indicate that it can be accomplished by our approach.

6. Conclusion

We have obtained expressions for the moments of coverage of the random ellipsoid $T_r(\bar{x}, S(X))$. These expressions lead to an approximation to the distribution function of the coverage $C_r(\bar{x}, S(X))$. This approximation leads to an approximation to the unique value of r which makes $T_r(\bar{x}, S(X))$ a β-content at confidence γ tolerance region. These approximations apply for any sample size.

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