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A ROBUSTNESS STUDY IN SAMPLING ON SUCCESSIVE OCCASIONS

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Summary

In this paper, we have undertaken an investigation covering three occasions in sampling on successive occasions with a view to examining efficiency robustness of the best linear unbiased estimator (BLUE) vis-a-vis certain other potentially conceivable estimators when the usual correlation model breaks down. We have inferred that the BLUE is, by and large, an efficiency robust estimate in the face of unforeseen deviations from the usual correlation model.

1. Introduction

In a population where the characters change with the passage of time, it pays dividends, if the population is repeatedly sampled, to exploit judiciously the information collected on all occasions for optimizing the estimation of the current population mean. If the population is sampled on more than two occasions, the estimation theory providing the best linear unbiased estimator (BLUE) for the current population mean is based on the assumption of a multiplicative correlation structure modelled as either of the following two:

$$\rho_{ij}=\rho^{|i-j|} \qquad \text{for all } i \text{ and } j \tag{1.1}$$

$$\rho_{ij}=\prod_{k=i}^{j-1}\rho_{k,k+1} \qquad \text{for all } i \text{ and } j\ (i<j) \tag{1.2}$$

where ρ_{ij} is the correlation coefficient between observations for the same units on ith and jth occasions. (1.2) simplifies to (1.1) when $\rho_{k,k+1}=\rho$. Since, practical experience at times reflects something far from what is embodied in a multiplicative correlation model, we need to reassess BLUE vis-a-vis other potentially conceivable estimators from the stand-point of a breakdown in the assumed model. A brief study,

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in this direction, relating to BLUE in multi-stage designs has been made by Singh and Srivastava [7]. Rao and Graham [6] have presented an empirical study of the composite estimators in rotation sampling by considering, besides (1.1), an arithmetic correlation pattern.

In this paper, an investigation covering three occasions has been undertaken with a view to evaluating the impact of a breakdown in multiplicative correlation model on the BLUE and three other known estimators to be discussed later. In other words, we intend to examine the BLUE from the standpoint of robustness, and are interested in monitoring its performance compared to three other estimators in the event of a model breakdown. To achieve this, we have conducted a three-fold study in Section 3.

In the context of our choice of three occasions for the present study, we would like to refer to a finding of Kulldorff [3] that points to the fact that, in respect of the BLUE, the loss of efficiency (after showing consideration to cost) incurred by using the information from the latest two or three occasions, instead of the whole sequence of occasions, is fairly small. Hence, the results for three occasions presented in this paper have a natural scope for application, if need be, to the latest three of a sequence of occasions.

Consider a population Ω of size N. Let y denote the characteristic under study, and y_{ij} be its value for the jth unit on occasion i ($j=1, 2, \cdots, N$; $i=1, 2, 3$). Let n_i denote the size of the sample T_i on the ith occasion, which consists of two simple random samples drawn on each occasion $i \geqq 2$, one being T_i' of n_i' 'matched' units, i.e., $T_i'=T_{i-1} \cap T_i$, and the other being T_i'' of n_i'' 'unmatched' units i.e., $T_i''=T_i-T_i'$. Having first selected a random sample T_1 from Ω, we deem it necessary to point out that the following selection schemes are available on the ith occasion ($i=2, 3$):

I. (i) Select a sample T_2' of n_2' units from T_1,
(ii) Select a sample T_3' of n_3' units from T_2,
(iii) Select a sample T_i'' ($i=2, 3$) of n_i'' units from $\Omega - \bigcup_{g=1}^{i-1} T_g$.

II. In this scheme, the steps (i) and (iii) are the same as in scheme I, the step (ii) being replaced by
(ii) (a) Select a sample of n_3' units from T_2'' if $n_3' \leqq n_2''$; otherwise,
(b) Select a sample of n_3' units, i.e., $T_2'' \cup$ (simple random sample from T_2'), if $n_3' > n_2''$.

III. This sampling scheme also differs from the schemes I and II in so far as the step (ii) of these schemes is concerned, i.e., we have
(ii) (a) Select a sample of n_3' units from T_2', if $n_3' \leqq n_2'$; other-

wise,

(b) Select a sample of n_3' units, i.e., $T_2' \cup$ (simple random sample from T_2''), if $n_3' > n_2'$.

We will examine the variance of the BLUE under these selection schemes with a view to selecting one of them for our robustness study.

The sample means based on n_i, n_i' and n_i'' units will be denoted by $\bar{y}_i$, $\bar{y}_i'$ and $\bar{y}_i''$, respectively. The arithmetic mean of the observations on occasion $i-1$ corresponding to the units in T_i' will be denoted by $\bar{x}_{i-1}'$.

2. The four estimators and their variances

For estimating the population mean μ_3 given by

$$\mu_3 = \frac{1}{N} \sum_{j=1}^{N} y_{3j} ,$$

we would consider four estimators, and find their variances to compare their performance when the usual correlation model breaks down. For the sake of convenience, we will assume that the population variances are equal from occasion to occasion, and the same will be denoted by $S^2\left(\text{i.e., } S^2 = \frac{1}{N-1} \sum_{j=1}^{N} (y_{ij} - \mu_i)^2,\ \mu_i = \frac{1}{N} \sum_{j=1}^{N} y_{ij},\ i=1, 2, 3\right)$.

