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THE GEOMETRIZATION OF BIOLOGY.

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The twentieth century has witnessed a geometrization of physics, that is, a reduction of the basic concepts of physics to geometric concepts. The topological approach to biology, recently proposed and to some extent developed by the author, is a small step in the direction of geometrization of biology, but is unable to achieve the main purpose of such a geometrization of biology, namely, the reduction to geometric concepts of such purely biological concepts as ingestion, digestion, assimilation, etc. To achieve this purpose we must find geometric structures or spaces, in which different geometric properties stand to each other in the same formal logical relation, as the different concepts of biology stand to each other. If this were possible, then a set of geometric theorems could be "translated" by an appropriate "glossary" into a set of biological laws.

While not offering a solution to this problem, the present paper illustrates the possibility of such an approach on several examples. Certain new types of topological spaces are introduced, which are used for illustration purposes only. It is shown, however, how from a theorem about such spaces a verifiable biological prediction could be made, if these spaces were to be taken seriously.

A possible application to biology of E. Artin's theory of braids is outlined.

The study of mechanics began much earlier than that of any other branch of physics. Although Newton was the father of contemporary analytical mechanics, yet different mechanical laws, like those of the levers or those of flotation of bodies, have been known and were studied since the days of Archimedes. This historical situation developed among physicists the attitude to consider mechanics as the basic branch of physics. The fundamental concepts, and even the fundamental laws of mechanics, became to be regarded as something self-evident. Newton referred to his laws as the "axioms or laws of motion," and it was only at the turn of this century that

Henri Poincaré pointed out that these were neither self-evident axioms nor experimental laws, but what he called useful conventions.

Of the other branches of physics, the phenomena of sound were relatively early and easily shown to be of a mechanical nature. Thus acoustics became a branch of mechanics. The second half of the nineteenth century witnessed a similar "mechanization" of thermodynamics. Attempts to "explain" in mechanical terms the phenomena of light and of electromagnetism, especially the latter, proved, however, to be unsuccessful. When it was demonstrated that propagation of light is a periodic phenomenon, it was natural to interpret it as a wave motion in some invisible and even "imponderable" medium, and the concept of universal ether was created. As is well known, this concept became more and more fraught with difficulties and self-contradictions.

Yet all the great physicists of the 19th century believed—nay, were almost convinced—that all phenomena of physics are mechanical in their nature. Faraday thought of the electromagnetic field in mechanical terms; Maxwell found his equations by a series of unsuccessful attempts at interpreting electromagnetic phenomena as manifestations of mechanical ones. The equations were established and proved their worth by their enormous predictive value only, however, to show that they were the inspirational guess of a genius rather than a logical consequence of a mechanistic picture. Yet, outstanding physicists like L. Boltzmann and others continued the unsuccessful efforts to reduce Maxwell's equations to Newton's laws of motion.

Inasmuch as Maxwell's work established the identity of optical and electromagnetic phenomena, showing the former to be only a branch of the latter, the difficulty of interpreting mechanically Maxwell's equations only further increased the difficulties of a mechanical interpretation of optical phenomena.

Towards the turn of the century the notion began to crystallize that perhaps electromagnetic phenomena not only cannot be reduced to mechanical ones, but that actually they are the basic phenomena of physics to which all other phenomena, including mechanics, may be reduced. This idea was especially suggested by the discovery of the electromagnetic mass of the electron, predicted theoretically by H. A. Lorentz and subsequently verified by several experimenters.

The notion, however, did not look to be a happy one to the 19th century classical physicists. The notions of electromagnetic field, electric and magnetic vectors did not have the property of immediate intuitive self-evidence, which the mechanical notions of motion, mass, acceleration, and force seem to possess. It was not understood that this self-evidence of the mechanical concepts was only an apparent one, due to the centuries of familiarity of the physicists with those concepts. This self-evidence was nothing more than a habit of thinking; but even the great scientists have frequently found it difficult to free themselves from long established habits of thinking. The laws of human psychology are very much the same for the scientist and for the layman!

It was in 1909 that Hermann Minkowski, inspired by the profound discovery by Einstein in 1905 of the relativity of space and time, indicated a way of reducing physical phenomena to something which is outside of physics, even outside of any other natural science, namely, pure geometry. Minkowski showed that the basic concept of mechanics, and possibly of all physics, the concept of motion, may be interpreted as an orthogonal transformation of a system of coordinates in a four-dimensional hyperspace with one imaginary coordinate. Inspired by Einstein, this discovery in its turn eventually led Einstein to the creation of the General Theory of Relativity, in which not only the concept of motion, but also concepts of acceleration, mass, and force, and especially the notion of gravitation received a geometrical interpretation in a four-dimensional non-Euclidean hyperspace.

True enough, a *complete* geometrization of physics is still a relatively distant goal. One of the stumbling stones is again Maxwell's equations. However, the difficulties now appear to be of a different kind; they are more of a mathematical than of a conceptual nature. Beginning with the ingenious—though unsuccessful attempt by Herman Weyl (1920)—attempts at a reduction of electromagnetic phenomena to geometric concepts have continued, and Einstein's own later contribution (Einstein, 1953) seems to carry a great deal of promise.

But even to the limited extent that geometrization of physics has been achieved, it proved to be important. Not only has it shown an unusual predictive value both in physics and astronomy, but it is different from the old attempts to reduce physics to one of its branches in that the reduction of actual natural phenomena is made to purely mathematical concepts. Those concepts are creations of the human mind, and basically are therefore much more intuitive than the physical ones, even though a visualization of multidimensional spaces may be barred to us. Basically, what modern physics does, is to map the observable physical phenomena isomorphically onto an abstract geometric structure. Certain concepts of physics, like motion, acceleration, etc., map on purely geometrical concepts of coordinates, curvature, etc. The theorems of geometry, which establish metric relations between different geometrical concepts, then lead directly to laws of physics by the use of a sort of glossary or dictionary which shows the name of the physical concept that is to be substituted for the name of a geometric concept. A "translation" of physical laws into geometric theorems and *vice versa* is thus possible.

