Applied Mathematics and Mechanics
(English Edition, Vol. 20, No. 2, Feb. 1999)

Published by SU,
Shanghai, China

WEAK FORMULATION OF MIXED STATE EQUATION AND BOUNDARY VALUE PROBLEM OF LAMINATED CYLINDRICAL SHELL *

Ding Kewei (丁克伟), Tang Limin (唐立民)

Research Institute of Engineering Mechanics, Dalian University of Technology,
Dalian 116023, P R China

(Received May 27, 1997; Revised July 12, 1998)

Abstract: *Weak formulation of mixed state equations including boundary conditions are presented in a cylindrical coordinate system by introducing Hellinger-Reissner variational principle. Analytical solutions are obtained for laminated cylindrical shell by means of state space method. The present study extends and unifies the solution of laminated shells.*

Key words: mixed state equation; state space; cylindrical shell; weak formulation; boundary value problem

Introduction

Based on three-dimensional elasticity theory, exact solution of homogeneous isotropic, orthotropic, and laminated plates and shells has been studied[1]~[6], respectively. But all of above papers adopted rigorous equilibrium and boundary conditions, and their solution can be relied only on special technique. Thus those methods would be difficult to be popularized. Ref. [7] clarified the importance of mixed state equation of elasticity, and first gave Hamilton canonical equation by modifying Hellinger-Reissner variational principle. At the same time Tang[7] pointed out that the study of solution of mixed state equation has still not been well developed though it has a wide spread application prospect. This is because Hamilton equation must be satisfied complex boundary condition in continuous medium mechanics. This is its characteristic and difficult point, too. In this paper, however, by introducing Hellinger-Reissner variational principle, universal weak formulation of mixed state equation and boundary condition are presented. Shells with different boundary condition can be dealt with by uniform formulas. Furthermore, for applying state space method, and analytical solutions are obtained of homogeneous and laminated shells. The present study extends and unifies the solution of laminated shells.

1 Hellinger-Reissner Variational Principle and Hamilton Canonical Equation

In the light of three-dimensional elasticity Hellinger-Reissner variational principle, three-dimensional composite function based on independent variable σ_x, σ_θ, σ_r, $\tau_{r\theta}$, τ_{rx}, $\tau_{x\theta}$, u, v, w

* Project supported by Provincial Natural Science Foundation of Anhui, China

can be shown to be of the form:

$$U = \iiint_V \Big[\sigma_x \frac{\partial u}{\partial x} + \sigma_\theta \left(\frac{w}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \right) + \sigma_r \frac{\partial w}{\partial r} + \tau_{x\theta} \left(\frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} \right) +$$
$$\tau_{r\theta} \left(\frac{1}{r}\frac{\partial w}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) + \tau_{rx} \left(\frac{\partial u}{\partial r} + \frac{\partial w}{\partial x} \right) - \frac{1}{2}\{\sigma\}^T\{c\}^{-1}\{\sigma\} \Big] \mathrm{d}V -$$
$$\iint_S [\lambda_x (p_x - \bar{p}_x) u + \lambda_\theta (p_\theta - \bar{p}_\theta) v + \lambda_r (p_r - \bar{p}_r) w + ((1 - \lambda_x)\bar{u} - u) p_x +$$
$$((1 - \lambda_\theta)\bar{v} - v) p_\theta + ((1 - \lambda_r)\bar{w} - w) p_r] \mathrm{d}S . \tag{1}$$

Collecting all formulae with $\partial(\quad)/\partial r$, one has

$$U = \iiint_V \left\{ \sigma_r \frac{\partial w}{\partial r} + \tau_{r\theta} \frac{\partial v}{\partial r} + \tau_{rx} \frac{\partial u}{\partial r} - H \right\} r \mathrm{d}r \mathrm{d}\theta \mathrm{d}x + \Gamma , \tag{2}$$

where

$$- H = \sigma_x \frac{\partial u}{\partial x} + \sigma_\theta \left(\frac{w}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \right) + \tau_{x\theta} \left(\frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta} \right) + \tau_{r\theta} \left(\frac{1}{r}\frac{\partial w}{\partial \theta} - \frac{v}{r} \right) +$$
$$\tau_{rx} \frac{\partial w}{\partial x} - \frac{1}{2}\{\sigma\}^T[c]^{-1}\{\sigma\} . \tag{3}$$