2.1. BLUE *and its variance*

If we assume (1.2) for three occasions, i.e.,

$$\rho_{13} = \rho_{12}\rho_{23} , \tag{2.1.1}$$

then, for any of the sampling designs described in Section 1, the best linear unbiased estimator (BLUE) of μ_3 is given by

$$\hat{\mu}_{3b} = r_3[\bar{y}_3' + \rho_{23}\{r_2(\bar{y}_2' + \rho_{12}(\bar{y}_1 - \bar{x}_1')) + (1-r_2)\bar{y}_2'' - \bar{x}_2'\}] + (1-r_3)\bar{y}_3'' , \tag{2.1.2}$$

where

$$r_i = 1 - n_i''/Q_i , \qquad (i=2, 3)$$

$$Q_i = \left(\frac{\rho_{i-1,i}^2}{Q_{i-1}} + \frac{1-\rho_{i-1,i}^2}{n_i'}\right)^{-1} + n_i'' , \qquad (i=2, 3)$$

and

$$Q_1 = n_1 .$$

The reader is referred, for a detailed discussion of the BLUE, to Patterson [4] and Kulldorff [3]. The variance of $\hat{\mu}_{3b}$ is known to be

$$V_0(\hat{\mu}_{3b})=\left(\frac{1}{Q_3}-\frac{1}{N}\right)S^2 . \tag{2.1.3}$$

As remarked earlier, the multiplicative correlation model is not borne out by practical experience and hence let us consider a model

$$\rho_{13}=(1+\delta)\rho_{12}\rho_{23} \tag{2.1.4}$$

where δ represents deviations in the model (2.1.1) and its range, in order that the associated correlation matrix be positive definite, is given by

$$|\delta|\leqq\sqrt{(1-\rho_{12}^2)(1-\rho_{23}^2)}/\rho_{12}\rho_{23} . \tag{2.1.5}$$

For each of the sampling designs described in Section 1, we will now obtain the variance of $\hat{\mu}_{3b}$ under (2.1.4). For this purpose, we need to evaluate certain variance and covariance terms. If C is the number of common units between T_2' and T_3' (and consequently between T_1 and T_3'), the conditional covariance between $\bar{y}_2'$ and $\bar{y}_3'$ would be obtained as

$$\mathrm{Cov}\,(\bar{y}_2', \bar{y}_3')=\left(\frac{C}{n_2'n_3'}-\frac{1}{N}\right)\rho_{23}S^2 .$$

By using the conditional expectations, the variance of $\hat{\mu}_{3b}$ under (2.1.4), whatever be the selection scheme out of the three described in Section 1, turns out, after some manipulation, as

$$V(\hat{\mu}_{3b})=\left(\frac{1}{Q_3}-\frac{1}{N}\right)S^2-2r_3^2r_2\rho_{12}^2\rho_{23}^2\delta\,\mathrm{E}\,(C)\frac{(n_1-n_2')}{n_2'n_3'}\,\frac{S^2}{n_1} . \tag{2.1.6}$$

For the sake of simplicity, we will, in future, assume that the total sample size is, say, n on each occasion. Evaluating $\mathrm{E}\,(C)$ under each of the three sampling schemes, we get the variance of $\hat{\mu}_{3b}$ for the corresponding scheme. It is worthwhile to point out that the scheme II or the scheme III has an advantage over the other two according as $\delta<0$ or $\delta>0$. However, it can be noted without much effort that, if we entertain both positive and negative deviations as symbolised by δ, all the three selection schemes spelt out in Section 1 are admissible in terms of the language of decision theory. Furthermore, each of these schemes will be a Bayes' rule if we assume a symmetric distribution (prior) for δ subject to (2.1.5). For our study we have listed out both positive and negative δ's, and we confine ourselves to the scheme I in what follows. Setting

$$\frac{n_2'}{n}=m_1 , \qquad \frac{n_3'}{n}=m_2 ,$$

and employing the scheme I, we can write from (2.1.6) the variance of $\hat{\mu}_{3b}$ under (2.1.4) as

$$V(\hat{\mu}_{3b})=\left[\frac{1-r_3}{1-m_2}-2r_3^2r_2\rho_{12}^2\rho_{23}^2(1-m_1)\delta\right]\frac{S^2}{n}-\frac{S^2}{N}, \tag{2.1.7}$$

where r_2 and r_3 can now be expressed as

$$r_2=\frac{m_1}{1-\rho_{12}^2(1-m_1)^2},$$

$$r_3=\frac{m_2\{1-\rho_{12}^2(1-m_1)^2\}}{\{1-\rho_{12}^2(1-m_1)^2\}\{1-\rho_{23}^2(1-m_2)^2\}-m_1m_2(1-m_1)(1-m_2)\rho_{12}^2\rho_{23}^2}.$$

It is clear from (2.1.7) that $V(\hat{\mu}_{3b})$ is decreasing with increase in δ. For $\delta=0$, the variances given by (2.1.3) and (2.1.7) are the same, while for $\delta>0$, the latter yields a lower value than the former.

An empirical study examining the effect of model breakdown on the BLUE will be presented in Section 3.

2.2. *Three other estimators and their variances*

A variant of the BLUE, proposed by Cochran ([1], p. 352) as a simplified version of the BLUE, is defined by

$$\hat{\mu}_{3v}=Q[\bar{y}_3'+\rho_{23}\{Q(\bar{y}_2'+\rho_{12}(\bar{y}_1-\bar{x}_1'))+(1-Q)\bar{y}_2''-\bar{x}_2'\}]+(1-Q)\bar{y}_3'', \tag{2.2.1}$$

where Q is a constant weight, $0\leqq Q\leqq 1$.

We find the variance of $\hat{\mu}_{3v}$ under (2.1.4), which, after some simplification, can be expressed as

$$\begin{aligned}V(\hat{\mu}_{3v})=\Bigg[&\frac{Q^4\rho_{23}^2}{m_1(1-m_1)}\{1-\rho_{12}^2(1-m_1)^2\}-\frac{2Q^3\rho_{23}^2}{(1-m_1)}\\ &\cdot\{1+\rho_{12}^2\delta(1-m_1)^2\}+\frac{Q^2}{m_2}\left\{\frac{1}{1-m_2}+\frac{(m_1+m_2-1)\rho_{23}^2}{1-m_1}\right\}\\ &-\frac{2Q}{1-m_2}+\frac{1}{1-m_2}\Bigg]\frac{S^2}{n}-\frac{S^2}{N},\end{aligned} \tag{2.2.2}$$

where m_1 and m_2 are matching proportions defined in Section 2.1. An increase in δ, for specified values of other parameters, leads to a decrease in the above variance.