Compared with physics in age, biology is almost a newborn baby. Whenever we experience outside of ourselves any manifestations of life we do it only through its physical manifestations. Certain introspective psychological experiences, as well as the more immediate contact with life than with the nonliving, may have led some biologists to assume that phenomena of biology are basically different from those of the nonliving physical world. An utterly useless and time-wasting argument ensued between the vitalists and the mechanists, which are now better called physicalists, as to whether phenomena of life are something *sui generis,* nonphysical in nature, or are basically reducible to the laws of physics. As remarked above, the only objectively scientific study of life can be made through the study of its physical manifestations. This holds also for the so-called mental phenomena. We can know what another person thinks only by his telling us, or writing it, which are both physical acts, or we may infer about his thoughts from some other of his overt behavior, which again must be physically manifested if we wish to observe it. Therefore the scientific study of biology has become a study of different very special physicochemical situations, with the applications of the methods of both experimental and theoretical physics. This approach has proved to be of tremendous success. As to whether all phenomena of biology can be explained in terms of contemporary (1955) physics is a question which cannot be answered by any general speculation. All we can say that attempts at such explanation, when made competently, have been hitherto crowned with success. It is, however, impossible to deny that a biological phenomenon may be discovered which cannot be explained in terms of physical laws known at present. But far from proving anything "unphysical" about life, such a discovery will merely mean the need of an extension of physics (Rashevsky, 1934, 1955b), just as the impossibility of explaining some spectroscopic phenomena in terms of classical physics has led to its extension by introducing quantum mechanics. No one will ever call quantum mechanics "unphysical."

If physics, in its present or in its extended form, is to be reduced to geometry, then it follows from all the above that eventually a geometrization will be also the fate of biology. When we come to consider this possibility more closely, we notice one very essential difference between some laws of biology and the laws of physics. All phenomena of physics are quantitative in nature. They not only *can* be measured, but even more, unless they are measured, very little if anything significant can be said about them. Many biological phenomena are also quantitative in nature, and the biologist is now becoming used to both measurement and mathematical analysis. But very many biological phenomena, and perhaps the most basic of them, are not quantitative but *relational.* Yet very definite statements can be made about them, and their importance is unquestionable.

As has been pointed out elsewhere (Rashevsky, 1954), the response to a stimulus by a paramecium, followed by a locomotion toward food; the ingestion of the food, followed by digestion; the resulting absorption and assimilation of the digested products on one hand, and the defecation of undigested residue on the other—all those individual biological processes stand in basically the same relation in the paramecium as they do in a human being. The processes of stimulation, locomotion, digestion, etc., in a higher organism are much more complicated than in a protozoan and consist of many more partial "subprocesses." But the corresponding complicated processes in a higher organism can be mapped in a many-to-one way onto the processes of a one-celled organism in such a manner that the basic relations remain invariant.

If we wish to describe mathematically the above situation, we need a different mathematical apparatus from the one used hitherto in physics or in mathematical biology. Such an apparatus is provided by topology, which is a geometry of relations rather than of quantities. To use this apparatus we must find some method of describing an organism in terms of some topological spaces or complexes. The word "organism" itself suggests that a graph used as an organization chart of the organism may be employed to advantage. Such considerations led us to the formulation of a general principle, the principle of bio-topological mapping (Rashevsky 1954):

> The graphs which represent the organization charts of different organisms are such that they all can be mapped on the graph or graphs of one, or at most very few, primordial organisms. The graphs of all organisms are obtained from the graph of the primordial organism by the same geometrical multiparametric transformation, the graphs of different organisms corresponding to the different choices of the parameters.

It has been shown that a topological biology developed on that basis with the addition of some special hypotheses has a definite predictive value. Not only is it possible to derive the known facts —that the more complex organisms possess a lesser regenerating ability for lost organs (Rashevsky, 1955c) and possess a greater adaptability (Rashevsky, 1955d), but it is also possible to derive the total number of possible different organisms (Rashevsky, 1954; 1955a). The number thus found is about 100 times the actual number of known species. Whether this means that nature has produced hitherto only about 1% of all possible organisms or that some of the special hypotheses must be modified within the general principle cannot be said at present. The important thing, however, is that topological biology predicts something that can be verified by observation and thus offers a stimulus to new experimental work. This is even better illustrated by the prediction of a relation between the number of tissues, the number of organs and the number of distinct biological functions (Rashevsky, 1955d), as well as by an estimation of the total number of hormones to be discovered in organisms of different degree of differentiation (Rashevsky, unpublished). An extension of combinatorial topological considerations to organic molecules (Rashevsky, 1955e; Karreman, 1955) has led to biochemical conclusions which are in principle verifiable.

The above approach is to some extent a move in the direction of a geometrization of biology. How does it relate to the geometrization of physics? Topological considerations have hitherto played no part in the discussion of the four-dimensional hyperspace onto which the physical phenomena are isomorphically mapped. The metric properties of the four-dimensional hyperspace is all that mattered. In many instances the metric of a space determines its topology. If this were true generally, we would expect that topological relations of the four-dimensional universe would be derived from its metric, such that they may lead to the description *modo geometrico* of the basic biological relations. We then would have the following possibility of reduction: biology → physics → geometry. However, as we have seen on a previous example (Rashevsky, 1954, p. 345), a situation may be conceivable in which life is a manifestation of "local" topology of the universe, in such a way, that while biological phenomena will always follow the laws of physics, they may not be reducible to those laws, while both physics and biology will be reducible to geometry.

However, the geometrization of biology, as attempted by topological biology, is still something very different from the geometrization of physics. In the former we do not interpret in geometrical terms any of the biological functions, such as digestion, assimilation, etc. We may perhaps say that we eliminate from biology the elusive concept of "organization," and substitute for it the geometrical notion of the topological properties of the graph of an organism. But except for that, we still manipulate with such concepts as digestion, assimilation, locomotion, stimulation, etc. We consider those concepts as given, and all we do is to study topological relations between those concepts, in an abstract space.

