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CONNECTIVITY OF RANDOM NETS

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The weak connectivity γ of a random net is defined and computed by an approximation method as a function of a, the axone density. It is shown that γ rises rapidly with a, attaining 0.8 of its asymptotic value (unity) for $a = 2$, where the number of neurons in the net is arbitrarily large. The significance of this parameter is interpreted also in terms of the maximum expected spread of an epidemic under certain conditions.

Numerous problems in various branches of mathematical biology lead to the consideration of certain structures which we shall call "random nets." Consider an aggregate of points, from each of which issues some number of outwardly directed lines (axones). Each axone terminates upon some point of the aggregate, and the probability that an axone from one point terminates on another point is the same for every pair of points in the aggregate. The resulting configuration constitutes a *random net.*

The existence of a *path* in a random net from a point A to a point B implies the possibility of tracing directed lines from A through any number of intermediate points, on which these lines terminate, to B.

We shall say that B is t axones removed from A, if t is the smallest number of axones contained in any of the paths from A to B. Point A itself is zero axones removed from A. All the other points upon which the axones of A terminate are one axone removed. The points upon which the axones from these latter points terminate, and which are not one or zero axones removed, are two axones removed, etc.

The notion of a random net may be generalized, if it is not assumed that the probability of direct connection between every pair of points in the net is the same. In that case it is necessary to define this probability for every pair of points. This can be done, for example, in terms of the distance between them or in some other way.

If the connections are not equiprobable, we shall speak of a net with a bias.

The following examples illustrate problems in which the concept of a net, defined by the probability of the connections among its points, seems useful.

1. *A problem in the theory of neural nets.* Suppose the points of a net are neurons. What is the probability that there exists a path between an arbitrary pair of neurons in the net? If the net has bias, what is the probability that there exists a path between a specified pair? In particular, what is the probability that a neuron is a member of a cycle (i.e., there exists a path from the neuron to itself through any positive number of internuncials)? Or, one may ask, what is the probability that there exists a path from a given neuron to every other neuron in the net?

2. *A problem in the theory of epidemics.* Suppose a number of individuals in a closed population contract a contagious disease, which lasts a finite time and then either kills them or makes them immune. If the probability of transmission is defined for each pair of individuals, what is the expected number of individuals which will contract the disease at a specified time? In particular, what is the

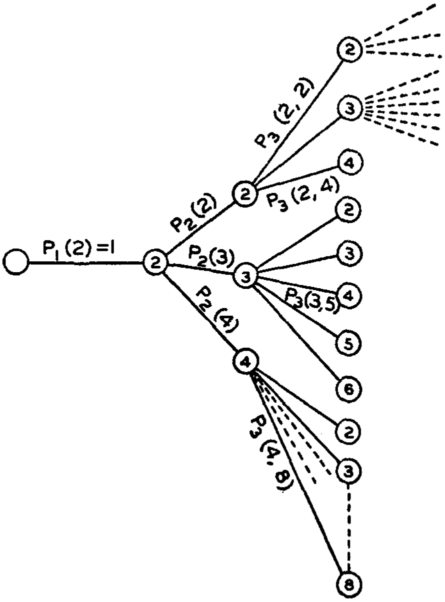

FIGURE 1. The probability tree for the number of ancestors of a single individual.

expected number of individuals which will eventually (after an infinite time) contract the disease? Or else, what is the probability that the entire population will succumb? Note that if the probability of transmission is the same for each pair of individuals, we are dealing with a random net.

3. *A problem in mathematical genetics.* Given the probability of mating between each pair of individuals in a population (as a function of their distance, or kinship, or the like), what is the expected number of ancestors of a given order for each individual? Clearly, the less the expected number of ancestors, the greater the genetic homogeneity of the population.