It is the quadratic form of the Hamilton function. Where $[c]$ is elastic stiffness matrix

$$[c] = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ & & c_{33} & c_{34} & c_{35} & c_{36} \\ & & & c_{44} & c_{45} & c_{46} \\ & \text{sym} & & & c_{55} & c_{56} \\ & & & & & c_{66} \end{bmatrix},$$

here c_{ij} is the elastic constant.

$$\Gamma = \iint_S [\lambda_x (p_x - \bar{p}_x) u + \lambda_\theta (p_\theta - \bar{p}_\theta) v + \lambda_r (p_r - \bar{p}_r) w + ((1 - \lambda_x)\bar{u} - u) p_x +$$
$$((1 - \lambda_\theta)\bar{v} - v) p_\theta + ((1 - \lambda_r)\bar{w} - w) p_r] \mathrm{d}S , \tag{4}$$

in which coefficients $\lambda_x , \lambda_\theta , \lambda_r$ can be determined,

$S \in S_\sigma$ (stress boundary) , $\lambda_x = \lambda_\theta = \lambda_r = 1$,

$S \in S_u$ (displacement boundary) , $\lambda_x = \lambda_\theta = \lambda_r = 0$,

$S \in S_{\sigma - u}$ (mixed boundary), for example $\lambda_x = 1$, when x direction is stress boundary, or else $\lambda_x = 0$.

It can be easily proved that true solution state corresponds to $\delta U = 0$ in elastic body.

Using $\delta U = 0$ the following relations can be obtained,

$$\left.\begin{aligned} &\iiint_V \left(\frac{\partial \tau_{rx}}{\partial r} + \frac{\partial H}{\partial u} \right) \delta u \mathrm{d}V + \iint_{S_\sigma} (\bar{p}_x - p_x) \delta u \mathrm{d}S = 0, \\ &\iiint_V \left(\frac{\partial \tau_{r\theta}}{\partial r} + \frac{\partial H}{\partial v} \right) \delta v \mathrm{d}V + \iint_{S_\sigma} (\bar{p}_\theta - p_\theta) \delta v \mathrm{d}S = 0, \\ &\iiint_V \left(\frac{\partial \sigma_r}{\partial r} + \frac{\partial H}{\partial w} \right) \delta w \mathrm{d}V + \iint_{S_\sigma} (\bar{p}_r - p_r) \delta w \mathrm{d}S = 0; \end{aligned}\right\} \tag{5a}$$

$$\left.\begin{aligned}&\iiint_V\left(\frac{\partial u}{\partial r}-\frac{\partial H}{\partial \tau_{rx}}\right)\delta\tau_{rx}\mathrm{d}V+\iint_{S_u}[(\bar{w}-w)n_x+(\bar{u}-u)n_r]\delta\tau_{rx}\mathrm{d}S=0,\\&\iiint_V\left(\frac{\partial v}{\partial r}-\frac{\partial H}{\partial \tau_{r\theta}}\right)\delta\tau_{r\theta}\mathrm{d}V+\iint_{S_u}[(\bar{w}-w)n_\theta+(\bar{v}-v)n_r]\delta\tau_{r\theta}\mathrm{d}S=0,\\&\iiint_V\left(\frac{\partial w}{\partial r}-\frac{\partial H}{\partial \sigma_r}\right)\delta\sigma_r\mathrm{d}V+\iint_{S_u}(\bar{w}-w)n_r\delta\sigma_r\mathrm{d}S=0;\end{aligned}\right\}\tag{5b}$$