The geometrization of physics goes much further. It interprets the basic concepts of physics in geometrical terms. It eliminates, in a sense, those concepts from physics. It does not merely seek formal geometric relations between different masses, velocities and accelerations. This has been done long ago by classical kinematics and does not constitute a mapping of physics onto geometry.

If we wish to achieve the same thing for biology, we must go much further than topological biology has done so far. It seems that combinatorial topology is hardly the adequate apparatus for our purposes and that recourse to set topology will have to be made (Rashevsky, 1955d).

The whole universe, physical and biological, is a set of "elements," the word element not being used in the chemical sense.

In order to avoid confusion with the chemical connotation of the word element, we shall use it in quotation marks when set theoretical connotation is implied, and without quotation marks when the chemical connotation is used. Perhaps the most basic way of looking at it is to consider the universe as a set of world lines. We may, however, consider as "elements" of the sets, groups of the world lines, or their points of intersections, or other configurations. Different chemical reactions may be considered as the "elements" of the set. In any case, however we define the "elements" of this set, the biological phenomena are a subset of it.

What are the characteristic properties which make us recognize an organism as such? Perhaps the most basic thing about an organism is that it selects certain "elements" from the inorganic environment even if those elements form parts of other subsets. Once a selection is made, those selected "elements" are organized in a definite pattern eventually becoming themselves "elements" of the organism, and leading to a duplication of the latter. In this case the "elements" are the different chemical molecules, radicals or complexes.

This process of selection is accompanied by a loss of some "elements" of the organism. This loss may be a complete breakdown into waste products, or it may be constituted by the secretion of a molecule of a digestive enzyme which "attacks" a food particle and selectively breaks off it the desired molecular configuration. Usually the secretion of a digestive enzyme is not considered on the same level as a catabolic breakdown. But *logically* they both resemble each other, and they both may be necessary for the process of selection. For while in some microorganisms, like nonmotile bacteria, the selection goes on at a purely chemical level; in higher organisms it is more complicated. The sight of possible food; the locomotion towards it; the breaking up of food either by teeth, or by hands, or by hand-made machinery; the picking up and consumption of the proper parts of the food—all this constitutes a process of selection, and for many such processes in higher organisms the energy released by catabolic processes is needed. Thus everywhere in the process of selection some loss of its constituents by the organisms seems essential.

Any selection process implies rejection of the unwanted material. Digestion, followed on one hand by absorption, on the other by defecation are only the logical aspects of the selection process.

Thus these three very basic biological functions may be described in terms of selection. And as we have just said, a number of other biological functions, which acquired typically biological names, are also basically logical aspects of the process of selection.

If we could interpret selection geometrically—or more specifically topologically—we would thus reduce some biological concepts to geometric ones.

Let us make the next logical step. Selection implies the division of the "elements" of some subset in two classes: those selected and those rejected. Division in two classes is not at all uncommon in topology. Thus a point of the one dimensional space of real numbers induces a Dedekind cut, and, therefore, a division of all other points in two classes. A closed Jordan curve in E^2 divides all points of E^2 in two classes: the inner and the outer ones. Both cases are examples of a subset M of a space S dividing $S - M$ in two classes. But those simple cases do not help us any. As we have seen, the division by the organisms of the set of "elements" which constitute the environment of an organism in two classes is contingent on the organism, considered as a set, losing some of its own "elements." Moreover, if one or more of the selected "elements" are added to the subset M which is the organism, the relation between that subset and the complement $S - M$ remains the same: the organism continues to select. That second property is exhibited in the following example. Let M be a subset of S not closed in S. Then M divides all the points of $S - M$ in two classes: those that are the limit points of M, and those that are not. Except for the very special case in which the first class is a degenerate subset, when some of the limit points of M are added to M the resulting subset M' still divides the points of $S - M'$ in two classes: those that are limit points of M', and those that are not.

A particular case of the above offers some remote analogy to the first necessary property, that of a loss by breakdown. Let S be the space of real numbers and let M be the subspace obtained from S by omitting all rational points in the closed interval $(0, 1)$. All rational points of S within that interval are limit points of M; those outside are not. The subspace M is not connected, however, and this loss of connectivity may be regarded as a remote analogy of a breakdown process. In this example there is no division by M of $S - M$ in two classes, that is, no selection, without loss of connectivity of M.

The above examples have, of course, no scientific value whatsoever. They are used only to illustrate a point which, if given an abstract formulation, may be difficult to make clear. Namely, if we can find a topological space, such that its different topological properties stand in the same logical relation to each other as do different basic biological concepts, then by substituting the names of those biological concepts for the names of the corresponding topological properties, we could, using again a glossary, translate every theorem about such a space into a corresponding biological law, just as it is done in the geometrization of physics. The purely mathematical study of the properties of such a properly chosen space would yield us all the laws of biology, both already known and those not yet discovered. If this ever becomes possible, it would indeed be a triumph of geometry, more generally, of pure mathematics.

This idea, suggested by the twentieth century physics, may appeal to some and not appeal to others. But even to those to whom it may appeal, it still is of no use. You cannot do much with a bare idea unless you at least show how, in principle, the idea *might* be realized. Here we must consider two logical possibilities.

Topology is still a young science; yet, it has already ramified in many directions, and a very great variety of its aspects have been studied. It is not precluded that the necessary types of spaces or other topological structures have already been studied and only need to be taken over, but that thus far we have failed to perceive the proper possibility. After all, the basic apparatus of non-Euclidean geometry has existed for a fairly long time; yet, it took Einstein to see that phenomena of physics are isomorphic to some of those already well known geometric properties. The rational problem-solving and the reduction of new situations to already known ones has psychologically and biophysically some similarity with the discovery of a "hidden picture" as has been shown elsewhere (Rashevsky, 1948; Chapt. XLII). Thus, we may hope that an inspiration by a mathematician or a mathematical biologist may indicate the solution, which will appear rather obvious after it is discovered.