Each of these problems can be formalized by constructing a "probability tree." As an example, a tree for the genetic problem is illustrated in **Figure 1.**

We note that the tree consists of "nodes" connected by lines. The nodes can be designated by "first order," "second order," etc., depending on their distance from the "root." The number at the node indicates a possible number of ancestors of a given order. The lines connecting the nodes are labeled with the corresponding probabilities. Thus $p_1(2) = 1$, since it is certain that an individual has exactly two ancestors of the first order (parents). However, the parents may have been siblings or half-siblings. Therefore it is possible that the number of grandparents is 2, 3, or 4. The corresponding probabilities are $p_2(2)$, $p_2(3)$, and $p_2(4)$. The probability of having a certain number of great-grandparents depends on how many grandparents one has had. Consequently, those probabilities must be designated by $p_3(i,j)$ where $i = 2, \cdots 4$ and $j = 2, \cdots 8$. In general, the probability of having a certain number of ancestors of order k will depend on how many ancestors of each of the smaller orders one has had. If, however, we simplify the problem by supposing that the probability of having a certain number of ancestors of the kth order depends only on how many ancestors of the $(k-1)$th order one has, then the probability that an individual has exactly n ancestors of the mth order will be given by

$$P_m(n) = \sum_{r=2}^{2^m} \cdots \sum_{j=2}^{8} \sum_{i=2}^{4} p_2(2,i) p_3(i,j) p_4(j,k) \cdots p_m(r,n). \qquad (1)$$

The expected number of ancestors of the mth order will then be

$$E(m) = \sum_{n=2}^{2^m} nP(n). \qquad (2)$$

Clearly, a similar tree can be constructed for the neural net problem. Here the numbers at the nodes of the kth order would designate the possible number of neurons k axones removed from a given neuron. The p's would designate the corresponding transition probabilities from a certain number of neurons $(k - 1)$ axones removed to a certain number k axones removed, etc. If N is the number of neurons in the aggregate, clearly, a neuron B is at most N axones removed from a neuron A , or else there exists no path from A to B . Hence $E(N)$ represents the expected number of neurons in the aggregate to which there exist paths from an arbitrary neuron, if the neurons are not in any way distinguished from each other. This expected number we shall call the *weak connectivity* of a random net and will designate it by γ .

The contagion problem could be formulated in similar terms. Here weak connectivity would represent the expected number of individuals which will contract the disease eventually. If we define Γ, the *strong connectivity* as the probability that from an arbitrary point in a random net there exist paths to every other point, then Γ will represent the probability that the entire population will succumb in the epidemic described above. In this case, the number of "axones" represents the number of individuals infected by a carrier before he recovers or dies.

The weak connectivity of a random net. We shall compute the weak connectivity of a neural net in terms of certain approximations whose justification will be given in subsequent papers. It will be assumed that:

1. The number of axones per neuron a is constant throughout the net. This constant (the axone density) need not be an integer, since it may equally well be taken as the average number of axones per neuron.

2. Connections are equiprobable, i.e., an axone synapses upon one or another neuron in the aggregate with equal probability.

A. Shimbel (1950) has formulated the problem in terms of the following differential-difference equation

$$dx/dt = [N - x(t)][x(t) - x(t - \tau)]. \tag{3}$$

Here $x(t)$ is a function related to the expected number of neurons t axones removed from an arbitrary neuron, and τ is related to the axone density. Then the problem of finding γ is equivalent to the problem of finding $x(\infty)$. A somewhat generalized form of equation (3) is given also by M. Puma (1939). The solution of the equation is, however, not given.

An approximate expression for γ where N is large was derived by one of the authors (Rapoport, 1948) where the number of axones per neuron is exactly one. This case will be generalized here to a axones per neuron, which are supposed constant through out the aggregate.

The axone-tracing procedure. Let us start with an arbitrarily selected neuron A and consider the set of all neurons removed by not more than t axones from A. Let x be the expected number of these neurons. Then evidently $x = x(N, a, t)$ depends on the total number of neurons in the net, on the axone density, and on t. Moreover, the weak connectivity of the net can be expressed as

$$\gamma(N, a) = x(N, a, N)/N. \tag{4}$$

Since N and a are fixed, we shall refer to the expected number of points removed from A by not more than t axones by $x(t)$. Note that t is a positive integer.