$$\left.\begin{aligned}&\iiint_V\left(\frac{\partial u}{\partial x}-\frac{\partial H}{\partial \sigma_x}\right)\delta\sigma_x\mathrm{d}V+\iint_{S_u}(\bar{u}-u)n_x\delta\sigma_x\mathrm{d}S=0,\\&\iiint_V\left(\frac{1}{r}\frac{\partial v}{\partial \theta}-\frac{\partial H}{\partial \sigma_\theta}\right)\delta\sigma_\theta\mathrm{d}V+\iint_{S_u}(\bar{v}-v)n_\theta\delta\sigma_\theta\mathrm{d}S=0,\\&\iiint_V\left(\frac{1}{r}\frac{\partial u}{\partial \theta}+\frac{\partial v}{\partial x}-\frac{\partial H}{\partial \tau_{x\theta}}\right)\delta\tau_{x\theta}\mathrm{d}V+\iint_{S_u}[(\bar{v}-v)n_x+(\bar{u}-u)n_\theta]\delta\tau_{x\theta}\mathrm{d}S=0.\end{aligned}\right\}\tag{5c}$$

We denoted:

$$q=(u,v,w)^T,\ p=(\tau_{rx},\tau_{r\theta},\sigma_r)^T,\ p_1=(\sigma_x,\sigma_\theta,\tau_{x\theta})^T.\tag{6}$$

Eq. (5) can be written in the form as follows:

$$\left.\begin{aligned}&\iiint_V\left(\frac{\partial p}{\partial r}+\frac{\partial H}{\partial q}\right)\delta q\mathrm{d}V+\Gamma_1=0,\\&\iiint_V\left(\frac{\partial q}{\partial r}-\frac{\partial H}{\partial p}\right)\delta p\mathrm{d}V+\Gamma_2=0,\\&\iiint_V\left(Dq-\frac{\partial H}{\partial p_1}\right)\delta p_1\mathrm{d}V+\Gamma_3=0,\end{aligned}\right\}\tag{7}$$

where

$$\Gamma_1=\begin{bmatrix}\iint_{S_\sigma}(\bar{p}_x-p_x)\delta u\mathrm{d}S=0\\\iint_{S_\sigma}(\bar{p}_\theta-p_\theta)\delta v\mathrm{d}S=0\\\iint_{S_\sigma}(\bar{p}_r-p_r)\delta w\mathrm{d}S=0\end{bmatrix},$$

$$\Gamma_2=\begin{bmatrix}\iint_{S_u}[(\bar{w}-w)n_x+(\bar{u}-u)n_r]\delta\tau_{rx}\mathrm{d}S=0\\\iint_{S_u}[(\bar{w}-w)n_\theta+(\bar{v}-v)n_r]\delta\tau_{r\theta}\mathrm{d}S=0\\\iint_{S_u}(\bar{w}-w)n_r\delta\sigma_r\mathrm{d}S=0\end{bmatrix},$$

$$\Gamma_3=\begin{bmatrix}\iint_{S_u}(\bar{u}-u)n_x\delta\sigma_x\mathrm{d}S=0\\\iint_{S_u}(\bar{v}-v)n_\theta\delta\sigma_\theta\mathrm{d}S=0\\\iint_{S_u}[(\bar{v}-v)n_x+(\bar{u}-u)n_\theta]\delta\tau_{x\theta}\mathrm{d}S=0\end{bmatrix},$$

$$D = \begin{bmatrix} -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{1}{r} \\ -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{\partial}{\partial x} & 0 \end{bmatrix}.$$

Eq. (7) can also be written in a simplified form,

$$\frac{\partial p}{\partial r} = -\frac{\partial H}{\partial q},\ \frac{\partial q}{\partial r} = \frac{\partial H}{\partial p}. \tag{8}$$

This is a classical Hamilton canonical equation.