But we cannot discount the other possibility, namely, that appropriate topological spaces have not yet been studied and that the needs of biology may provide a stimulus to the topologist to make new discoveries in his field, just as the needs of physics have provided in the past the stimulus for new mathematical discoveries. Gauss has actually credited physics with many of his purely mathematical inspirations.

But here again we must be somewhat more definite and therefore, even without hoping to formulate at present the necessary purely mathematical problem and to describe the necessary type of topological space, give at least a few examples of the kind of thinking that may be required. Those examples we shall give again *as an illustration only*. Their actual biological and mathematical values are nil. But they may serve, as the French say, "pour fixer les idées."

Let us return again to the basic logical property of the organism. Certain "elements" of the environment are selected, that is, begin to belong to one of two classes if some "elements" of the organism itself are detached from the latter. Considering again the organism as a subspace M in the space S and considering the "elements" as points in these spaces, we may translate the above logical property into topological language. In order that M should induce a division of the points of $S - M$ in two classes, it is necessary that some points of $S - M$ would possess some property P with respect to M, which others do not possess. In order that the division of the points of $S - M$ in two classes becomes possible only after M loses some "elements," some points of $S - M$ must acquire the property P with respect to M only after certain points of M are removed. The simplest example of such a situation is offered by a closed Jordan curve, from which one point, α (Figure 1), is removed. Let such a Jordan curve with a point removed be the subspace M in the space $S = E^2$. It is actually a Jordan arc, the ends of which are separated by the point α only. Thus M is a single connected subspace of E^2. The point α does not belong to M, but belongs to $S - M$. Let property P consist of making M connected. Since M is already connected, therefore α does not possess property P. But if we remove from M any other point, say β (Figure 1), then M becomes non-connected, and α acquires the property P.

Instead of a closed Jordan curve with one point removed, consider the structure shown in Figure 2, in which the points α_1, α_2, and α_3 are removed. As a subspace M of $S = E^2$, it is connected. If, however, we remove the point β from M, it breaks up into four components, and the *set of points* $(\alpha_1, \alpha_2, \alpha_3)$ acquires the property P. Thus the removal of one point, β, makes M divide the different subsets of $S - M$ in two classes: one class, consisting only of one element—the subset $(\alpha_1, \alpha_2, \alpha_3)$—has the property P; the other does not. The subsets of the two different "classes" are not all disjoined. Thus the subset $(\alpha_1, \alpha_2, \alpha_4)$ does not have property P, but $(\alpha_1, \alpha_2, \alpha_4) \cap (\alpha_1, \alpha_2, \alpha_3) \neq 0$.

In a similar space like the one shown in Figure 2, but having instead of 4 lines $n + 1$ lines connected by the point β, the removal of the point β will make a set of n points of $S - M$ acquire the property P, where n is an arbitrary positive integer.

We shall call such subspaces of E^2, as those shown in Figure 2, λ_n- spaces. Now connect an infinite number of λ_n- spaces in a manner shown in Figure 3, in which all $\lambda_n = \lambda_2$. Considered as subspace M of $E^2 = S$ the space thus obtained is connected, noncompact, dense in itself and nowhere dense in S, and not closed in E^2, since all the α points are its limit points. It satisfies the four Hausdorf axioms. It has the property that the removal of any β_i makes a set of points $(\alpha_i^1, \alpha_i^2, \ldots \alpha_i^n)$ acquire the property P. Any subspace of S that has this property we shall call a λ - space. The number of points β_i is countably infinite, and so is the number of sets of $S - M$ that can acquire the property P.

We can make the λ - space shown in Figure 3 bounded in $S = E^2$ and derive a more complicated λ - space. To achieve this, take the spiral with the equation in polar coordinates ρ, θ:

$$\rho = \frac{\theta - 1}{\theta}; \quad \theta \geq 1. \tag{1}$$

At equal, or unequal intervals, along the line (1), which begins at $\rho = 0$ and approaches asymptotically the circle $\rho = 1$, attach to it λ_n- spaces, as they are attached in Figure 3 to the line AB, (Figure 4), making the size of each λ_n- space smaller, as the windings of the spiral become closer, so that no λ_n- space would be intersected by the line (1). Then take spirals given by:

$$\rho = \frac{\theta - \left(1 + \frac{\pi}{m}\right)}{\theta}; \quad \theta \geqslant 1 + \frac{\pi}{m}, \tag{2}$$

where m is an arbitrary integer different from zero. The spirals (2), obtained for different values of m, lie all between the windings of spiral (1) (Figure 5). Attach to each spiral (2) a λ_n - space,

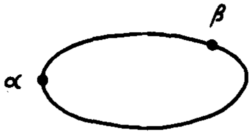

FIGURE 1.

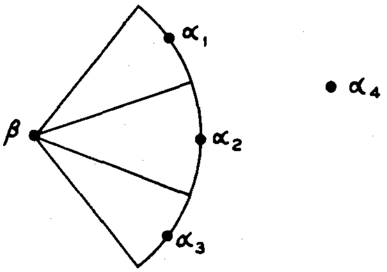

FIGURE 2.

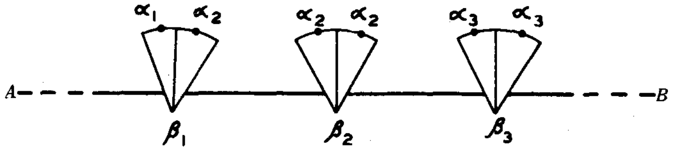

FIGURE 3.

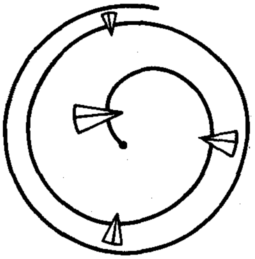

FIGURE 4.