For three occasions, the composite estimator due to Hansen et al. ([2], p. 272) could be expressed as

$$\hat{\mu}_{3c}=K[K(\bar{y}_1+\bar{y}_2'-\bar{x}_1')+(1-K)\bar{y}_2+\bar{y}_3'-\bar{x}_2']+(1+K)\bar{y}_3, \tag{2.2.3}$$

where K is a constant weight, $0\leqq K\leqq 1$. After some algebra, the variance of $\hat{\mu}_{3c}$ can be obtained as

$$(2.2.4)\qquad V(\hat{\mu}_{3c})=\Big[2K^4\frac{(1-m_1)}{m_1}\{1-(1-m_1)\rho_{12}\}-2K^3(1-m_1)(1-m_2)\rho_{13}$$
$$+2K^2\Big\{\frac{(1-m_2)}{m_2}(1-(1-m_2)\rho_{23})-(1-m_1)m_2\rho_{13}\Big\}$$
$$-2K(1-m_2)\rho_{23}+1\Big]\frac{S^2}{n}-\frac{S^2}{N},$$

where m_1 and m_2 are matching proportions defined in Section 2.1.

The third estimator (vide Raj [5], p. 276) employing the technique of difference estimation could be termed as difference estimator. For three occasions, it can be written as

$$(2.2.5)\qquad \hat{\mu}_{3d}=W[\bar{y}_3'+W(\bar{y}_2'+(\bar{y}_1-\bar{x}_1'))+(1-W)\bar{y}_2''-\bar{x}_2']+(1-W)\bar{y}_3'',$$

where W is a constant weight, $0\leqq W\leqq 1$. This estimator coincides with $\hat{\mu}_{3b}$ if $\rho_{12}=\rho_{23}=1$. After some simplification, the variance of $\hat{\mu}_{3d}$, taking n as the sample size on each occasion, is found to be

$$(2.2.6)\qquad V(\hat{\mu}_{3d})=\Big[W^4\Big\{\frac{2(1-m_1)}{m_1}(1-\rho_{12})+1+\frac{1}{1-m_1}\Big\}+2W^3\Big\{(1-m_1)$$
$$\cdot(\rho_{12}-\rho_{13})-\frac{1}{1-m_1}\Big\}+W^2\Big\{2\frac{(1-m_2)}{m_2}(1-\rho_{23})$$
$$+\frac{1}{1-m_1}+\frac{1}{1-m_2}\Big\}-\frac{2W}{1-m_2}+\frac{1}{1-m_2}\Big]\frac{S^2}{n}-\frac{S^2}{N},$$

where m_1 and m_2, as before, are the matching fractions.

We will investigate in the next section as to how the estimates $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$, besides $\hat{\mu}_{3b}$, are affected if, instead of the model given in (2.1.1), the one given in (2.1.4) operates. It is, however, evident that, under (2.1.4), the variances of $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$ would decrease with increase in δ, assuming that $\operatorname{sgn}\rho_{12}=\operatorname{sgn}\rho_{23}$.

3. A comparison of the four estimators

Using a CDC Cyber 730, we have computed the percent gains in precision for the estimators $\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$ relative to the simple arithmetic mean $\bar{y}_3$. In computing the gains via (2.1.7), (2.2.2), (2.2.4) and (2.2.6), we have ignored the term $-S^2/N$. Now, if we denote the coefficients of S^2/n in (2.1.7), (2.2.2), (2.2.4) and (2.2.6) by B, V, C and D, respectively, then their reciprocals (taken in the respective order) would mean the relative efficiencies of the BLUE, the variant of BLUE, the composite estimator and the difference estimator with respect to the simple arithmetic mean $\bar{y}_3$. We will denote the percent gains in the three cases (in the same order as described in the preceding sentence) by

$$G_b=100\left(\frac{1}{B}-1\right), \quad G_v=100\left(\frac{1}{V}-1\right), \quad G_c=100\left(\frac{1}{C}-1\right)$$

and

$$G_d=100\left(\frac{1}{D}-1\right).$$

In the computations of G_b, G_v, G_c and G_d, we have considered: (i′) $m_1=m_2=m$ (say), (ii′) the common matching fraction m is optimally chosen, (iii′) m_1 and m_2 are optimally chosen. In the three cases (i′), (ii′) and (iii′), the weights Q, K and W are optimally determined, in the sense that they have the values that minimise the variances $V(\hat{\mu}_{3v})$, $V(\hat{\mu}_{3c})$ and $V(\hat{\mu}_{3d})$ under (2.1.1), respectively. In conducting the empirical investigation, we have chosen, for the different combinations of ρ_{12} and ρ_{23}, those values of δ out of the set $\delta=-0.4, -0.2, -0.1, 0, 0.1, 0.2, 0.4$, that satisfy (2.1.5). For ρ_{12} and ρ_{23}, we have selected moderately large values (i.e., $\rho_{12}, \rho_{23}=0.6, 0.7, 0.8$) and large values (i.e., $\rho_{12}, \rho_{23}=0.9, 0.95$). For the case (i′) we have chosen $m=0.4, 0.6$.

To evaluate the performance of $\hat{\mu}_{3b}$ vis-a-vis the other three competing estimators, we intend to examine their efficiency robustness to model breakdown, meaning thereby that we are interested in finding the effect of model deviations on the efficiency of $\hat{\mu}_{3b}$ vis-a-vis the other three estimators with a view to determining whether the former retains its superiority or not for different δ's.

3.1. *Equal matching fractions* ($m_1=m_2$)

It is convenient in practical application to keep the proportions matched constant instead of changing them on every occasion, (see Cochran [1]). Accordingly we take $m_1=m_2=m$ (say), and choose different values for the common matching fraction.

It should be made clear that the weights Q, K and W have been optimised under the model (2.1.1) for each combination of m, ρ_{12} and ρ_{23}, and then, using these weights, the percent gains G_b, G_v, G_c and G_d have been computed for different δ's (i.e., when the model (2.1.4), instead of the model (2.1.1), operates).

The following features—illustrated by Table 1—emerged from our numerical investigation.

(1) If (i) ρ_{23} is higher than ρ_{12} or (ii) ρ_{12} and ρ_{23} are large, then $\hat{\mu}_{3b}$, from a robustness viewpoint, maintains its bestness among the estimators being considered in the event of a model breakdown.