We seek a recursion formula for $x(t)$ which will give us an approximate determination of that function. To give a rigorous treatment of the problem, one would need to deal with distribution functions instead of expected values. For example, $p(i,t)$, denoting the probability that there are *exactly* i neurons not more than t axones removed from A, would determine the distribution for t. Successive distributions (for $t + 1$, etc.) would then depend on previous *distributions,* instead of merely upon the first moments of these distributions (expected values). The "probability tree" method does take these relations into account. An "exact" approach to the problem will be given in a subsequent paper. Meanwhile, however, we shall develop an approximation method in which it will be assumed that the expected value $x(t)$ depends only upon previous expected values, and, of course, upon the parameters of the net.

The recursion formula. We now seek an expression for $x(t + 1) - x(t)$. This is evidently the expected number of neurons *exactly* $(t + 1)$ axones removed from A. We shall make use of the following formula, which may be readily verified. Let s marbles be placed independently and at random into N boxes. Then the expected number of boxes occupied by one or more marbles will be given by

$$N[1 - (1 - 1/N)^s]. \tag{5}$$

In our axone-tracing procedure there are $a[x(t) - x(t-1)]$ axones of the *newly* contacted neurons to be traced on each step. Then the total number of neurons contacted on the $(t+1)$th tracing will be, according to formula (5),

$$N[1-(1-1/N)^{a[x(t)-x(t-1)]}]\,. \tag{6}$$

But of these neurons the fraction $x(t)/N$ has already been contacted. Hence the expected number of newly contacted neurons will be given by

$$x(t+1)-x(t)=[N-x(t)][1-(1-1/N)^{a[x(t)-x(t-1)]}]\,, \tag{7}$$

which is our desired recursion formula.

Determination of γ. Let us set

$$y(t)=N-x(t). \tag{8}$$

Then equation (7) may be written as

$$y(t+1)=y(t)\,(1-1/N)^{a[y(t-1)-y(t)]}, \tag{9}$$

or

$$y(t+1)\,(1-1/N)^{ay(t)}=y(t)\,(1-1/N)^{ay(t-1)}. \tag{10}$$

Hence

$$y(t+1)\,(1-1/N)^{ay(t)}=\text{constant}=K\,. \tag{11}$$

We proceed to evaluate K. We have

$$y(t+1)=K(1-1/N)^{-ay(t)}. \tag{12}$$

But $y(t)$ represents the expected number of uncontacted points in the tth step. Since before the tracing began one point constituted the set of contacted points, therefore we have

$$y(0)=N-1\,, \tag{13}$$

and using formula (5),

$$y(1)=(N-1)^{a+1}N^{-a}. \tag{14}$$

Letting $t=0$ in (12), we obtain

$$K=N^{-aN}(N-1)^{aN+1}. \tag{15}$$

Furthermore, since $y(1) \leqq y(0)$ and $(1-1/N)^{-a} > 1$, we have $y(2) \leqq y(1)$, etc., so that $y(t)$ is a non-increasing function of t (this is also intuitively evident from the definition of y). Since $y \geq 0$ for all t, $y(t)$ must approach a limit as t grows without bound. Hence

$$\lim_{t\to\infty} y(t+1) = \lim_{t\to\infty} y(t) = Y. \tag{16}$$

Note that $\gamma = x(N)$ may also be considered as $\lim_{t\to\infty} x(t)/N$. This is so since contacting no new neurons on any tracing implies that no new neurons will be contacted on any subsequent tracings. If we continue to carry out tracings "symbolically," it is evident that at some tracing not greater than the Nth no new neurons will be contacted, and all subsequent tracings will be "dummy" tracings.