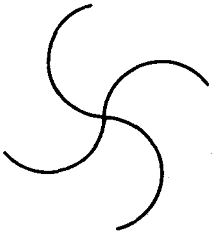

FIGURE 5.

at finite intervals, decreasing the size of λ_n - spaces as the m increases and as the spirals come closer to each other. If m increases to infinity, in the limit, we obtain a λ - space, which lies all within the circle of radius 1. It has a countably infinite number of β points and there is a countably infinite number of sets of points of $S - M$ ($M = \lambda$; $S = E^2$) which can acquire property P. The λ - space in question is not compact because a sequence of equidistant points along any of the spiral lines does not have a limit point. As subspace M of E^2, the λ - space is connected, dense in itself and nowhere dense in $E^2 = S$. It satisfies the four Hausdorf axioms.

In the λ - space considered above, the cardinal number of the set of β - points, as well as of the set of sets that can acquire the property P, is $\aleph_0$, whereas the cardinal number of the points of the whole λ - space is $\aleph$, that of a continuum.

We now shall construct a different space M, in which the removal of one point makes an infinite set of points of $S - M$ acquire property P. Consider the family of circles in E^2:

$$(x - \alpha)^2 + y^2 = \alpha^2, \tag{3}$$

where $\alpha = \frac{1}{2}$ or α is an irrational number between 0 and $\frac{1}{2}$.

The cardinal number of the set of such circles is $\aleph$. Each circle intersects the line $y = 0$ at $x = 0$ and at the point $x = 2\alpha$. Remove in each circle the point $x = 2\alpha$. The space M obtained by this removal from the family of circles (3), considered as a subspace of $S = E^2$, is connected. Removal of the point 0, which we shall designate by β, from M makes it not connected, and in order to make it connected with the point 0 removed, we must add to it all the points $x = 2\alpha$. Hence the set of all irrational points in the interval (0,1), whose cardinal number is $\aleph$, acquires now property P. We shall call this space M a λ' - space.

Let us now consider the family of circles (3) not in the plane $z = 0$, but in a plane $z =$ irrational number, and $0 < z < 1$. Remove in each plane all the points of intersection $x = 2\alpha$ and connect the outermost circles, those corresponding to $\alpha = \frac{1}{2}$, by any line l parallel to the z - axis, except the z - axis itself, passing through that outermost circle.

Such a set of circles with points removed as indicated, considered as a subspace M of E^3, is connected and bounded in E^3. The removal of any point β with irrational z - coordinate on the line l destroys the connectedness of M, and the set of points 2α which correspond to that z - coordinate acquires the property P. The space so constructed we shall call the λ'' - space.

The following theorems can be readily proved about the λ, λ' and λ'' - spaces.

The intersection of two λ, (λ', λ'') spaces is not a λ, λ' or λ'' space, and is always not connected.

As we have emphasized already, the above examples of "fancy" topological spaces are of no value either biologically or topologically and are used only to illustrate a point. We shall now use them for such an illustration.

An organism, as we have seen, selects from the environment some "elements" which have some special properties with respect to that organism. The selection of a set of "elements" becomes possible only after the organism loses some other set of its own "elements," and, in general, a specific set must be lost in order to make the selection of a given set possible. The selection, as we also have seen, is logically equivalent to inducing the division of all the "elements" of the environment in two classes: those that possess certain properties, and those that do not.

The λ, λ', and λ'' - spaces are, from a logical point of view, selecting the specific sets from the environment $S - M$ upon losing other specific sets. We have here a description of a biological phenomenon in topological terms. Or we may put it this way. If we make the "elements" of an organism, which must be lost in order to make a selection from the environment, correspond to the β - points of the λ, λ', or λ'' - space and the selected "elements" to the α - points, then the logical relation between the loss of "elements" by an organism and its selection activities are mapped isomorphically on the topological relations between the α and β points.

In an organism the loss of an "element" is compensated by the addition of the properly selected "elements" of the environment. If, after we remove a β - point, we add the corresponding α - points, we compensate by this addition for the loss of the β - point and restore the original connectedness of the λ, λ', or λ'' - space.

Now, to continue the illustration, let us just for a moment imagine that the λ, λ' and λ'' - spaces would be used not only for illustration, but so to say, taken seriously.

The selection of "elements" of the environment through loss of some of its own "elements" is a basic property of the organism. We have seen that an intersection of two λ, λ', or λ'' - spaces is not a connected space, and it does not possess the property of selection which the λ, λ', and λ'' - spaces possess. Let us translate the theorem into biological language. It reads thus: the common part of two organisms is not an organism and does not itself select the proper "elements" from the surroundings. Since a cell is an organism, we may restate the above as follows: The common part of two cells does not possess the ability of selecting proper elements from the environment by losing some of its own elements.

But the only possible common part of two cells could be the cell membrane which separates two adjoining cells. Hence from topological considerations, we reach the biological conclusion that the cell membrane in such cases does not make a selection of proper material from the environment in a manner which the cell as a whole, and possibly some of its parts, do. Whether this is true or not, it is certainly an experimentally verifiable conclusion.

The above example illustrates how the approach suggested in this paper, if properly made, can lead to verifiable predictions.

Very many objections can be raised against the λ, λ', and λ'' - spaces. Inasmuch as they are used only for illustration, we do not need to worry about the numerous possible objections. One point, however, may be mentioned perhaps as possible suggestion for the directions future research may take.

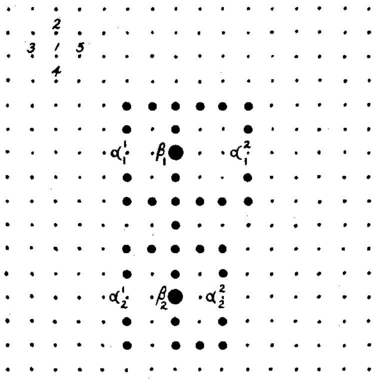

FIGURE 6.

