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APPLICATIONS OF MATRIX ALGEBRA TO COMMUNICATION NETS

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A "generic" problem amenable to matrix algebraic treatment is outlined. Several examples are given and one, a communication system, is studied in some detail.

A typical structure matrix is used to describe the channels of communication and a "status" matrix is used to describe the distribution of information in the system at any time.

A theorem is proved relating the status matrix at any time t to the tth power of the structure matrix.

The elements of the communication system are interpreted as individuals who can send messages to each other. For the individuals attempting to solve a "group problem" certain relations are derived between the structure and status matrices and time of solution.

The structure of the communication system is permitted to vary with time. A general theorem is proved relating the status matrix to the matrix product of the series of structure matrices representing the changing structure of the system.

Some suggestions are made for further generalizations. In particular, it is suggested that so-called "higher order" information transmission can be similarly treated.

Introduction. A variety of problems which have been (or could be) studied by the use of matrix representations have some interesting properties in common. The kinds of problems which we refer to are exemplified by: the mathematics of radiation and cosmic ray counters, the theory of neutron production and absorption, mathematical genetics, experimental design, the mathematics of peck right, epidemiology, the ontogeny of neural nets, stereochemistry, communication systems, and a host of others.

This wide range of problems (or significant aspects of them) may be regarded as special cases of the following "generic" problem:

A certain number (n) of essentially "equivalent" objects are the elements of a system. Associated with every *ordered pair* of elements (which may be regarded as distinguishable) is the *affirmation or negation* of k relations. We may symbolize the affirmation of the pth relation between the ith and jth elements by the expression iR_pj. This, of course, need not imply that jR_pi, that is, the relation R_p need not be symmetric.

The negation of the pth relation between the ith and jth elements may be written as $iR_p^{-1}j$.

In addition to the *relations* associated with the ordered pairs of the system each *element* may be associated with one or more of m *intrinsic* properties. Thus the affirmation (of possession) of intrinsic property q by element i can be symbolized by the expression iP_q, and its negation by iP_q^{-1}.

The relations and intrinsic properties of the system may, in general, imply a dynamics which is reflected by a quantized temporal shift of relational and intrinsic property *affirmations* to *negations* or vice versa.

Matrix Representation. For any given problem of this kind our attention may be focused upon the "structures" implied by the relations between the elements or upon the dynamics of the system. In either case matrix representation can be used to advantage. Such a system can be described by matrices in the following way:

First, number the elements of the system from one to n in some arbitrary fashion. Assume for illustration that only one type of relation will be considered. In other words, for each ordered pair of elements i,j we will say either iRj or $iR^{-1}j$, i.e., i is related to j, or i is not related to j.

The set of all such relations can be conveniently described by an $n \times n$ matrix whose elements are either R or R^{-1}. Thus if a typical element of the matrix e_{ij} is R, then we will say iRj. If, however, the typical matrix element e_{ij} is R^{-1}, then we will say $iR^{-1}j$. Such a matrix can be called a *structure* matrix insofar as the relations between the elements imply a structure of the system. The matrix representation can be extended to include any number, say k, of different kinds of relations. In this case it would be necessary to use an $nk \times n$ matrix or $kn \times n$ matrices to represent the totality of relationships.

In a similar way the distribution of intrinsic properties among the elements of the system can also be represented by a matrix. If we consider a system possessing m different classes of intrinsic properties, then an $n \times m$ matrix will do the job. Such a matrix can be called a *status* matrix insofar as the distribution of properties among the elements of the system describes a status of the elements as compared to each other.

An Example. For the sake of illustration, consider the following problem. *How many structurally different compounds are represented by the formula $C_\mu H_\nu$* (only saturated hydrocarbons considered)?

If we picture such a compound in a conventional steric diagram, it becomes evident that every carbon atom in the molecule (except for the special case of methane) *is chemically bound to* one or more other carbon atoms. In the most general case such bonds may be single, double, or triple, but for saturated compounds only single bonds occur.

Now, if we let the integer 1 represent the relation *"has a single bond with"* and if we let 0 represent its negation, that is, *"is not bound to,"* then, by arbitrarily numbering the carbon atoms from 1 to μ we can write a series of propositions of the form $i1j$ and $p0q$. In words, these expressions mean "carbon atom i *has a single bond with* carbon atom j" and "carbon atom p *is not bound to* carbon atom q."

