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ENTROPY AND THE COMPLEXITY OF GRAPHS: II. THE INFORMATION CONTENT OF DIGRAPHS AND INFINITE GRAPHS

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In a previous paper (Mowshowitz, 1968), a measure $I_g(X)$ of the structural information content of an (undirected) graph X was defined, and its properties explored. The class of graphs on which I_g is defined is here enlarged to include directed graphs (digraphs). Most of the properties of I_g observed in the undirected case are seen to hold for digraphs. The greater generality of digraphs allows for a construction which shows that there exists a digraph having information content equal to the entropy of an arbitrary partition of a given positive integer.

The measure I_g is also extended to a measure defined on infinite (undirected) graphs. The properties of this extension are discussed, and its applicability to the problem of measuring the complexity of algorithms is considered.

1. Introduction. In this paper we will extend the definition of information content (Mowshowitz, 1968) to finite directed graphs, and explore the implications of the measure in the case of infinite graphs. Most of the results of the earlier paper will be seen to carry over to the directed case. In addition, the greater generality of digraphs will allow us to show that for any partition of an integer, there exists a digraph whose information content equals the entropy of the partition.

The generalization of the measure to finite digraphs is immediate, since the automorphisms of a digraph form a group as in the undirected case. In fact, we could consider objects of a more general nature, such as graphs with loops and parallel lines. However, the structure of a digraph is sufficiently rich with respect to the information measure as to render unwarranted the added difficulty of exposition which would be incurred by such an extension.

The case of infinite graphs (or digraphs) poses a more difficult problem. There does not appear to be any "natural" way of extending the information measure to the infinite case since it is not clear how to assign probabilities to the orbits of the group of an infinite graph. However, we will examine the behavior of the measure on sequences of finite graphs whose sum is an infinite graph.

As in the case of undirected graphs, the absence of standardized terminology and notation in the literature compels us to present a great many definitions.* We will define the general concepts here; specialized definitions will be given as the need arises.

A *directed graph* (or *digraph*) X is an irreflexive binary relation on a finite set whose elements are called the *points* (or *vertices*) of X. The ordered pairs of points in the relation are the (directed) *lines* (or *edges*) of X. As before we will use $V(X)$ and $E(X)$ to denote the set of points and the set of lines, respectively. If $V(X) = \emptyset$, X will be called the *trivial digraph*. A point x is said to be *adjacent to* y if the line $(x, y) \in E(X)$; x is said to be *adjacent from* y if $(y, x) \in E(X)$. X is called a *symmetric digraph* if $(x, y) \in E(X)$ when and only when $(y, x) \in E(X)$.

For $x \in V(X)$ we denote the set of points adjacent to x and the set of points adjacent from x by $V_i(X; x)$ and $V_o(X; x)$, respectively; that is to say, $V_i(X; x) = \{y \in V(X) \mid (y, x) \in E(X)\}$ and $V_o(X; x) = \{y \in V(X) \mid (x, y) \in E(X)\}$. The *indegree* $id(x)$ and the *outdegree* $od(x)$ of a point x are given by $id(x) = |V_i(X; x)|$ and $od(x) = |V_o(X; x)|$. The *total degree* $td(x)$ of x is the sum $id(x) + od(x)$. We shall call a digraph X *regular of degree* k if $id(x) = od(x) = k$ for every $x \in V(X)$.

A sequence S of points $x_i \in V(X)$ $(0 \le i \le n + 1)$ and lines $l_i \in E(X)$ $(0 \le i \le n)$ given by $S = (x_0, l_0, x_1, l_1, \ldots, x_n, l_n, x_{n+1})$ where $l_i \in \{(x_i, x_{i+1}), (x_{i+1}, x_i)\}$ is called a *semisequence* with *initial point* x_0 and *endpoint* x_{n+1}. S is called a *sequence* if $l_i = (x_i, x_{i+1})$ for $0 \le i \le n$. A (semisequence) sequence is called a (*semipath*) *path* if no line occurs more than once in it. A (*semicycle*) *cycle* is a (semipath) path whose initial point and endpoint are identical.

Two points x and y are said to be *weakly connected* if $x = y$ or there exists a semipath with initial point x and endpoint y; x and y are *strongly connected* if $x = y$ or there is a path with initial point x and endpoint y, and one with initial point y and endpoint x. A digraph is (*weakly*) *strongly connected* if any two points are (weakly) strongly connected. The relation of being weakly connected and that of being strongly connected are both equivalence relations on the points of a digraph.

* The definitions used here are largely those of Harary, Norman and Cartwright (1965).

A digraph X is said to be *disconnected* if it is not weakly connected, and *totally disconnected* if it contains no lines. X is called *complete* if for every x and y in $V(X)$, $(x, y) \in E(X)$ or $(y, x) \in E(X)$. We will use the notation K_n and $\bar{K}_n$ to denote, respectively, the complete symmetric and totally disconnected digraphs of $n(>0)$ points.