Using equations (12) and (15), we see that Y satisfies the transcendental equation

$$Y = (N-1)(1-1/N)^{a(N-Y)}. \tag{17}$$

For large N, this can be approximated by

$$Y \sim N \operatorname{Exp}\{a(Y/N-1)\}. \tag{18}$$

Hence, for large N,

$$Y/N \sim \operatorname{Exp}\{a(Y/N-1)\}. \tag{19}$$

But $\gamma = x(\infty)/N = 1 - Y/N$. Substituting this value into (19), we obtain the transcendental equation which defines γ implicitly as a function of a, namely,

$$\gamma = 1 - e^{-a\gamma}. \tag{20}$$

We note that for $\gamma = 0$, every a is a solution of (20). If $\gamma \neq 0$, then equation (20) can be solved explicitly for a giving

$$a = \frac{-\log(1-\gamma)}{\gamma}. \tag{21}$$

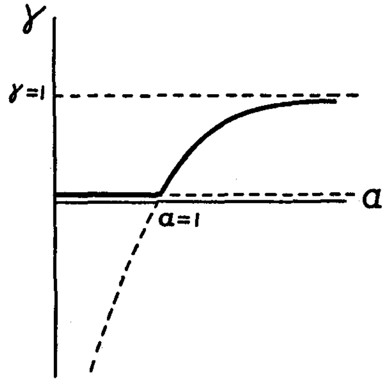

FIGURE 2. Weak connectivity as a function of axone density.

The right side of (21) is analytic in every neighborhood of the origin and tends to unity as γ approaches 0. Expanding that function in powers of γ, we have

$$a = 1 + \gamma/2 + \gamma^2/3 \cdots, \tag{22}$$

which allows us to plot a against γ (cf. Fig. 2). This graph consists of two branches, namely, the entire a-axis and the function (21). Negative values of γ, being physically meaningless, must be discarded. Thus in the region $0 \leqq a \leqq 1$, we have $\gamma \equiv 0$, as is intuitively evident. We must show, however, that for $a > 1$, γ follows the nonzero branch of the graph, otherwise we get the unlikely result that for sufficiently large N the fraction of individuals eventually infected in an epidemic will be negligible, regardless of the number of individuals infected by each carrier of the disease. Actually, the solution $\gamma \equiv 0$ is extraneous for $a > 1$ and appears in our equation because we have let N increase without bound *before* determining the relation between a and γ. In any physical situation N is finite. Hence a physically meaningful procedure is to determine γ as a function of a and N and *then* allow N to increase without bound. Such a function is given by equation (17). Proceeding from that equation we obtain

$$Y/(N-1) = (1-1/N)^{a(N-Y)}, \tag{23}$$

$$\log Y - \log(N-1) = a(N-Y)\log(1-1/N), \tag{24}$$

$$a = \frac{\log Y - \log(N-1)}{(N-Y)\,[\log(N-1) - \log(N)]}. \tag{25}$$

Let us write $Y = N - \phi(N) = N[1 - \phi(N)/N]$. Then equation (25) may be written as

$$\begin{aligned} \log N - \log(N-1) &+ \log[1-\phi(N)/N] \\ &= a\,\phi(N)\,[\log(N-1) - \log N]. \end{aligned} \tag{26}$$

Since $\phi(N) < N$ for all N, we may expand the last term of the left side of (26) and obtain

$$\begin{aligned} \log N - \log(N-1) &- \phi(N)/N - \tfrac{1}{2}[\phi(N)/N]^2 \\ &- 1/3[\phi(N)/N]^3 \cdots = a\,\phi(N)\,[\log(N-1) - \log N]. \end{aligned} \tag{27}$$

We now expand $\log(N-1) - \log N$ which appears in the right side of (27) and after rearrangements obtain

$$\begin{aligned}\log N - \log(N-1) \\ &= \frac{\phi(N)}{N}\left[1-a+\frac{\phi(N)-a}{2N}+\frac{[\phi(N)]^2-a}{3N^2}+\cdots\right] \quad (28)\\ &< \frac{\phi(N)}{N}\left[1-a+(1-\phi(N)/N)^{-1}\right].\end{aligned}$$

Now if a is fixed and greater than unity, the limit of $\phi(N)/N$ cannot be zero as N increases without bound, because otherwise for N sufficiently large the right side of (28) becomes negative, while the left side is always positive, a contradiction of inequality (28). Therefore, the limit of Y/N, as N increases without bound, cannot be unity for $a > 1$. But this means that $\gamma \neq 0$ if $a > 1$. Hence, for $a > 1$, the non-zero branch of our curve is the only meaningful one.