According to the procedure outlined above, the totality of relations among the carbon atoms can be represented by a $\mu \times \mu$ *structure* matrix.

For the trivial case of CH_4 (methane) we have a 1×1 matrix whose only element is 0. In other words, the only carbon atom in the molecule has no bonds with itself. Ethane (C_2H_6) is represented by the 2×2 *structure* matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The elements of the major diagonal of such a structure matrix will always be 0 since the statement "atom i has a single bond with atom i" is meaningless. The relation "has a single bond with" as used here is always reciprocal, so it follows that the structure matrix will always be symmetric around the major diagonal. Also note that the sum of the elements of the structure matrix is exactly twice the number of carbon atoms used in forming the "backbone" of the molecule. It follows then that

$$4\mu - 1/2 \sum_{j=1}^{\mu} \sum_{j=1}^{\mu} e_{ij} = \nu . \qquad (1)$$

The class of all pure saturated hydrocarbons having the formula $C_\mu H_\nu$ can, therefore, be represented by a class of $\mu \times \mu$ matrices. It must be noted, however, that the total number of isomers of the form $C_\mu H_\nu$ is considerably smaller than the number of different $\mu \times \mu$ matrices which can be used to represent them. This is due to the fact that the numbering of the carbon atoms is arbitrary and that renumbering does not change the structure. It is further complicated by the fact that renumbering also may leave the matrix unchanged.

For these reasons many matrices are equivalent in the sense that they represent the same substance. The original problem, therefore, of finding the number of different substances represented by the formula $C_\mu H_\nu$ reduces to the problem of *counting the number of subclasses of equivalent matrices making up the class of* $\mu \times \mu$ *matrices representing the formula* $C_\mu H_\nu$.

The fact is that this particular problem has been attacked and partially solved by number and group theoretical methods (Polya, 1936). These results are, of course, translatable into the language of matrices and, therefore, should have a direct bearing on all of the other phenomena subsumed by the aforementioned "generic" problem.

Other Examples. In the above example no *status* matrix is involved and the attention is focused on the *structure* of the system. However, as was mentioned in the introduction, the existence of a structure matrix and status matrix describing the same system frequently implies a dynamics. Accordingly, the dynamics of a neural net conforming to the postulates of McCulloch and Pitts has been described and analyzed by the use of matrix algebra (Landahl and Runge, 1946).

In this case the elements of the system are neurons. The *relations* appearing in the *structure* matrix are "can stimulate" and "can inhibit." The intrinsic properties associated with the status vector are "is active" and "is not active." In this paper it is shown that consecutive products of the status vector with the structure matrix give consecutive status vectors which represent the "state" of the net for different moments.

In a subsequent paper H. D. Landahl (1947) extends the results of the first paper and treats the "reverse problem," i.e., instead of describing the behavior of a given net he starts out with a given dynamics and defines the kind of net which will exhibit such behavior.

As a final example A. Rapoport (1949) and H. G. Landau (1950) use matrix theory in the study of the dynamics of chicken "societies" organized by the relation of "peck right."

The foregoing is not intended as a proof for the wide applicability of matrix representation, for this is well known. The point is that in the cases cited above the structure matrix represents, in one way or another, a connected system, a network, an organization, or "lacework" of simple relations. The structure matrix can, therefore, be interpreted as an important invariant of what may be loosely termed "group order." An extensive study of square matrices, therefore, with an eye to the structures they imply may lead to useful definitions of such vague concepts as organization, group stability, and so on.

Some important steps in this direction appear in recent literature (Luce and Perry, 1949; Luce, 1950). The authors define an organizational concept which they call a *clique* and the more general notion of an n-*clique*. Roughly speaking, a clique is a certain ordering of relations in a subgroup of a system. R. D. Luce and A. D. Perry show certain relations between the existence of cliques and certain properties of the matrices representing the system.

In addition, these authors apply matrix algebra to a simple "communication system" in a manner quite analogous to the Landahl-Runge treatment of neural nets. They arrive at a recursion formula for consecutive states of the system which is also analogous to the formula of Landahl and Runge.

Group Communication. The Problem. The following problem, although in a way more general than that approached by Landahl and Runge, Luce and Perry, is nevertheless treated in a similar manner. The statement of the problem and some of the definitions may, therefore, be expected to overlap somewhat.