(2) If (i) ρ_{12} is larger than ρ_{23} or (ii) ρ_{12} and ρ_{23} are moderately large and equal, then, viewed in terms of robustness against deviations

from the model (2.1.1), $\hat{\mu}_{3b}$ performs better than $\hat{\mu}_{3v}$, let alone $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$, except possibly when $\delta \leqq -0.1$ in the case (i), or when $\delta \geqq 0.1$ in the case (ii). However, we have noted from this investigation that, under the most unfavourable configurations of m, ρ_{12}, ρ_{23} and δ, the maximum loss in efficiency of $\hat{\mu}_{3b}$, relative to its closest rival $\hat{\mu}_{3v}$ is approximately 0.75%.

(3) If we look at each of the estimators ($\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$) relative to the simple arithmetic mean $\bar{y}_3$, then each of them except $\hat{\mu}_{3d}$ appears to be fairly robust (as measured by the gain in efficiency) to a model breakdown. In the case of $\hat{\mu}_{3d}$, the preceding comments would apply when $\rho_{12} \geqq 0.7$ and $\rho_{23} \geqq 0.8$.

3.2. *Optimum matching fractions*

In this section we present Tables 2 and 3 which have been extracted from a larger numerical investigation. The distinguishing feature of these tables compared with Table 1 is the determination of (a) optimum m when $m_1 = m_2 = m$ (b) optimum m_1 and m_2, assuming that optimally matched portions of the sample are available. The optimum m_1 and m_2 are those four pairs of values corresponding to the four estimators, that minimize the four variances, viz., $V(\hat{\mu}_{3b})$, $V(\hat{\mu}_{3v})$, $V(\hat{\mu}_{3c})$ and $V(\hat{\mu}_{3d})$ under the model (2.1.1). We will denote optimum percent matchings by (m_{1b}^*, m_{2b}^*) for $\hat{\mu}_{3b}$, (m_{1v}^*, m_{2v}^*) for $\hat{\mu}_{3v}$, (m_{1c}^*, m_{2c}^*) for $\hat{\mu}_{3c}$ and (m_{1d}^*, m_{2d}^*) for $\hat{\mu}_{3d}$. When $m_1 = m_2 = m$, these four pairs would reduce to four singlets, viz., m_b^* for $\hat{\mu}_{3b}$, m_v^* for $\hat{\mu}_{3v}$, m_c^* for $\hat{\mu}_{3c}$ and m_d^* for $\hat{\mu}_{3d}$. The weights Q, K and W are determined optimally as before. Finally, the gains G_b, G_v, G_c and G_d for the cases (a) and (b) are, respectively, computed in Tables 2 and 3.

Case (a): Consequent upon optimisation of the common matching fraction m, certain reflections (illustrated by Table 2) are noted below:

(1) If (i) $\rho_{23} > \rho_{12}$ or (ii) ρ_{12} and ρ_{23} are large or (iii) $\rho_{12} \geqq 0.95$ and $\rho_{23} \geqq 0.6$, then $\hat{\mu}_{3b}$ maintains its bestness, under a model breakdown, in the bouquet of competing estimators considered here.

(2) For combinations of ρ_{12} and ρ_{23} other than those noted in (1) above, the estimator $\hat{\mu}_{3b}$, broadly speaking, tends to sustain, under a model breakdown, its better performance relative to any of the other three estimators.

(3) Each of the estimators ($\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$) relative to the simple arithmetic mean $\bar{y}_3$ appears to be quite robust (as measured by the gain in efficiency) to model deviations.

Case (b): As a result of optimisation of both m_1 and m_2, the following features (borne out by Table 3) emerge:

(1) For large ρ_{12} and ρ_{23} (i.e., $\rho_{12}, \rho_{23} \geqq 0.9$), the estimator $\hat{\mu}_{3b}$ maintains its superiority (as measured by its efficiency) vis-a-vis the other

Table 1. Percent gains in precision for $\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$