The number of "elements" of which an organism is composed is very large but finite, whereas the number of points in the λ, λ', and λ'' - spaces is infinite. Moreover in the λ - space the cardinal number of all points is $\aleph$, while the cardinal number of β - and α - points is $\aleph_0$. We may say that only an infinitesimal fraction of the total number of points of the λ - space take part in the "selection" process. This is certainly not so in a biological system.

It seems to be desirable to study spaces with a finite number of points. As an example used again for illustration purposes only, let S be the space formed by the points in a plane arranged in an infinite square lattice (Figure 6) and in which the neighborhoods are defined as follows: the neighborhood of a point consists of that point and of the four adjacent points which lie on the lines of the lattice. Thus the neighborhood of point 1 in Figure 6 is constituted by the points 1, 2, 3, 4, 5. With neighborhoods so defined, S is connected because there exists no partition $S = A|B$ such that $A \cap \bar{B} = \bar{A} \cap B = 0$. The subspace M, which consists of the points marked by heavy dots, is also connected and has a finite number of points. The removal of point β_1 from M makes the points α_1 and α_2 of $S - M$ acquire the property P; while the removal of the point β_2 makes the points α_2^1 and α_2^2 of $S - M$ acquire the property P. More complicated spaces of that kind can be readily constructed.

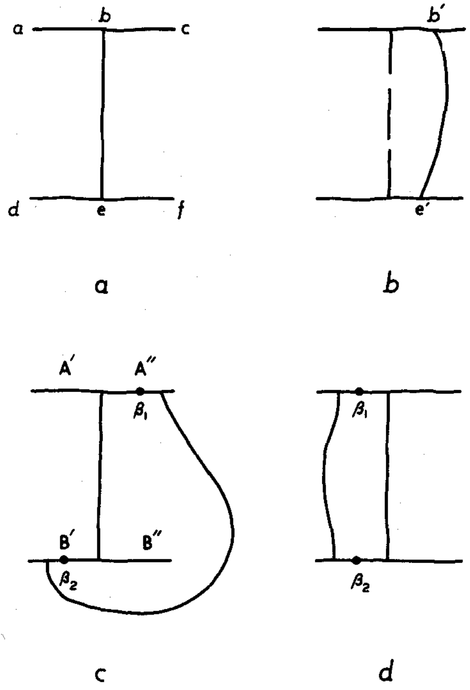

FIGURE 7.

The following last example shows the possibility of having spaces which, as a result of a selection followed by addition of the selected "elements" (assimilation in biology), may *duplicate themselves*. The biological interest of such spaces is quite obvious.

Consider in $S = E^2$ the space M shown in Figure 7a. From the point of view of combinatorial topology, M is a tree with the bicenters b and e (Konig 1936, p. 64). We shall consider it, however, from a set topological point of view as a subspace of $S = E^2$. M is irreducibly connected about its subspace

$$C = A \cup B\,, \tag{4}$$

where A and B denote respectively the segments ac and df:

$$A = ac; \quad B = df\,. \tag{5}$$

Hence the segment be, which we shall denote by D, may be defined as

$$D = M - C\,. \tag{6}$$

The end points of D are cut points of A and B respectively. They divide A and B each in two components: A', A'', and B', B''. If we remove from M either a point or a nondegenerate subset of D which does not contain the cut points of A and B, then M loses its connectedness, and any Jordan arc in $S = E^2$ which joins the sets A and B has the property P. In Figure 7b, line $b'e'$ has, for example, the property P.

Let us, however, remove from M two points β_1 and β_2, which are not end points of M and which are contained in A', A'', B', or B'', and such that β_1 and β_2 are not contained both in the same one of the above four subspaces. Thus in Figure 7c, β_1 is contained in A'', β_2 in B'; in Figure 7d, β_1 is contained in A', β_2 in B'. The removal of β_1 and β_2 from M separates M into three components, one of which contains D [expression (6)] and the other two which do not contain it. Denote the first component by M', the other two by M'' and M'''. The component M' is homeomorph with M; the components M'' and M''' are not.

Now, after the removal of β_1 and β_2 from M, any Jordan arc in $S = E^2$ which connects M'' and M''' and whose end points are cut points of M'' and M''' has the property of reconstituting a second space M_1, which is homeomorph to M. After performing thus a specified operation, which involves the loss by M of two properly specified points β_1 and β_2 and the addition of a Jordan arc from a set of arcs selected by the loss of β_1 and β_2, we now have two spaces M' and M_1, each homeomorph to the original M. In Figures 7c and 7d such arcs are, for example, $b'e'$.

Thus, removal of two properly located points from M imposes on certain Jordan arcs of $S - M$ the property of duplicating the space M. The process can be repeated again with M' and M_1 indefinitely.

It is true that the sizes of the subspaces which correspond to A', A'', B', and B'' decrease with each "reproduction." However, we are interested here not in the metric, but only in the topological properties. And topologically all the "daughter spaces" are homeomorph to the original space. Moreover, in spite of the reduction in size, the cardinal number of the points contained in the "daughter spaces" is always $\aleph$.

By study of appropriate topological spaces, it thus may be possible to map the properties of the simplest conceivable organism onto a proper topological space. Using then the principle of biotopological mapping, it may be possible to construct, by proper transformations, more complicated spaces, which map continuously in a many-to-one manner on the simpler space, and the study of the topological properties of those spaces may lead us to the discovery of new biological properties of multicellular organisms.

Continuing our fantastic excursion into the possible future of topological biology, we shall now take a look at another branch of topology which, perhaps, may also be destined to contribute its share to mathematical biology.

With the present-day reduction of physics to geometry, the physical events are represented geometrically by the intersections of the world lines of different particles. The metric characteristics of those intersections, or as the physicists call them, space-time coincidences, describe also the metric characteristics of the physical events. Inasmuch as an organism is composed of physical particles and obeys the laws of physics, the living phenomena are also representable by the intersections of world lines of the particles of which the organism is built. However, the characteristic and basic properties of life being of a relational character rather than of a metric one, it is the topological relations between the intersections of world lines that are important now. The branch of topology which comes close to the study of such types of relations is the theory of knots (Reidemeister, 1932) and especially its subdivision, E. Artin's theory of braids (Artin, 1925).