Suppose that a group of n persons are asked to play the following game. Each member is told at the outset that he must send "messages" to certain other *specified* members of the group. On the other hand, he is not told who the recipients of the other members' "messages" are. In fact, at the beginning of the game he does not even know who, if anyone, will send messages to him. The point of the game is for each member to know the complete structure of their communication system. In other words, each member must know the destination of every other member's messages.

Accordingly, the messages themselves must contain such information. A typical message will read "The destination of C's messages is D, the destinations of B's messages are A and D, and so on." The expression "the destinations of B's messages are A and D" is actually a statement of all the information which member B possessed at the beginning of the game. Accordingly we shall refer to such expressions as *primary elements* of information.

Now since there are n members in the group, there must also be n primary elements of information. The game will end, therefore, when every member possesses n different primary elements of information.

The Method. If we number the members of the group from 1 to n in some arbitrary fashion we can proceed in the usual manner to construct a structure matrix which will represent the communication system. The matrix can be constructed by rows. To write down the jth row of the matrix we ask "Does member j send messages to member i"? If the answer is *yes,* then the element e_{ij} of the structure matrix is one. If the answer is *no,* then the element e_{ij} is zero. By repeating this procedure n^2 times the complete structure matrix can be written down.

For example, see the three member group diagrammed below in Figure 1 and a corresponding structure matrix as follows:

$$\begin{Bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{Bmatrix}. \qquad (2)$$

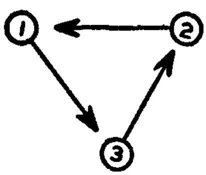

FIGURE 1

Notice that the diagonal elements of the structure matrix are all 1's. This would be as though each member of the group sends himself messages. This notation is useful in the operations which follow. We can think of it as though each number repeatedly "reminds himself of" or "remembers" the information which he has already obtained.

A status matrix describing the distribution of primary information throughout the group can also be constructed. This can be done columnwise. To write down the jth element of the ith column we must ask: "Does member i possess the primary element of information associated with member j"? If the answer is *yes,* then the jth element of the ith column is one. If the answer is *no,* then the jth member of the ith column is zero. This process defines the status matrix.

At the beginning of the game each member of the group will possess only the primary information associated with himself. In other words, all that he will know is where he sends his own messages. The status matrix at the beginning of the game will, therefore, be the identity matrix. The matrix will have 1's along the major diagonal and zeros everywhere else.

Successive States. As the game proceeds the information will begin to shuttle from person to person according to the pathways of the system. This will result in a constantly changing status matrix. Now, if the rate of flow of messages were in no way controlled, certain differences between the various people playing the game would begin to show themselves. Some members of the group may find themselves, for example, sending only one-third as many messages per unit time as other members. In order to remove such differences between the individuals from our considerations we will stipulate that each member must send messages along all of his permissible channels exactly once per unit time. The problem is now well defined and we can ask the first question, namely, "How long will the game last"?

Before going on to answer this question it should be noted that a slight change in the definition of the status matrix makes it possible for us to prove a useful theorem. To this end we will say that if the general element of the status matrix e_{ij} is some *positive integer,* then member i possesses the primary element of information associated with member j. However, as before, if e_{ij} is zero, then we shall say that the member i is ignorant of the primary element of information associated with member j.

Fundamental Theorem. The status matrix, *representing the distribution of primary elements of information among the members of the system after* t *units of time, is given by the* t*th power of the* structure matrix *in the sense of ordinary matrix multiplication.*

Proof: *Lemma.* If S denotes the structure matrix and I_t denotes the status matrix after t units of time then,

$$I_{t+1} = I_t S \,. \tag{3}$$

Let $e_{ij}^{(t)}$ denote the typical element of the status matrix after t units of time. Also, let s_{ij} denote the typical element of the structure matrix. According to the definition of ordinary matrix multiplication, we have

$$p_{ij} = s_{i1}e_{1j}^{(t)} + s_{12}e_{2j}^{(t)} + \cdots + s_{in}e_{nj}^{(t)}, \tag{4}$$

where p_{ij} is the typical element of the matrix product $I_t S$.