2 Weak Formulation of Mixed State Equation and Boundary Condition and Its Solution

Selecting $F = (p, q)^T$ constitute state vector, Eq. (7) can be expressed as follows:

$$\iiint_V G_1 \frac{\partial}{\partial r} F \mathrm{d}V = \iiint_V G_1 HF \mathrm{d}V + \iiint_V G_1 D_1 p_1 \mathrm{d}V + \iint_S G_1 S_1 \mathrm{d}S, \tag{9}$$

$$\iiint_V G_2 D_2 F \mathrm{d}V + \iiint_V G_2 B p_1 \mathrm{d}V + \iint_S G_2 S_2 \mathrm{d}S = 0. \tag{10}$$

Eqs. (9) and (10) are weak formulation of mixed state equation and boundary condition. where

$$G_1 = \begin{bmatrix} \delta u & & & & & \\ & \delta v & & & [0] & \\ & & \delta w & & & \\ & & & \delta \tau_{rx} & & \\ & [0] & & & \delta \tau_{r\theta} & \\ & & & & & \delta \sigma_r \end{bmatrix},\quad S_1 = \begin{bmatrix} p_x - \bar{p}_x \\ p_\theta - \bar{p}_\theta \\ p_r - \bar{p}_r \\ (w - \bar{w}) n_x + (u - \bar{u}) n_r \\ (w - \bar{w}) n_\theta + (v - \bar{v}) n_r \\ (w - \bar{w}) n_r \end{bmatrix},$$

$$H = \begin{bmatrix} -\frac{1}{r} & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2}{r} & 0 & 0 & 0 & 0 \\ -\frac{\partial}{\partial x} & -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{1}{r} & 0 & 0 & 0 \\ S_{55} & S_{54} & S_{53} & 0 & 0 & -\frac{\partial}{\partial x} \\ S_{45} & S_{44} & S_{43} & 0 & \frac{1}{r} & -\frac{1}{r}\frac{\partial}{\partial \theta} \\ S_{35} & S_{34} & S_{33} & 0 & 0 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} -\frac{\partial}{\partial x} & 0 & -\frac{1}{r}\frac{\partial}{\partial \theta} \\ 0 & -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{\partial}{\partial x} \\ 0 & \frac{1}{r} & 0 \\ S_{51} & S_{52} & S_{56} \\ S_{41} & S_{42} & S_{46} \\ S_{31} & S_{32} & S_{36} \end{bmatrix},$$

$$G_2 = \begin{bmatrix} \delta \sigma_x & 0 & 0 \\ 0 & \delta \sigma_\theta & 0 \\ 0 & 0 & \delta \tau_{x\theta} \end{bmatrix},\ D_2 = \begin{bmatrix} S_{15} & S_{14} & S_{13} & -\frac{\partial}{\partial x} & 0 & 0 \\ S_{25} & S_{24} & S_{23} & 0 & -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{1}{r} \\ S_{65} & S_{64} & S_{63} & -\frac{1}{r}\frac{\partial}{\partial \theta} & -\frac{\partial}{\partial x} & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{21} & S_{22} & S_{26} \\ S_{61} & S_{62} & S_{66} \end{bmatrix}, \quad S_2 = \begin{bmatrix} (u - \bar{u})n_x \\ (v - \bar{v})n_\theta \\ (u - \bar{u})n_\theta + (v - \bar{v})n_x \end{bmatrix}.$$

In order to solve Eqs. (9) and (10), we assume function

$$F = F(r)N_1(x,\theta), \quad p_1 = p(r)N_2(x,\theta). \tag{11}$$

Substituting Eq. (11) into Eqs. (9) and (10), respectively. After finishing dual integral in x, θ plane, the following formula can be obtained:

$$\frac{d}{dr}F(r) = AF(r) + S(r). \tag{12}$$

The solution to Eq. (12) is

$$F(r) = e^{Ar}F(0) + \int_0^r e^{A(r-\tau)}S(\tau)d\tau. \tag{13}$$

Solution of composite laminated shell can easy be obtained by means of Eq. (13) and the continuity of state vector F at the two interfaces and state transfer matrix method.

