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CONCERNING THE SPREAD OF NEWS IN A POPULATION OF INDIVIDUALS WHO NEVER FORGET

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In a deterministic model for the spread of news in a closed homogeneously mixing population of individuals who never forget (or, of an epidemic without recovery), it is shown that the fractions π_i of the population first hearing the news (contracting the disease) i^{th} hand are given by the terms of a truncated Poisson distribution.

1. Introduction and summary. This note could equally well be entitled "A note on the spread of a deterministic epidemic with no recovery." We have preferred the given title because the aspect of the problem to be investigated is *the fraction* π_i *of the population hearing the news (for the first time)* i^{th} *hand*, and this is more relevant when there may be distortion of information in successive generations of transmissions than in the epidemic context. The appropriate probabilistic interpretation of the deterministic result obtained in Section 2 below is that $\pi_i = Pr$ (a given individual hears the news for the first time i^{th} hand) is approximately given by a truncated Poisson distribution (one with the term corresponding to π_0 deleted) with mean determined by the relative rate at which transmission from the source occurs compared with transmission between individuals.

The problem is related in particular to Section 2, Chapter 8 of Bartholomew (1967), to whom I am indebted for allowing me to see his material prior to publication. There are connections also with work on neural networks (cf. Cane, 1966 and Rapoport, 1963).

2. The model and its mathematical analysis. We suppose that at time $t = 0$ the whole population is ignorant of a certain news item which is spread both from a source and by contact between individual members of the population. At time t the population can be divided into the following mutually exclusive and exhaustive categories, the proportions $X(t)$ or $Y_i(t)$ both being labels for the categories and measures of their size:

$X(t)$ is the fraction of the population which has not heard the news by time t;

$Y_1(t)$ is the fraction of the population which by time t has heard the news for the first time of hearing direct from the source;

$Y_{i+1}(t)$ $(i = 1, 2, \ldots,)$ is the fraction of the population which by time t has heard the news for the first time of hearing from the proportion $Y_i(t)$.

We assume a closed population, so

$$X(t) + \sum_{i=1}^{\infty} Y_i(t) = 1. \tag{1}$$

We assume that the rate at which a proportion X of the population has contact with the source is αX; that there is homogeneous mixing of the population; that the news is always passed on when individuals meet; and that the rate at which disjoint fractions X and Y of the population meet is βXY. By choosing the time scale appropriately we can (and do) make $\beta = 1$: this leads to a new α equal to α/β.

We can thus write immediately

$$\dot{Y}_1(t) = \alpha X(t), \tag{2a}$$

$$\dot{Y}_{i+1}(t) = Y_i(t)X(t), \quad (i = 1, 2, \ldots,), \tag{2b}$$

$$\dot{X}(t) = -\alpha X(t) - [1 - X(t)]X(t). \tag{2c}$$

It is easy to solve (2c) to give

$$X(t) = \frac{1 + \alpha}{1 + \alpha e^{(\alpha+1)t}}, \quad (0 \leq t), \tag{3}$$

so $X(t)$ is a strictly monotonic decreasing function of t on $0 \leq t < \infty$, with $X(0) = 1$ and $\lim_{t\to\infty} X(t) = 0$. It is clear that the inverse function $t(x)$ is well-defined for $0 < x \leq 1$ with $t(x) \to \infty$ for $x \to 0$. By setting $y_i(x) = Y_i[t(x)]$ and $y_i'(x) = (d/dx)y_i(x) = \dot{Y}_i(t)/\dot{X}(t)$, equations (2a) and (2b) reduce to

$$y_1'(x) = -\frac{\alpha}{\alpha + 1 - x}, \tag{4a}$$

$$y_{i+1}'(x) = -\frac{y_i(x)}{\alpha + 1 - x}, \quad (i = 1, 2, \ldots,). \tag{4b}$$

Recalling that $y_i(1) = Y_i(0) = 0$ $(i = 1, 2, \ldots,)$, (4b) can be integrated so as to give

$$\begin{aligned} y_{i+1}(x) &= \int_x^1 \frac{y_i(u)}{\alpha + 1 - u}\, du; \\ &= \int_x^1 \frac{du_i}{\alpha + 1 - u_i} \int_{u_i}^1 \frac{y_{i-1}(u_{i-1})}{\alpha + 1 - u_{i-1}}\, du_{i-1}; \\ &\vdots \\ &= \int \cdots \iint_{x \le u_i \le \cdots \le u_1 \le u_0 \le 1} \frac{du_i}{\alpha + 1 - u_i} \cdots \frac{du_1}{\alpha + 1 - u_1} \cdot \frac{du_0}{\alpha + 1 - u_0}; \\ &= \frac{\alpha}{(i+1)!} \left(\int_x^1 \frac{du}{\alpha + 1 - u} \right)^{i+1}; \\ &= \alpha \{\log [1 + \alpha^{-1}(1 - x)]\}^{i+1}/(i+1)!. \end{aligned} \tag{5}$$

Equivalently,

$$Y_i(t) = \frac{\alpha}{i!} \left[\log \left(\frac{\alpha + 1}{\alpha + e^{-(\alpha+1)t}} \right) \right]^i, \quad (i = 1, 2, \ldots,). \tag{6}$$

In particular, from (6),

$$\pi_i = \lim_{t \to \infty} Y_i(t) = \alpha[\log (1 + \alpha^{-1})]^i/i!, \quad (i = 1, 2, \ldots,), \tag{7}$$

$$\sum_{i=1}^{\infty} \pi_i = 1,$$

and our assertion is proved.

3. Concluding remark. Suppose that the spread of news is observed in a population (of ignorants) of size N, and the numbers of individuals in the categories after a sufficiently long time are $(Z_1, Z_2, \ldots,)$. Then we can write

$$Z_i = \sum_{n=1}^{N} U_n(i)$$

where $U_n(i)$ $(n = 1, \ldots, N;\ i = 1, 2, \ldots,)$ is the indicator random variable associated with the n^{th} individual in the population and is equal to zero except when the n^{th} individual belongs to the fraction Y_i when it equals one. Then the set $\{U_n(i)\}$ of random variables is *exchangeable*, but it is not a set of *independent* random variables. Therefore, to compare $(Z_1, Z_2, \ldots,)$ with the approximations $(N\pi_1, N\pi_2, \ldots,)$ implied by the analysis of the deterministic model above by means of any goodness of fit test (such as χ^2) for independent random variables, is not necessarily appropriate, even asymptotically.

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