Since none of the elements s_{ik} or $e_{kj}^{(t)}$ is ever negative, it follows that neither the right-hand member of expression (4) nor any of its terms $s_{ik}e_{kj}^{(t)}$ can ever be negative. This implies that if the right-hand member of expression (4) is zero then all of its terms

$s_{ik}e_{kj}^{(t)}$ are individually zero. However, $s_{ik}e_{kj}^{(t)} = 0$ means either $s_{ik} = 0$, $e_{kj}^{(t)} = 0$, or both. But, according to the definition of the structure matrix, $s_{ik} = 0$ tells us that member i never receives messages from member k. From the definition of the status matrix $e_{kj}^{(t)} = 0$ tells us that member k (at the time t) *does not* possess the primary element of information associated with member j. If *either* (or both) of these conditions hold, it follows that member i *could not* obtain the primary element of information associated with member j *at the time* $t + 1$ *from member* k. If this is true for *all of the terms* in the right-hand member of expression (4), then it follows that member i *could not* obtain the primary element of information associated with member j *at the time* $t + 1$ *from any other member.* We must conclude then that, given these conditions, the general element of the status matrix at the time $t + 1$ must be zero, i.e., $e_{ij}^{(t+1)} = 0$. But, given these conditions, we see from expression (4) that p_{ij} is zero. Hence, for this case we find that $p_{ij} = e_{ij}^{(t+1)}$.

The only remaining possibility is for the right-hand member of expression (4) to be a positive integer. This means that $s_{ik}e_{kj}^{(t)}$ (for at least one k) is a positive integer. In order for this to be true we must have $s_{ik} = 1$ and $e_{kj}^{(t)} = +$ integer. But $s_{ik} = 1$ means that member i *does* receive messages from member k. Also, $e_{kj}^{(t)} = +$ integer means that member k (at the time t) *does* possess the primary element of information associated with member j. From this we must conclude that member i, at the time $t + 1$, *will* possess the primary element of information associated with member j. This implies that $e_{ij}^{(t+1)}$ must be a positive integer. But under these conditions p_{ij} is a positive integer, as a glance at expression (4) will show. Hence again $e_{ij}^{(t+1)} = p_{ij}$. Thus the lemma is proved. In other words, expression (3) is a valid recursion formula.

Now let us recall that the status matrix $\boldsymbol{I}_0$ (at the beginning of the game) is simply the identity matrix. Using expression (3), we get $\boldsymbol{I}_1 = \boldsymbol{I}_0\boldsymbol{S}$. But $\boldsymbol{I}_0\boldsymbol{S} = \boldsymbol{S}$; hence $\boldsymbol{I}_1 = \boldsymbol{S}^1$. It follows by induction that the fundamental theorem is true, that is,

$$\boldsymbol{I}_t = \boldsymbol{S}^t . \tag{5}$$

Expression (5) makes it possible for us to give at least a formal solution to the original problem, namely: "How long will the game last"?

If any element of the status matrix $\boldsymbol{I}_t$ is zero, then we know that at least one member of the group does not yet possess the primary information associated with at least one other member, and so the game is not finished. The game will end at the first moment when every element of the status matrix is non-zero. *The game will end, therefore after* T *units of time, where* T *is the smallest power of the structure matrix containing no zero elements.*

Adequate Systems. Not all groups will be able to end their game. Their channels of communication may be inadequate. Such inadequacy may be due to an insufficient number of channels or an inadequate arrangement of those which are available. We shall refer to the structure matrix of such an inadequate communication system as an *inadequate* matrix. Similarly, the structure matrix of an inadequate communication system can be called an *adequate* matrix. Some interesting details relating to these notions of adequate and inadequate matrices are developed by Luce (1950) in the section on *connectivity in a structure.*

In order for a structure to be adequate, it is *necessary* for every member to send messages to at least one other member and to receive messages from at least one other member. In fact, there must exist at least one path of channels from any given member of the group to every other member. If this were not so, that is, if no such path existed between member i and member j, for example, then the primary element of information associated with member i would never be available to member j. In terms of the structure matrix this means that there must exist some sequence of integers k_1, k_2, k_3, $\cdots$ k_m (not necessarily all different) such that s_{ik_1}, $s_{k_1k_2}$, $s_{k\,k_3}$ $\cdots$ s_{k_mj} are all 1's.

The Cycle. Since every member must send messages to at least one other member, it follows that *at least n* channels are *necessary* in order to construct an adequate system. In fact, this number is sufficient if the channels are laid out in a simple closed chain which includes all of the members of the group. We shall refer to the matrix of such a structure as a *minimal-cyclic* matrix.