m	ρ_{12}	ρ_{23}	δ	G_b	Q	G_v	K	G_c	W	G_d
.4	.60	.70	−.4	16.96	.4888	16.84	.2398	11.24	.4508	9.07
.4	.60	.70	−.2	17.61	.4888	17.52	.2398	11.48	.4508	10.18
.4	.60	.70	−.1	17.93	.4888	17.86	.2398	11.61	.4508	10.74
.4	.60	.70	0.0	18.26	.4888	18.21	.2398	11.73	.4508	11.31
.4	.60	.70	.1	18.59	.4888	18.55	.2398	11.86	.4508	11.88
.4	.60	.70	.2	18.93	.4888	18.90	.2398	11.98	.4508	12.46
.4	.60	.70	.4	19.60	.4888	19.60	.2398	12.23	.4508	13.64
.4	.70	.60	−.4	11.28	.4675	11.30	.1934	7.36	.4260	3.02
.4	.70	.60	−.2	11.83	.4675	11.84	.1934	7.50	.4260	3.85
.4	.70	.60	−.1	12.11	.4675	12.11	.1934	7.57	.4260	4.28
.4	.70	.60	0.0	12.39	.4675	12.38	.1934	7.64	.4260	4.78
.4	.70	.60	.1	12.68	.4675	12.66	.1934	7.71	.4260	5.13
.4	.70	.60	.2	12.96	.4675	12.93	.1934	7.78	.4260	5.56
.4	.70	.60	.4	13.53	.4675	13.49	.1934	7.91	.4260	6.44
.4	.70	.70	−.4	17.08	.4947	17.05	.2466	11.40	.4607	9.66
.4	.70	.70	−.2	18.03	.4947	18.01	.2466	11.71	.4607	11.07
.4	.70	.70	−.1	18.51	.4947	18.50	.2466	11.87	.4607	11.78
.4	.70	.70	0.0	19.00	.4947	18.99	.2466	12.02	.4607	12.50
.4	.70	.70	.1	19.48	.4947	19.49	.2466	12.18	.4607	13.23
.4	.70	.70	.2	19.98	.4947	19.99	.2466	12.34	.4607	13.98
.4	.70	.70	.4	20.98	.4947	21.00	.2466	12.65	.4607	15.49
.4	.80	.70	−.4	17.23	.5020	17.28	.2540	11.58	.4718	10.30
.4	.80	.70	−.2	18.59	.5020	18.61	.2540	11.96	.4718	12.05
.4	.80	.70	−.1	19.28	.5020	19.28	.2540	12.15	.4718	12.94
.4	.80	.70	0.0	19.98	.5020	19.96	.2540	12.35	.4718	13.85
.4	.80	.70	.1	20.69	.5020	20.65	.2540	12.54	.4718	14.77
.4	.80	.70	.2	21.40	.5020	21.35	.2540	12.73	.4718	15.71
.4	.80	.70	.4	22.86	.5020	22.76	.2540	13.12	.4718	17.63
.4	.80	.80	−.4	25.88	.5377	25.74	.3192	17.60	.5146	19.65
.4	.80	.80	−.2	28.30	.5377	28.21	.3192	18.46	.5146	22.72
.4	.80	.80	−.1	29.54	.5377	29.47	.3192	18.89	.5146	24.32
.4	.80	.80	0.0	30.81	.5377	30.77	.3192	19.32	.5146	25.96
.4	.80	.80	.1	32.10	.5377	32.09	.3192	19.76	.5146	27.64
.4	.80	.80	.2	33.42	.5377	33.44	.3192	20.20	.5146	29.37
.4	.80	.80	.4	36.14	.5377	36.21	.3192	21.09	.5146	32.97
.4	.90	.90	−.2	46.61	.5992	46.16	.4157	29.81	.5872	42.62
.4	.90	.90	−.1	50.26	.5992	49.87	.4157	31.10	.5872	46.74
.4	.90	.90	0.0	54.10	.5992	53.77	.4157	32.41	.5872	51.10
.4	.90	.90	.1	58.14	.5992	57.88	.4157	33.75	.5872	55.73
.4	.90	.90	.2	62.39	.5992	62.22	.4157	35.11	.5872	60.66
.4	.90	.95	−.1	65.36	.6236	63.82	.4589	39.42	.6161	61.36
.4	.90	.95	0.0	71.29	.6236	69.73	.4589	41.44	.6161	67.86
.4	.90	.95	.1	77.65	.6236	76.09	.4589	43.53	.6161	74.90
.4	.95	.90	−.1	52.75	.6126	52.59	.4302	32.16	.6021	49.99
.4	.95	.90	0.0	57.55	.6126	57.44	.4302	33.69	.6021	55.20
.4	.95	.90	.1	62.66	.6126	62.60	.4302	35.27	.6021	60.79
.4	.95	.95	−.1	69.17	.6397	68.12	.4769	41.11	.6335	66.35
.4	.95	.95	0.0	76.73	.6397	75.67	.4769	43.53	.6335	74.33
.4	.95	.95	.1	84.99	.6397	83.94	.4769	46.04	.6335	83.12