3 Numerical Examples

Example 1 A thick three-plied laminated shell is loaded by a liquid pressure q at the interior surface:

$$q = -\gamma b(\cos\theta - \cos\varphi), \quad \text{when } \theta < \varphi,$$
$$q = 0, \quad \text{when } \theta \geqslant \varphi,$$

where γ is the density of the liquid, $\varphi = 2\pi/3$. The shell has the following geometrical parameters: $h_1 = h_3 = 0.1h, h_2 = 0.8h, l = 2\pi R_0$, l is the length of the shell, R_0 is radius of middle surface. For each ply, the material properties are orthotropic

$$\begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_r \\ \tau_{r\theta} \\ \tau_{rx} \\ \tau_{x\theta} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_\theta \\ \varepsilon_r \\ \gamma_{r\theta} \\ \gamma_{rx} \\ \gamma_{x\theta} \end{bmatrix},$$

where

$$c_{22}/c_{11} = 0.543\,103, \quad c_{12}/c_{11} = 0.246\,269,$$
$$c_{23}/c_{11} = 0.115\,017, \quad c_{13}/c_{11} = 0.083\,172,$$
$$c_{33}/c_{11} = 0.530\,172, \quad c_{44}/c_{11} = 0.266\,810,$$
$$c_{55}/c_{11} = 0.159\,914, \quad c_{66}/c_{11} = 0.262\,931.$$

Introducing $c_{11}^{(1)}$ and $c_{11}^{(2)}$, which denote c_{11} corresponding to the outer and middle ply, respectively, and $\rho = c_{11}^{(1)}/c_{11}^{(2)}$.

Let state vector function of Eq. (11) to be expressed as

$$u = \sum_m \sum_n u_{mn}(r)\cos\frac{m\pi x}{l}\cos(n\theta), \quad \tau_{rx} = \sum_m \sum_n \tau_{rx,mn}(r)\cos\frac{m\pi x}{l}\cos(n\theta),$$

$$v = \sum_m \sum_n v_{mn}(r)\sin\frac{m\pi x}{l}\sin(n\theta), \quad \tau_{r\theta} = \sum_m \sum_n \tau_{r\theta,mn}(r)\sin\frac{m\pi x}{l}\sin(n\theta),$$

$$w = \sum_m \sum_n w_{mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta),\ \sigma_r = \sum_m \sum_n \sigma_{r,mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta),$$
$$\sigma_x = \sum_m \sum_n \sigma_{x,mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta),\ \sigma_\theta = \sum_m \sum_n \sigma_{\theta,mn}(r)\sin\frac{m\pi x}{l}\cos(n\theta),$$
$$\tau_{x\theta} = \sum_m \sum_n \tau_{x\theta,mn}(r)\cos\frac{m\pi x}{l}\sin(n\theta).$$

The boundary conditions for this problem, $v = w = \sigma_x = 0$, on $x = 0, l$; are satisfied. The numerical results are given in Table 1 with $m = 1,3,\cdots,29; n = 0,1,\cdots,15$.

Table 1 Deflection and stresses of shell ($h/R_0 = 1.0$)*

	$x = l/2, \theta = 0$	$\rho = 1$			$\rho = 5$		
		$wc_{11}/(\gamma bh)$	$\sigma_x/(\gamma b)$	$\sigma_\theta/(\gamma b)$	$wc_{11}^{(2)}/(\gamma bh)$	$\sigma_x/(\gamma b)$	$\sigma_\theta/(\gamma b)$
Present study	1+	14.136	6.569 8	14.410	7.466 2	18.126	38.670
	1−	13.847	0.754 0	0.738 8	7.397 7	2.836 0	2.695 4
	2+	13.847	0.754 0	0.738 8	7.397 7	0.468 4	0.402 4
	2−	12.897	2.290 8	0.481 4	6.698 8	1.013 2	0.132 8
	3+	12.897	2.290 8	0.481 4	6.698 8	5.140 5	0.767 4
	3−	12.848	2.631 4	0.727 5	6.675 9	6.154 5	1.695 8
Ref. [5]	3−	12.830	2.629 0	0.727 0	6.668 9	6.149 0	1.695 0

* 1+ : internal ply at internal surface, 1− : internal ply at outer surface; 2: middle ply; 3: outer ply