Artin studies braids in a three-dimensional space, but an extension to four dimensions seems to be natural and worth trying. Since again we shall use here some elementary notions of the theory of braids only for purposes of illustration, we shall confine ourselves to the three-dimensional case, studied by Artin. In fact, we shall consider even for simplicity only a degenerate two-dimensional case.

A braid is basically a set of lines interwoven in a regular manner. Consider the three lines of Figure 8. Let them first run parallel and then, at a certain point, begin an orderly interweaving. Such an interweaving constitutes, as Artin has shown, a group. The elements of that group are defined in the following manner.

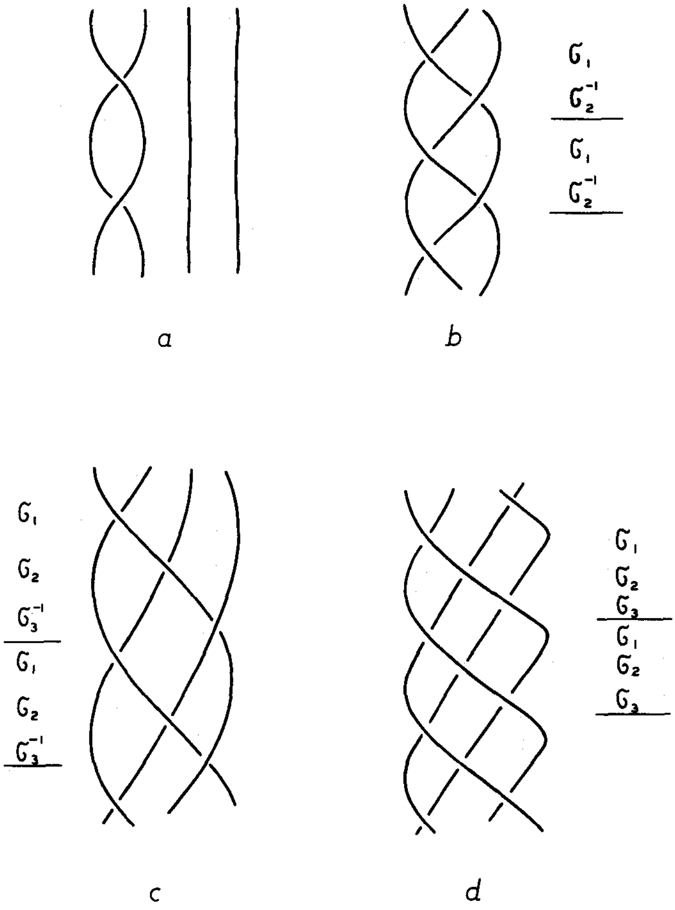

FIGURE 8.

Let us number at any place of the braid the lines from left to right, 1, 2, ... If the first line crosses the second *over* it, as in the upper crossing of the Figure 8a, we denote this process by σ_1. If the first line crosses the second *underneath,* as in the second crossing in Figure 8a, we denote this process by σ_1^{-1}. Clearly σ_1^{-1} is the inverse of σ_1, for, as seen from Figure 8a, a successive application of the two, $\sigma_1\sigma_1^{-1}$, results in the reestablishment of a situation which is homotop to the initial situation. It is seen from Figure 8a that a mere homotopic deformation reduces 8a to just a set of parallel lines which do not cross at all. In a similar manner the relation between the second and the third line are denoted by σ_2 and σ_2^{-1}, etc. If there are n lines, the number of σ's is $n-1$, the last being σ_{n-1}, which describes the relation between the $(n-1)$st and nth lines. The σ's constitute the generators of a group, the braid group, and every braid can be described as a power of proper products of σ's. Thus what Artin refers to graciously as the ordinary ladies' braid is given by

$$(\sigma_1\sigma_2^{-1})^m. \tag{7}$$

That this is so is intuitively clear from Figure 8b.

Figure 8c represents a braid; the group-theoretical expression for which is

$$(\sigma_1\sigma_2\sigma_3^{-1})^m, \tag{8}$$

while Figure 8d represents a braid

$$(\sigma_1\sigma_2\sigma_3)^m. \tag{9}$$

The two examples show that the braid group is noncommutative. It is readily seen that braids of any complexity can thus be represented.

We shall considerably simplify the examples to follow by considering a special case, discussed by Artin, namely, when it does not matter whether one line is above or below the other. Then the σ_k^{-i}'s are the same as σ_k^{i}'s. The braid of Figure 8b now becomes

$$(\sigma_1\sigma_2)^m \tag{10}$$

and is represented in Figure 9.

Artin considers all the lines of a braid as indistinguishable, which is the case when they are just abstract lines. If, however, each line represents the world line of a particle, then if the particles are distinguishable physically, the lines must be distinguishable also. In a drawing we may represent them in different colors, or by full, broken, and dotted lines, etc.

A braid may be deformed homotopically and the metric relations in it, such as the actual distance along a line between points of intersection, will change. But the topological properties expressed by (7) – (10) remain invariant. It is, therefore, rather natural to consider the relational properties of the organism as corresponding to the topological properties of the four-dimensional braid of world lines of the particles, of which the organism is composed. A braid being a special kind of a knot, it may well turn out to be literally true that life is a knotty problem!

It is, however, readily seen that the concept of an organism as a simple braid of world lines is utterly inadequate. An organism undergoes continuous catabolism and anabolism. While the chemical constitution of an organism may remain relatively constant, no constituent atom or molecule remains in the organism for any length of time. Some molecules are continuously lost and are just as continuously replaced by other similar molecules. In terms of a braid, we have a braid which is constantly unwoven and rewoven from other threads, something like a "Belgian lace." This, at first glance, seems to complicate the topological problem tremendously. Actually it does not.