.6	.60	.70	−.4	15.11	.6547	15.03	.2887	8.89	.6252	5.59
.6	.60	.70	−.2	16.15	.6547	16.09	.2887	9.16	.6252	7.45
.6	.60	.70	−.1	16.68	.6547	16.63	.2887	9.30	.6252	8.40
.6	.60	.70	0.0	17.22	.6547	17.17	.2887	9.44	.6252	9.38
.6	.60	.70	.1	17.75	.6547	17.72	.2887	9.58	.6252	10.37
.6	.60	.70	.2	18.30	.6547	18.27	.2887	9.72	.6252	11.38
.6	.60	.70	.4	19.40	.6547	19.39	.2887	10.00	.6252	13.45
.6	.70	.60	−.4	10.53	.6451	10.54	.2453	6.17	.6136	.42
.6	.70	.60	−.2	11.47	.6451	11.47	.2453	6.35	.6136	2.01
.6	.70	.60	−.1	11.95	.6451	11.95	.2453	6.44	.6136	2.82
.6	.70	.60	0.0	12.43	.6451	12.42	.2453	6.54	.6136	3.65

Table 1. (Continued)

m	ρ_{12}	ρ_{23}	δ	G_b	Q	G_v	K	G_c	W	G_d
.6	.70	.60	.1	12.91	.6451	12.90	.2453	6.63	.6136	4.49
.6	.70	.60	.2	13.40	.6451	13.39	.2453	6.72	.6136	5.34
.6	.70	.60	.4	14.39	.6451	14.37	.2453	6.90	.6136	7.09
.6	.70	.70	−.4	15.29	.6599	15.26	.2972	9.03	.6344	6.46
.6	.70	.70	−.2	16.77	.6599	16.74	.2972	9.38	.6344	8.78
.6	.70	.70	−.1	17.52	.6599	17.50	.2972	9.55	.6344	9.97
.6	.70	.70	0.0	18.28	.6599	18.27	.2972	9.73	.6344	11.20
.6	.70	.70	.1	19.05	.6599	19.05	.2972	9.90	.6344	12.45
.6	.70	.70	.2	19.83	.6599	19.83	.2972	10.08	.6344	13.73
.6	.70	.70	.4	21.43	.6599	21.44	.2972	10.43	.6344	16.38
.6	.80	.70	−.4	15.50	.6661	15.51	.3065	9.19	.6443	7.39
.6	.80	.70	−.2	17.52	.6661	17.52	.3065	9.61	.6443	10.22
.6	.80	.70	−.1	18.55	.6661	18.56	.3065	9.83	.6443	11.70
.6	.80	.70	0.0	19.61	.6661	19.61	.3065	10.04	.6443	13.21
.6	.80	.70	.1	20.68	.6661	20.68	.3065	10.26	.6443	14.77
.6	.80	.70	.2	21.77	.6661	21.77	.3065	10.47	.6443	16.37
.6	.80	.70	.4	24.02	.6661	24.01	.3065	10.91	.6443	19.71
.6	.80	.80	−.4	21.95	.6838	21.84	.3633	13.01	.6675	15.10
.6	.80	.80	−.2	25.12	.6838	25.04	.3633	13.78	.6675	19.28
.6	.80	.80	−.1	26.77	.6838	26.70	.3633	14.17	.6675	21.49
.6	.80	.80	0.0	28.46	.6838	28.40	.3633	14.57	.6675	23.78
.6	.80	.80	.1	30.20	.6838	30.15	.3633	14.96	.6675	26.16
.6	.80	.80	.2	31.98	.6838	31.95	.3633	15.36	.6675	28.63
.6	.80	.80	.4	35.70	.6838	35.70	.3633	16.17	.6675	33.88
.6	.90	.90	−.2	38.07	.7143	37.78	.4400	20.10	.7066	34.61
.6	.90	.90	−.1	41.80	.7143	41.51	.4400	20.97	.7066	38.89
.6	.90	.90	0.0	45.74	.7143	45.45	.4400	21.85	.7066	43.44
.6	.90	.90	.1	49.90	.7143	49.61	.4400	22.74	.7066	48.30
.6	.90	.90	.2	54.30	.7143	54.02	.4400	23.65	.7066	53.51
.6	.90	.95	−.1	50.59	.7245	49.85	.4710	24.89	.7200	47.99
.6	.90	.95	0.0	55.82	.7245	55.02	.4710	26.05	.7200	53.81
.6	.90	.95	.1	61.42	.7245	60.55	.4710	27.24	.7200	60.09
.6	.95	.90	−.1	43.83	.7215	43.63	.4506	21.52	.7147	41.53
.6	.95	.90	0.0	48.51	.7215	48.31	.4506	22.52	.7147	46.72
.6	.95	.90	.1	53.50	.7215	53.30	.4506	23.53	.7147	52.29
.6	.95	.95	−.1	53.27	.7325	52.67	.4831	25.63	.7287	51.40
.6	.95	.95	0.0	59.54	.7325	58.88	.4831	26.95	.7287	58.08
.6	.95	.95	.1	66.35	.7325	65.61	.4831	28.31	.7287	65.39