Table 2 Stresses of a homogeneous and three-ply shell*

		$\rho = 1(h/R_0 = 0.4)$				$\rho = 5(h/R_0 = 0.6)$			
		$\sigma_x/\gamma b$	$\sigma_\theta/\gamma b$	$\sigma_x^{(0)}/\gamma b$	$\tau_{rx}/\gamma b$	$\sigma_x/\gamma b$	$\sigma_\theta/\gamma b$	$\sigma_x^{(0)}/\gamma b$	$\tau_{rx}/\gamma b$
Present study	1+	1.468	1.701	45.44	0.000	2.623	4.570	54.34	0.000
	1−	1.947	1.951	6.044	2.654	3.570	4.859	−24.11	2.047
	2+	1.947	1.951	6.044	2.654	0.589	0.798	−4.822	2.047
	2−	4.953	3.444	−12.58	2.284	1.297	0.817	1.112	1.180
	3+	4.953	3.444	−12.58	2.284	6.620	4.270	5.562	1.180
	3−	5.396	3.636	−45.13	0.000	7.468	4.514	−45.64	0.000
SAP5	1+	1.442	1.279	9.499	4.002	2.752	5.162	9.432	6.798
	1−	2.009	1.811	5.307	3.876	3.454	5.269	−0.518	6.650
	2+	2.009	1.811	5.307	3.876	0.593	0.974	−0.104	1.330
	2−	5.668	4.616	−13.17	3.692	1.302	0.878	−1.914	1.037
	3+	5.668	4.616	−13.17	3.692	6.837	4.657	−9.571	5.185
	3−	6.116	4.895	−17.16	3.746	7.526	4.798	−18.16	5.252

* σ_x, σ_θ at $x = l/2, \theta = 0$. $\sigma_x^{(0)}, \tau_{rx}$ at $x = 0, \theta = 0$.

1+ : internal ply at internal surface, 1− : internal ply at outer surface; 2: middle ply; 3: outer ply

Example 2 Consider three-ply shell with two clamped edges($x = 0, l$). Geometrical parameters, elastic constants and loaded q are identical to Example 1. Selecting state vector functions are the same as those of Example 1. Partial boundary conditions can not be satisfied. Change the clamped edges into simply supported ones, and add the reactions of the original clamped edges. The remaining boundary conditions can be satisfied by means of $\iint_S (u - \bar{u}) n_x \mathrm{d}S = 0$ on $x = 0,\ l$. The numerical results are given in Table 2.

4 Conclusion

Weak solution for mixed state equation is efficient in this paper, and analytical solution is given by combining it with state space method. The principle and method suggested here have clear physical concepts, and extends and unifies solution of laminated shells. The present method can also be employed to study vibration problem of laminated shells.

References

[1] Soldators K P, Hadjigeorgiou V P. Three-dimensional solution of the free vibration problem of homogeneous isotropic cylindrical shells and panels [J]. *J Sound Vibration*, 1990, **137**(3):369 ~ 384

[2] Hawkes T D, Soldators K P. Three-dimensional axisymmetric vibrations of orthotropic and cross-ply laminated hollow cylinders [J]. *AIAA J*, 1992,**30**(4):1089 ~ 1098

[3] Kardomateas G A. Buckling of thick orthotropic cylindrical shells under external pressure [J]. *ASME J Appl Mech*, 1993,**60**(1):195 ~ 202

[4] Srinivas S, Rao A K. Bending vibration and buckling of simply supported thick orthotropic rectangular plates and laminates [J]. *Internat J Solids Structures*, 1970,**6**(12):1463 ~ 1481

[5] Fan J, Ding K. Analytical solution for thick closed laminated cylindrical shells [J]. *Internat J Mech Sci*, 1993,**35**(8):657 ~ 668

[6] Ding Keiwei, Fan Jiarang. Exact solution for axisymmetric problem of thick closed laminated continuous cylindrical shells[J]. *Engineering Mechanics*, 1994, **11**(2):8 ~ 19(in Chinese)

[7] Tang Limin. Mixed formulation and Hamilton canonical equation of theory of elasticity [J]. *Comput Struct Mech Appl*, 1991,**8**(4):343 ~ 350(in Chinese)