An examination of the meaningful part of the graph of equation (20) shows that as long as the axone density does not exceed one axone per neuron, $\gamma = 0$, i.e., for very large N, the number of neurons to which there exist paths from an arbitrary neuron is negligible compared with the total number of neurons in the net. On the other hand, as the axone density increases from unity, γ increases rather rapidly, starting with slope 2. Already for $a = 2$, γ reaches about 0.8 of its asymptotic value (unity) and is within a fraction of one per cent of unity for quite moderate a (say > 6). This means that no matter how large the net is, it is practically certain that there will exist a path between two neurons picked at random, provided only the axone density is a few times greater than unity. The interpretation in terms of an epidemic with equiprobable contacts is entirely analogous.

The case a $= 1$. This case was treated by one of the authors (Rapoport, 1948) by a different method. It was shown that for large N, the probability that a neuron was member of a cycle was given by $\sqrt{\pi/2N}$. This gives the probability of the existence of a path from a neuron over any number of internuncials greater than one to itself. But under the assumption of equiprobable connections, this may well represent the probability of the existence of a path from the given neuron to any *other* neuron in the net. Therefore we should have for large N, in the case $a = 1$,

$$\gamma \sim \sqrt{\pi/2N}. \quad (29)$$

For $N = \infty$, γ reduces to zero, as it should according to equation (20). We shall, however, examine the asymptotic behavior of γ for large N deduced from our approximate method, in order to compare it with the asymptotic behavior (29) deduced from an exact treatment of the special case. Dividing both sides of (17) by N, we may write for $a = 1$

$$Y/N = [(N-1)/N]^{N-Y+1}, \tag{30}$$

whence, since $Y/N = 1 - \gamma$,

$$\begin{aligned} 1-\gamma &= [(N-1)/N]^{N\gamma+1} \\ &= \operatorname{Exp}\{\ln(1-1/N) + N\gamma \ln(1-1/N)\}. \end{aligned} \tag{31}$$

We let $z = N^{-1}$ and examine the behavior of γ for small values of z. Expanding the right side of (31) by power series and retaining only terms of the second order (note that z and γ vanish together), we obtain

$$\begin{aligned} 1-\gamma = 1 &+ [-z - z^2/2 \cdots] + [-\gamma - \gamma z/2 - \cdots] \\ &+ z^2/2 + \gamma^2/2 + \gamma z + \cdots. \end{aligned} \tag{32}$$

Hence,

$$0 = -z + \gamma^2/2 + \gamma z/2 + \cdots. \tag{33}$$

Differentiating with respect to γ, we get

$$dz/d\gamma = \gamma + \gamma/2 \cdot dz/d\gamma + z/2 + \cdots, \tag{34}$$

$$dz/d\gamma \sim (\gamma + z/2)/(1-\gamma/2). \tag{35}$$

Therefore $dz/d\gamma$ vanishes at $z = 0$, $\gamma = 0$. Differentiating once again with respect to γ, we obtain

$$\left.\frac{d^2z}{d\gamma^2}\right|_{\substack{z=0\\ \gamma=0}} = 1. \tag{36}$$

Hence the power series representing z as a function of γ begins as follows:

$$z = \gamma^2/2 + \cdots. \tag{37}$$

Thus

$$\gamma^2 \sim 2z = 2/N, \tag{38}$$

$$\gamma \sim \sqrt{2/N} \cong 1.41\sqrt{N}. \tag{39}$$

The "exact" result as expressed by (22) gives

$$\gamma \sim 1.2/\sqrt{N}.$$

Thus the approximate method applied to the case $a = 1$ implies an asymptotic behavior of γ for large N which does not depart too sharply from that deduced by the exact method. The limiting value for γ is zero in both cases. The question of how well the limiting values of γ are approached by the approximate method for $a > 1$ remains open.

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