three estimators in the event of a model breakdown.

(2) For combinations of ρ_{12} and ρ_{23} other than those implied in (1) above, the estimator $\hat{\mu}_{3b}$, even in the face of deviations from the model (2.1.1), sustains its better performance relative to any of the other three estimators for $\delta \geqq -0.2$. However, for $\delta = -0.4$, the estimator $\hat{\mu}_{3b}$, under the most unfavourable configurations of ρ_{12} and ρ_{23} (e.g. when $\rho_{12} \gg \rho_{23}$) suffers a maximum loss in efficiency of 5.5% relative to $\hat{\mu}_{3v}$, while it gains, over the remaining values of δ, about 10% in efficiency relative to $\hat{\mu}_{3v}$.

(3) Each of the estimators ($\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$) relative to the simple arithmetic mean $\bar{y}_3$ appears to be fairly robust (as measured by the gain in efficiency) to model deviations.

Thus, viewed in totality against the background of the observations made in this section, we may infer that the estimator $\hat{\mu}_{3b}$, which

Table 2. Optimum percent matched and percent gains in precision for $\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$

ρ_{12}	ρ_{23}	δ	m_b^*	G_b	m_v^*	Q	G_v	m_c^*	K	G_c	m_d^*	W	G_d
.60	.60	−.4	48.83	11.28	48.82	.5439	11.27	42.38	.2058	7.47	33.67	.3596	3.22
.60	.60	−.2	48.83	11.80	48.82	.5439	11.80	42.38	.2058	7.69	33.67	.3596	3.70
.60	.60	−.1	48.83	12.07	48.82	.5439	12.07	42.38	.2058	7.80	33.67	.3596	3.93
.60	.60	0.0	48.83	12.34	48.82	.5439	12.33	42.38	.2058	7.91	33.67	.3596	4.18
.60	.60	.1	48.83	12.60	48.82	.5439	12.60	42.38	.2058	8.02	33.67	.3596	4.42
.60	.60	.2	48.83	12.87	48.82	.5439	12.88	42.38	.2058	8.13	33.67	.3596	4.66
.60	.60	.4	48.83	13.42	48.82	.5439	13.42	42.38	.2058	8.35	33.67	.3596	5.15
.70	.80	−.4	43.43	25.42	43.49	.5537	25.08	36.02	.3116	18.41	41.58	.5117	18.27
.70	.80	−.2	43.43	27.27	43.49	.5537	26.99	36.02	.3116	19.51	41.58	.5117	20.78
.70	.80	−.1	43.43	28.21	43.49	.5537	27.97	36.02	.3116	20.07	41.58	.5117	22.07
.70	.80	0.0	43.43	29.16	43.49	.5537	28.96	36.02	.3116	20.63	41.58	.5117	23.39
.70	.80	.1	43.43	30.13	43.49	.5537	29.97	36.02	.3116	21.20	41.58	.5117	24.74
.70	.80	.2	43.43	31.12	43.49	.5537	30.99	36.02	.3116	21.78	41.58	.5117	26.12
.70	.80	.4	43.43	33.13	43.49	.5537	33.09	36.02	.3116	22.94	41.58	.5117	28.97
.80	.80	−.4	44.79	25.54	44.72	.5742	25.40	37.16	.3340	18.84	44.14	.5475	19.08
.80	.80	−.2	44.79	28.25	44.72	.5742	28.16	37.16	.3340	20.34	44.14	.5475	22.50
.80	.80	−.1	44.79	29.65	44.72	.5742	29.58	37.16	.3340	21.10	44.14	.5475	24.29
.80	.80	0.0	44.79	31.09	44.72	.5742	31.04	37.16	.3340	21.87	44.14	.5475	26.13
.80	.80	.1	44.79	32.55	44.72	.5742	32.53	37.16	.3340	22.65	44.14	.5475	28.02
.80	.80	.2	44.79	34.05	44.72	.5742	34.05	37.16	.3340	23.44	44.14	.5475	29.98
.80	.80	.4	44.79	37.15	44.72	.5742	37.20	37.16	.3340	25.05	44.14	.5475	31.07
.90	.60	−.4	52.50	11.49	53.20	.5970	11.49	45.64	.2395	7.76	51.45	.5519	2.69
.90	.60	−.2	52.50	13.03	53.20	.5970	12.96	45.64	.2395	8.22	51.45	.5519	4.58
.90	.60	−.1	52.50	13.82	53.20	.5970	13.70	45.64	.2395	8.45	51.45	.5519	5.56
.90	.60	0.0	52.50	14.61	53.20	.5970	14.46	45.64	.2395	8.68	51.45	.5519	6.55
.90	.60	.1	52.50	15.42	53.20	.5970	15.22	45.64	.2395	8.92	51.45	.5519	7.56
.90	.60	.2	52.50	16.24	53.20	.5970	16.00	45.64	.2395	9.15	51.45	.5519	8.59
.90	.60	.4	52.50	17.91	53.20	.5970	17.58	45.64	.2395	9.63	51.45	.5519	10.71
.90	.80	−.2	45.76	29.61	45.82	.5960	29.62	38.31	.3609	21.29	46.09	.5801	24.61
.90	.80	−.1	45.76	31.64	45.82	.5960	31.65	38.31	.3609	22.33	46.09	.5801	27.01
.90	.80	0.0	45.76	33.74	45.82	.5960	33.74	38.31	.3609	23.39	46.09	.5801	29.50
.90	.80	.1	45.76	35.91	45.82	.5960	35.90	38.31	.3609	24.47	46.09	.5801	32.70
.90	.80	.2	45.76	38.16	45.82	.5960	38.14	38.31	.3609	25.56	46.09	.5801	34.79
.90	.90	−.2	38.74	46.78	38.56	.5902	46.35	31.14	.4262	36.12	38.83	.5797	42.80
.90	.90	−.1	38.74	50.37	38.56	.5902	49.99	31.14	.4262	38.43	38.83	.5797	46.85
.90	.90	0.0	38.74	54.13	38.56	.5902	53.81	31.14	.4262	40.82	38.83	.5797	51.13
.90	.90	.1	38.74	58.09	38.56	.5902	57.84	31.14	.4262	43.29	38.83	.5797	55.67
.90	.90	.2	38.74	62.26	38.56	.5902	62.08	31.14	.4262	45.85	38.83	.5797	60.50
.90	.95	−.1	32.30	67.50	32.51	.5825	65.70	25.48	.4591	53.53	32.76	.5756	63.15
.90	.95	0.0	32.30	72.95	32.51	.5825	71.23	25.48	.4591	57.41	32.76	.5756	69.21
.90	.95	.1	32.30	78.77	32.51	.5825	77.15	25.48	.4591	61.48	32.76	.5756	75.75
.95	.90	−.1	38.71	52.87	38.50	.6036	52.73	31.32	.4486	40.44	38.79	.5948	50.10
.95	.90	0.0	38.71	57.59	38.50	.6036	57.49	31.32	.4486	43.40	38.79	.5948	55.24
.95	.90	.1	38.71	62.61	38.50	.6036	62.55	31.32	.4486	46.48	38.79	.5948	60.73
.95	.95	−.1	31.99	71.58	31.72	.5981	70.53	25.09	.4874	57.62	31.93	.5923	68.66
.95	.95	0.0	31.99	78.74	31.72	.5981	77.74	25.09	.4874	62.70	31.93	.5923	76.24
.95	.95	.1	31.99	86.52	31.72	.5981	85.59	25.09	.4874	68.11	31.93	.5923	84.55

is the best linear unbiased estimator under (2.1.1), is an efficiency robust estimator, in the sense that it sustains, by and large, its bestness vis-a-vis the competing estimators considered in the face of unforeseen model deviations that are envisaged via the model (2.1.4) and are reflected through the values of δ displayed in Tables 1, 2 and 3.

Table 3. Optimum percent matched and percent gains in precision for $\hat{\mu}_{3b}$, $\hat{\mu}_{3v}$, $\hat{\mu}_{3c}$ and $\hat{\mu}_{3d}$