Changing Structure. The game can be complicated somewhat by permitting the members to shift their channels of communication in some prescribed temporal order. The structure matrix, under these conditions, would change from time to time.

In order to describe such a system it would be necessary to write down a series of structure matrices S_1, S_2, $\cdots$, S_t, $\cdots$, where S_t represents the particular communication structure being used at the time t. We shall refer to such a series of matrices as a ***structure** series.*

The General Theorem. The status matrix, representing the dis-

tribution of primary information among the members of the system after t *units of time, is given by the matrix product* $S_1S_2S_3 \cdots S_t$, *in the sense of ordinary matrix multiplication.*

Proof: In proving recursion formula (3) we assumed that the structure matrix S was constant with time. But this assumption in no way affects the argument, since it deals only with the tth exchange of information. This permits us to write the slightly more general recursion formula:

$$I_{t+1} = I_t S_t . \tag{6}$$

But since $I_1 = I_0 S_1 = S_1$, the general theorem follows by a simple induction on expression (6).

Adequate Structure Series. We have already shown that certain trivial structure series (constant adequate structure) results in a finite game. We may now ask for more general conditions on the structure series such that it will be *adequate.*

Theorem. The maximum number of different structure matrices representing possible communication systems between n *labeled members is given by the expression* $2^{(n-1)n}$.

Proof: The 1's appearing in the ith row of a structure matrix tell us (by definition) to whom member i sends his messages. It may be that he sends no messages to anyone, in which case the only 1 appearing in the ith row would be element s_{ii}. But this can happen in only one way. It may be that member i *does* send messages to exactly one other member, in which case a 1 would appear elsewhere in the row. This could happen in $n-1$ different ways. In general, member i may send messages to k other members. This could happen in $\binom{n-1}{k}$ different ways, where $\binom{n-1}{k}$ is the number of ways in which we can choose k objects from $n-1$ objects, all different. The number of ways N in which a given row can be "disposed of" is given by the expression

$$N = 1 + (n-1) + \binom{n-1}{2} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-1} = 2^{n-1}. \tag{7}$$

It follows that the number of ways in which all of the rows can be disposed of together is $N^n = 2^{(n-1)n}$, which proves the theorem.

Definition: Any structure series which indefinitely repeats a subseries of structure matrices will be called a *cyclic structure series.*

Theorem. Any cyclic structure series which contains one or more adequate matrices is an adequate structure series.

Proof: If a positive integer appears in some element of the status matrix at some time t, then, by virtue of the fact that each member "reminds himself of" his already acquired information (every diagonal element of any structure matrix is 1), that particular element of the status matrix can never again be zero, for any time after t. It follows that if a given adequate matrix is applied to the changing status matrix every k units of time, then every element of the status matrix will be a positive integer in at most kT units of time, where T is the minimum power of the adequate structure matrix containing no zeros. Such a cyclic structure series must, therefore, be adequate.

Since the game always ends at the moment when zeros no longer appear in the status matrix, it follows that the game will last T units of time, where T is the *minimum* subscript of the structure matrix for which the product $S_1S_2S_3 \cdots S_T$ contains no zeros. That this product contains no zeros for *some* T is, in fact, the definition of an adequate structure series.

It is possible for every matrix of a structure series to be inadequate and still form an adequate structure series. For example, consider the structure series $S_1S_2S_1S_2 \cdots$, where

$$S_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } S_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

It is readily seen that the element e_{21} of structure matrix S_1 and element e_{12} of structure matrix S_2 will remain zeros for all powers of these matrices. This means that both S_1 and S_2 are *inadequate* matrices. The product S_1S_2, however, contains no zeros; hence, the *structure series is adequate.*

Efficiency. For a group of any given size certain structure matrices or structure series end the game faster than others. The *solution time* T is, therefore, a kind of measure of the efficiency of the communication system. The various adequate structure matrices and series can, therefore, be ordered in a preferential hierarchy.