At any point of a braid we may "pull out" a thread or line homotopically without disturbing the topological properties of the braid. Thus the braids of Figures 8b and 10a are topologically the same, being homeomorph. We may make the loop larger and larger, and in the limit move the point a (Figure 10a) into infinity, still preserving the topological properties. In the limit we obtain the situation shown in Figure 10b, where the braid loses a line (l_1) and gains one (l_2). This can be done to any line of the braid at any point, except a point of intersection. In a finite four-dimensional world even the removal of the point a to infinity is not necessary. A mere removal to a sufficient distance is enough.

Thus in our following discussion we may disregard the constant unweaving and reweaving of the braid and consider the classical form of Artin.

An organism usually grows when it assimilates more molecules than it loses to the environment. In terms of a braid this means that more lines are woven in than are unwoven. In the situation shown in Figure 10, for each line lost there is one gained. We need, therefore, consider only the weaving in of the excess lines. This can be done in many different ways. Since, as we said above, the lines now are all physically distinguishable, we may, for example, postulate that every line which is at any given position on the outside of the braid "catches" a similar line unless there are already two lines of that kind in the braid. Such a "caught" line then remains adjacent to the other, making a "double thread" of identical lines until all lines are duplicated. The process is illustrated for three lines in Figure 11. At *a* the full and the dotted lines each "catch" a similar one; at *b* the broken line which here becomes an outside line, "catches" a similar line. After those two steps, all lines are duplicated. If there are more than three lines, there will be more than two steps, and the number of steps, in general, depends on the structure of the braid, that is, on the structure of its group. Each time a new line is woven in, in this way, a new generating element σ is introduced into the braid group. When each line is duplicated, we must make some further assumption as to what happens. For example, we may assume that, from that moment on, the braid is determined by the group of all 6! permutations of the six threads of which it now consists, the permutations being obtained by a prescribed sequence of transpositions. After a definite number of steps the situation will be reached in which we shall have the sequence: full, broken, dotted;

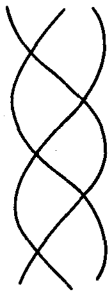

FIGURE 9.

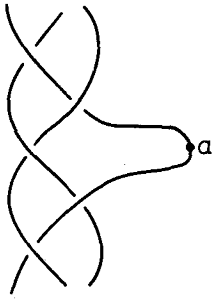

FIGURE 10a.

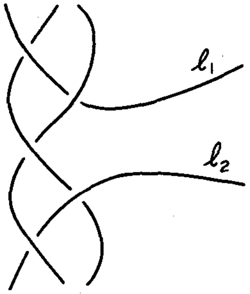

FIGURE 10b.

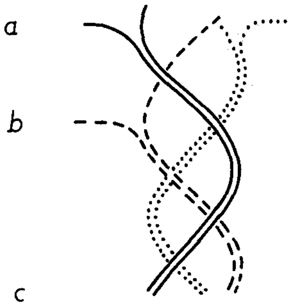

FIGURE 11.

full, broken, dotted. In other words, the original braid is now duplicated in two similar ones. From this moment the interweaving may continue only amongst the first three and the second three lines, and thus the process begins all over again with two identical braids instead of one. We have here a topological model of the duplication and continued multiplication of an organism through assimilation.

Instead of the above assumptions about the method of "catching" and weaving in of new lines, we could have made an infinite number of different ones which lead to an eventual duplication of the braid. The important part is that each set of assumptions leads, if translated into biological terms, to different verifiable conclusions. The number of steps between the beginning of the process and the duplication of every line depends, as we have said, on the assumption made. So does the number of steps between stage *c* of Figure 11, and the stage at which the actual multiplication begins. But each step would correspond to some biological process, since each step represents an intersection of two world lines, a space time coincidence, that is a physically observable phenomenon. Thus the number of physically observable processes between the different stages of the life cycle of an organism are predicted by the topological assumptions made.

Instead of assuming that the interweaving in separate groups of three begins only when the situation

full, broken, dotted; full, broken, dotted (11)

is reached, we may consider the case in which it begins at the stage

full, full, dotted; broken, broken, dotted. (12)

This stage will also eventually be reached after a definite number of steps. We have now a case which biologically corresponds to an unequal division with differentiation. One of the new braids has now an excess of full lines, the other an excess of broken lines. We may now consider braids of second order, in which the role of lines is played by the braids of the first order. We may, in particular, consider the case in which a braid with an excess of full lines interweaves with one that has an excess of broken lines, in the same way as the full line interweaves with the broken line in a first order braid. In the second-order braid, which may represent a multicellular organism, some of the individual first-order braids will stand to each other in the same relation as their corresponding components (lines) stand to each other in the first-order braid. But this is just the kind of relation which is found in biology (Rashevsky, 1955c).

In a braid of the second order there may also be a sort of division of functions. In two dimensions only two first-order braids will be "on the outside." Only those two can "catch" new lines from outside, and, as in the case of Figure 11, those new lines may "move inward" by a process of interweaving. When such a line has moved completely inward so as to become adjacent to an inward second-order braid, only then can the line be woven into that inner second-order braid.

Thus the outer second-order braids may be said, to use biological terminology, to specialize in ingesting outside lines and to pass them along to the inner braids.

Again we repeat that all the above examples are not to be taken seriously in any way. They are used only to illustrate a point which is undoubtedly very suggestive for different possible directions of future research both in mathematical biology and in pure mathematics. The λ - spaces seem to be new in topology; the theory of braids is an established branch of the latter. As remarked on p. 40, we cannot be sure that some already well developed branch of topology does not carry in it the solution of the problem of geometrization of biology.

Should such a geometrization, as anticipated in this paper, ever become a reality, it will be the greatest triumph of geometry and of pure mathematics in general. The doors to every science can then carry the inscription which appeared over Plato's Academy:

Μηδειζ αγεωμετρικοζ εισιτω.

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