ρ_{12}	ρ_{23}	δ	m^*_{1b}	m^*_{2b}	G_b	m^*_{1v}	m^*_{2v}	Q	G_v	m^*_{1c}	m^*_{2c}	K	G_c	m^*_{1d}	m^*_{2d}	W	G_d
.60	.60	−.4	44.44	49.38	11.26	48.82	48.82	.5439	11.27	31.57	43.75	.2080	7.41	36.06	33.85	.3640	3.24
.60	.60	−.2	44.44	49.38	11.80	48.82	48.82	.5439	11.80	31.57	43.75	.2080	7.69	36.06	33.85	.3640	3.72
.60	.60	−.1	44.44	49.38	12.07	48.82	48.82	.5439	12.07	31.57	43.75	.2080	7.82	36.06	33.85	.3640	3.96
.60	.60	0.0	44.44	49.38	12.35	48.82	48.82	.5439	12.33	31.57	43.75	.2080	7.96	36.06	33.85	.3640	4.20
.60	.60	.1	44.44	49.38	12.62	48.82	48.82	.5439	12.60	31.57	43.75	.2080	8.10	36.06	33.85	.3640	4.44
.60	.60	.2	44.44	49.38	12.90	48.82	48.82	.5439	12.88	31.57	43.75	.2080	8.24	36.06	33.85	.3640	4.68
.60	.60	.4	44.44	49.38	13.45	48.82	48.82	.5439	13.42	31.57	43.75	.2080	8.51	36.06	33.85	.3640	5.17
.70	.80	−.4	41.66	43.75	25.39	47.10	42.80	.5549	25.43	34.37	36.34	.3116	18.35	48.12	40.80	.5215	19.05
.70	.80	−.2	41.66	43.75	27.25	47.10	42.80	.5549	27.24	34.37	36.34	.3116	19.48	48.12	40.80	.5215	21.43
.70	.80	−.1	41.66	43.75	28.20	47.10	42.80	.5549	28.16	34.37	36.34	.3116	20.06	48.12	40.80	.5215	22.66
.70	.80	0.0	41.66	43.75	29.17	47.10	42.80	.5549	29.10	34.37	36.34	.3116	20.64	48.12	40.80	.5215	23.91
.70	.80	.1	41.66	43.75	30.15	47.10	42.80	.5549	30.05	34.37	36.34	.3116	21.23	48.12	40.80	.5215	25.19
.70	.80	.2	41.66	43.75	31.14	47.10	42.80	.5549	31.02	34.37	36.34	.3116	21.82	48.12	40.80	.5215	26.49
.70	.80	.4	41.66	43.75	33.18	47.10	42.80	.5549	33.00	34.37	36.34	.3116	23.02	48.12	40.80	.5215	29.19
.80	.80	−.4	37.50	46.88	25.28	44.72	44.72	.5742	25.40	28.01	39.99	.3397	18.43	47.14	43.67	.5508	19.42
.80	.80	−.2	37.50	46.88	28.20	44.72	44.72	.5742	28.16	28.01	39.99	.3397	20.26	47.14	43.67	.5508	22.74
.80	.80	−.1	37.50	46.88	29.71	44.72	44.72	.5742	29.58	28.01	39.99	.3397	21.19	47.14	43.67	.5508	24.46
.80	.80	0.0	37.50	46.88	31.25	44.72	44.72	.5742	31.04	28.01	39.99	.3397	22.14	47.14	43.67	.5508	26.24
.80	.80	.1	37.50	46.88	32.83	44.72	44.72	.5742	32.53	28.01	39.99	.3397	23.11	47.14	43.67	.5508	28.07
.80	.80	.2	37.50	46.88	34.45	44.72	44.72	.5742	34.05	28.01	39.99	.3397	24.09	47.14	43.67	.5508	29.95
.80	.80	.4	37.50	46.88	37.81	44.72	44.72	.5742	37.20	28.01	39.99	.3397	26.10	47.14	43.67	.5508	33.88
.90	.80	−.2	30.36	52.23	29.56	39.34	47.17	.5981	29.45	21.07	47.63	.3950	21.32	43.62	46.44	.5782	24.50
.90	.80	−.1	30.36	52.23	32.14	39.34	47.17	.5981	31.74	21.07	47.63	.3950	23.13	43.62	46.44	.5782	26.99
.90	.80	0.0	30.36	52.23	34.82	39.34	47.17	.5981	34.12	21.07	47.63	.3950	24.99	43.62	46.44	.5782	29.57
.90	.80	.1	30.36	52.23	37.61	39.34	47.17	.5981	36.58	21.07	47.63	.3950	26.91	43.62	46.44	.5782	32.26
.90	.80	.2	30.36	52.23	40.53	39.34	47.17	.5981	39.14	21.07	47.63	.3950	28.89	43.62	46.44	.5782	35.06
.90	.90	−.2	30.36	42.28	46.57	38.56	38.56	.5902	46.35	23.44	34.81	.4334	35.81	41.03	38.26	.5808	43.11
.90	.90	−.1	30.36	42.28	50.54	38.56	38.56	.5902	49.99	23.44	34.81	.4334	38.57	41.03	38.26	.5808	47.05
.90	.90	0.0	30.36	42.28	54.72	38.56	38.56	.5902	53.81	23.44	34.81	.4334	41.45	41.03	38.26	.5808	51.21
.90	.90	.1	30.36	42.28	59.14	38.56	38.56	.5902	57.84	23.44	34.81	.4334	44.45	41.03	38.26	.5808	55.62
.90	.90	.2	30.36	42.28	63.83	38.56	38.56	.5902	62.08	23.44	34.81	.4334	47.59	41.03	38.26	.5808	60.29
.90	.95	−.1	30.36	33.14	67.45	37.95	30.46	.5839	66.80	25.65	25.39	.4591	53.55	39.42	30.29	.5788	64.72
.90	.95	0.0	30.36	33.14	73.00	37.95	30.46	.5839	71.98	25.65	25.39	.4591	57.41	39.42	30.29	.5788	70.36
.90	.95	.1	30.36	33.14	78.93	37.95	30.46	.5839	77.49	25.65	25.39	.4591	61.47	39.42	30.29	.5788	76.39
.95	.90	−.1	23.80	46.27	53.84	32.34	40.02	.6049	52.71	17.47	40.16	.4763	41.62	35.35	39.52	.5937	50.01
.95	.90	0.0	23.80	46.27	59.88	32.31	40.02	.6049	58.00	17.47	40.16	.4763	46.17	35.35	39.52	.5937	55.40
.95	.90	.1	23.80	46.27	66.11	32.34	40.02	.6019	63.66	17.17	40.16	.4763	51.03	35.35	39.52	.5937	61.19
.95	.95	−.1	23.80	36.27	71.88	31.72	31.72	.5981	70.53	18.80	28.83	.4943	57.76	33.30	31.51	.5926	68.83
.95	.95	0.0	23.80	36.27	79.88	31.72	31.72	.5981	77.74	18.80	28.83	.4943	63.67	33.30	31.51	.5926	76.29
.95	.95	.1	23.80	36.27	88.66	31.72	31.72	.5981	85.59	18.80	28.83	.4943	70.55	33.30	31.51	.5926	84.43

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REFERENCES

[1] Cochran, W. G. (1977). *Sampling Techniques*, Third edition, John Wiley and Sons, New York.

[2] Hansen, M. H., Hurwitz, W. N. and Madow, W. G. (1953). *Sample Survey Methods and Theory*, Vol. 2, John Wiley and Sons, New York.

[3] Kulldorff, G. (1979). Optimum allocation for sampling on many occasions, *Invited paper, 42nd Session of the ISI*, Manila.

[4] Patterson, H. D. (1950). Sampling on successive occasions with partial replacement of units, *J. R. Statist. Soc.*, B, **12**, 241-255.

[5] Raj, D. (1965). On a method of using multiauxiliary information in sample surveys, *J. Amer. Statist. Assoc.*, **60**, 270-277.

[6] Rao, J. N. K. and Graham, J. E. (1964). Rotation design for sampling on repeated occasions, *J. Amer. Statist. Assoc.*, **59**, 492-509.

[7] Singh, D. and Srivastava, A. K. (1974). Successive sampling and its applications, Unpublished paper presented at the International symposium 'Recent trends of research in Statistics' in memory of the late Professor P. C. Mahalanobis, Calcutta, Dec. 16-27, 1974.