If every member of a group could communicate directly with every other member of the group, only *one* interchange of information would end the game. However, such a communication system would require $n(n-1)$ channels. On the other hand, a minimal-cyclic structure has a solution time of $T = n - 1$, but it has only n channels. For some purposes we may be interested in the efficiency of a communication system as reflected by its economical use of channels. The number C of channels in a given structure is simply the sum of the elements of the structure matrix minus the number of elements in the major diagonal; thus we have

$$C = \sum_{i=1}^{n} \sum_{j=1}^{n} s_{ij} - n. \tag{8}$$

Now let us return to the structure matrix (2) of the three member group example. The status matrices associated with this structure are as follows:

$$I_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 2 & 1. \end{pmatrix} \tag{9}$$

The solution time of this structure is 2 units, since I_2 is the first status matrix containing no zeros. Notice, however, that three of the elements of I_2 are 2's. This means that each member of the group received one primary element of information on two different occasions.

For some purposes such a repetition of information may be considered an advantage insofar as information received many times may perhaps be more likely accurate.

We may, however, be more interested in an economy of messages. This may be particularly true when the system is such that each message is very likely to be accurate. In this case it would be ideal if the solution matrix contained *only* 1's.

From the above considerations it would seem that the ratio $\mathcal{M}$ of the number of elements in the solution matrix (n^2) to the sum of the elements of that matrix could serve as a useful measure of the system's efficiency in regard to message economy; thus

$$\mathcal{M} = \frac{n^2}{\sum_{i,j=1}^{n} e_{ij}}. \tag{10}$$

It should be noted, however, that the sum of the elements of the solution matrix does not give the total number of messages sent. This sum is usually larger than the number of actual messages. To interpret the meaning of the operations between the structure and status matrices accurately, we must say that a given member who has received a primary element of information k times by the time t will *repeat* that element of information k times in all of the messages he sends at the time $t + 1$. Actually, this simply implies that the sum of the elements of the solution matrix reflects a "weighting" of the evils of redundance. It is as though this sum were saying: "It is bad to send the same information three times, but it is more than twice as bad to send it six times."

So far we have three possible measures of efficiency: the inverse of the solution time T^{-1}, the number of channels per member C/n, and the redundancy ratio $\mathcal{M}$.

Minimal-Cyclic Structure. In a minimal-cyclic structure the most remote member from any given member is $n-1$ steps away. This means that the solution time T of a minimal-cyclic structure is $n-1$ units. The efficiency of such a system (in regard to speed) drops linearly with increasing size of the group.

This structure has exactly 1 channel per member so that in this sense it is perfectly efficient.

By elementary considerations we find that the redundancy ratio $\mathcal{M}$ of a minimal-cyclic structure is given by the expression

$$\mathcal{M} = n/2^{n-1}. \tag{11}$$

This means that the redundancy rises sharply with increasing size of the group.

Theorem. The redundancy ratio $\mathcal{M}$ *of any series structure* $S_1 S_2 S_3 \cdots$ *which consists entirely of minimal-cyclic matrices is always less than* 1.

Proof: In order for the redundancy ratio $\mathcal{M}$ to be 1 it is necessary that all of the elements of the solution matrix also be 1. The determinant of such a matrix, however, is zero. This means that the ideal solution matrix is *singular.*

It can be readily shown that a minimal-cyclic matrix is never singular. But the solution matrix is given by a product of the form $S_1 S_2 \cdots S_T$ in which all of the factors are non-singular matrices. The theorem follows from the fact that the product of a non-singular matrix with a non-singular matrix is also non-singular.

Partial Solution. An interesting variation of this game would be to stop it after a designated playing time. We would undoubtedly find that some structures had progressed further toward solution than others (as measured by the number of zeros in the final status matrix).

It may be that under certain circumstances a communication system of this kind is in constant danger of being completely blocked. It may also be that certain structures which are highly efficient according to the previously mentioned standards actually transmit most of the information just near solution time.

Such a structure would not be very useful in such "dangerous" circumstances, since its disruption before solution time would leave most of the members ignorant of most of the information.

Other structures not quite as efficient according to the previous standards may exhibit a rapidly growing spread of information in the system. If it were disrupted before solution time at least much of the information would have already been transmitted.

Also in this connection, if we permit negative numbers in the status matrices to represent false information and in the structure matrices to represent "chronic" liars, we would have situations analogous to the spreading and checking of false "rumors." Perhaps simple matrix operations can be devised to represent these phenomena.

Higher Order Information. So far the players of the game were all interested in only one question, namely, "What is the structure of the system"? In a *higher order* game the participants might be interested not only in what they know about the structure of the system, but also in *what everyone else knows* about the system.

The distribution of information in such a system can be represented by a matrix whose elements are also matrices. A discussion of such higher order games will appear in a subsequent paper.

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