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DIFFERENTIAL OPERATORS ASSOCIATED WITH ZONAL POLYNOMIALS. II

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Summary

Let $C_\kappa(S)$ be the zonal polynomial of the symmetric $m\times m$ matrix $S=(s_{ij})$, corresponding to the partition κ of the non-negative integer k. If $\partial/\partial S$ is the $m\times m$ matrix of differential operators with (i, j)th entry $((1+\delta_{ij})\partial/\partial s_{ij})/2$, δ being Kronecker's delta, we show that $C_\kappa(\partial/\partial S)C_\lambda(S)=k!\delta_{\lambda\kappa}C_\kappa(I)$, where λ is a partition of k. This is used to obtain new orthogonality relations for the zonal polynomials, and to derive expressions for the coefficients in the zonal polynomial expansion of homogeneous symmetric polynomials.

1. Introduction

Let κ be a partition of the non-negative integer k (for brevity, we write this as $\kappa\vdash k$), and $C_\kappa(S)$ be the zonal polynomial, corresponding to κ, of the $m\times m$ symmetric matrix $S=(s_{ij})$. In Part I of this series [7], I defined a differential operator corresponding to $C_\kappa(S)$, which was then used to solve certain problems appearing in the theory of zonal polynomials. As I pointed out, the key to the definition of the operators was the orthogonality relations (James [5]) for the coefficients appearing in the power sum expansion of the zonal polynomials.

Here, the reasoning is reversed. Starting with the differential operators $C_\kappa(\partial/\partial S)$, $\partial/\partial S$ being the matrix of differential operators having (i, j)th entry $((1+\delta_{ij})\partial/\partial s_{ij})/2$, we proceed to show that the coefficients appearing in the expansion of $C_\kappa(S)$ in terms of the elements of S also satisfy orthogonality properties. The results are applied to show how we can obtain new expressions for various coefficients which have been previously studied in [1]-[4], [6], [7].

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2. Notation and main results

We shall need some multi-index notation. If $S=(s_{ij})$ is as before and $\alpha=(\alpha_{ij})$ is an $m\times m$ symmetric matrix of non-negative integers, we define $s^\alpha=\prod_{i\leq j}^m s_{ij}^{\alpha_{ij}}$, $|\alpha|=\sum_{i\leq j}^m \alpha_{ij}$, $|\alpha|'=\sum_{i<j}^m \alpha_{ij}$, $\alpha!=\prod_{i\leq j}^m \alpha_{ij}!$. For $\kappa\vdash k$, we define coefficients $c_{\kappa\alpha}$ by the expansion

$$C_\kappa(S)=\sum_{|\alpha|=k} c_{\kappa\alpha}s^\alpha . \tag{2.1}$$

We now have

THEOREM 1. *Let $\kappa\vdash k$, $\lambda\vdash l$. Then,*

$$C_\kappa(\partial/\partial S)C_\lambda(S)=\begin{cases} 0\,, & l<k \\ k!\delta_{\kappa\lambda}C_\kappa(I)\,, & l=k \\ \dfrac{l!}{(l-k)!}C_\lambda(I)\sum_{\tau\vdash l-k} b^\lambda_{\kappa,\tau}\dfrac{C_\tau(S)}{C_\tau(I)}\,, & l>k\end{cases} \tag{2.2}$$

where the $b^\lambda_{\kappa,\tau}$ coefficients are defined ([4], [6], [7]) *by the expansion*

$$C_\kappa(S)C_\tau(S)=\sum_{\lambda\vdash l} b^\lambda_{\kappa,\tau}C_\lambda(S)\,, \qquad \kappa\vdash k,\ \tau\vdash l-k\,. \tag{2.3}$$

PROOF. The first part of (2.2) really has nothing to do with zonal polynomial theory; it follows from $C_\kappa(S)$ being of higher degree then $C_\lambda(S)$. However, all three parts of (2.2) can be given a unified proof, if we use an idea of Bingham [1]. Let T be another $m\times m$ symmetric matrix. Then, a generating function for $C_\kappa(\partial/\partial S)C_\lambda(S)$ can be taken to be

$$\sum_{l=0}^\infty\sum_{\lambda\vdash l}\{C_\kappa(\partial/\partial S)C_\lambda(S)\}C_\lambda(T)/l!C_\lambda(I)=C_\kappa(\partial/\partial S)_0F_0(S,T) \tag{2.4}$$

where

$${}_0F_0(S,T)=\sum_{l=0}^\infty\sum_\lambda C_\lambda(S)C_\lambda(T)/l!C_\lambda(I)=\int_{O(m)}\exp(\mathrm{tr}\,SHTH')dH\,, \tag{2.5}$$

dH being the invariant (normalized) Haar measure on $O(m)$, the compact group of orthogonal $m\times m$ matrices. Since

$$C_\kappa(\partial/\partial S)\exp(\mathrm{tr}\,SHTH')=C_\kappa(T)\exp(\mathrm{tr}\,SHTH')\,, \tag{2.6}$$

it follows via (2.4) and (2.5) that our generating function equals

$$C_\kappa(T)\sum_{j=0}^\infty\sum_{\tau\vdash j} C_\tau(S)C_\tau(T)/j!C_\tau(I)\,. \tag{2.7}$$

Multiplying $C_\kappa(T)C_\tau(T)$ in (2.7) using (2.3) and comparing coefficients of $C_\lambda(T)$ with those in (2.4) establishes (2.2).

COROLLARY 1. *Let* $\kappa\vdash k$, $\lambda\vdash k$. *Then,*

$$\sum_{|\alpha|=k} \alpha! c_{\kappa\alpha} c_{\lambda\alpha}/2^{|\alpha|'} = k!\delta_{\kappa\lambda} C_\kappa(I) \ . \tag{2.8}$$

PROOF. Let $\partial^{|\alpha|}/\partial s^\alpha$ denote $\partial^{\alpha_{11}+\alpha_{12}+\cdots}/\partial s_{11}^{\alpha_{11}}\partial s_{12}^{\alpha_{12}}\cdots$. Then by (2.2),

$$\begin{aligned} k!\delta_{\kappa\lambda}C_\kappa(I) &= C_\kappa(\partial/\partial S)C_\lambda(S) \\ &= \sum_{|\alpha|=k} 2^{-|\alpha|'} c_{\kappa\alpha}\partial^k/\partial s^\alpha \sum_{|\beta|=k} c_{\lambda\beta}s^\beta \\ &= \sum_{|\alpha|=k} \alpha! 2^{-|\alpha|'} c_{\kappa\alpha}c_{\lambda\alpha} \ . \end{aligned} \tag{2.9}$$

3. Applications

We now derive expressions for the $b^\lambda_{\kappa,\tau}$ coefficients in terms of the $c_{\kappa\alpha}$'s.

THEOREM 2. *If the coefficients* $b^\lambda_{\kappa,\tau}$ *are as defined in* (2.3), *then*

$$b^\lambda_{\kappa,\tau} = \sum \sum_{\alpha=\beta+\gamma} \sum \alpha! c_{\lambda\alpha} c_{\kappa\beta} c_{\tau\gamma}/2^{|\alpha|'} l! C_\kappa(I) \ . \tag{3.1}$$

PROOF. This is easily established on applying the operator $C_\lambda(\partial/\partial S)$ to both sides of (2.3) and using (2.1).

Again using the operator $C_\lambda(\partial/\partial S)$, we can proceed exactly as in [7] to obtain results analogous to (3.1) for the generalised binomial coefficients studied by Constantine [2] and Bingham [1]. Since the method and results are by now transparent, we skip the details.

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