As in the undirected case, a digraph Y will be called a *subdigraph* of X (written $Y \subset X$) if $V(Y) \subset V(X)$ and $E(Y) \subset E(X)$; and if $V' \subset V(X)$, the digraph $Y = X(V')$ given by $V(Y) = V'$ and $E(Y) = \{(x, y) \in E(X) \mid x$ and $y \in V'\}$ will be called a *section digraph* of X.

Suppose X and Y are digraphs. Then, ϕ is an *isomorphism* of X onto Y if ϕ is a one-one mapping of $V(X)$ onto $V(Y)$ such that $(x, y) \in E(X)$ iff $(x, y)\phi = (x\phi, y\phi) \in E(Y)$. An *automorphism* of a labelled digraph X is an isomorphism of X onto itself, and the set of all automorphisms of X forms a group which we shall denote by $G(X)$. It is clear that from the standpoint of the automorphism group, a symmetric digraph is just an undirected graph. So, when we discuss symmetric digraphs, we will use the terminology of a previous paper (Mowshowitz, 1968). If X is a digraph we will call X' given by $V(X') = V(X)$ and $E(X') = E(X) \cup \{(x, y) \mid (y, x) \in E(X)\}$ the (undirected) *graph associated with* X.

2. *The information content of digraphs.* Since the automorphisms of a digraph form a group, it is immediately evident that the measure I_g given in Mowshowitz (1968, §3) is defined for digraphs. Moreover, it is clear that Theorems 4.1 and 5.1 (and, thus, 4.6) of the earlier paper hold for digraphs. Figure 1 shows the groups and information content of digraphs with three points.†

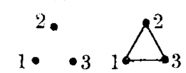
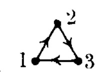
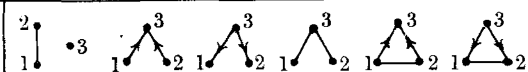
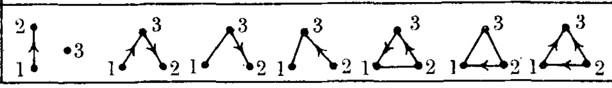
Group	Information	Digraphs with three Points
S_3	0	
{e, (123,) (132)}	0	
{e, (12)}	$\log 3 - \frac{2}{3}$	
{e}	$\log 3$	

Figure 1. The information content of 3-point digraphs

† In illustrations, the points of a digraph will be represented as points in the plane; an edge (x, y) will be indicated by a continuous line segment with an arrow pointing from x to y. If both (x, y) and (y, x) are edges, a line without an arrow will be used, as in the undirected case.

We begin our investigation of the information content of digraphs by defining operations analogous to those considered in the previous paper (Mowshowitz, 1968) for undirected graphs. Let X_1 and X_2 be digraphs with $V_i = V(X_i)$ and $E_i = E(X_i)$ for $i = 1, 2$. The *complement* of X_1 is the digraph $\bar{X}_1$, with $V(\bar{X}_1) = V_1$ and $E(\bar{X}_1) = \{(x, y) \mid (x, y) \notin E_1, x \neq y, x, y \in V_1\}$.

The *sum* of X_1 and X_2 is the digraph $X_1 \cup X_2$ given by $V(X_1 \cup X_2) = V_1 \cup V_2$ and $E(X_1 \cup X_2) = E_1 \cup E_2$. As in the undirected case, the relation of being weakly connected partitions the set of points of a digraph into equivalence classes. Thus, if X is a digraph, $V(X) = \bigcup_{i=1}^{n} V_i$ $(V_i \cap V_j = \phi$ for $i \neq j)$ and $V_i = V(X_i)$ where the $X_i = X(V_i)$ are section digraphs of X. Moreover, when $x \neq y$, x and y are in the same V_i iff (x, y) or (y, x) is in $E(X)$. The section digraphs X_i are called the *weakly connected components* of X.

The *join* of X_1 and X_2 is the digraph $X_1 + X_2$ defined by $V(X_1 + X_2) = V_1 \cup V_2$ and $E(X_1 + X_2) = E(X_1 \cup X_2) \cup \{(x, y) \mid x \in V_1, y \in V_2\}$.

The *cartesian product* of X_1 and X_2 is the digraph $X_1 \times X_2$ defined by $V(X_1 \times X_2) = V_1 \times V_2$ and $E(X_1 \times X_2) = \{(x, y) = [(x_1, x_2), (y_1, y_2)] \mid x_1, y_1 \in V_1, x_2, y_2 \in V_2,$ and either $x_1 = y_1$ and $(x_2, y_2) \in E_2$ or $x_2 = y_2$ and $(x_1, y_1) \in E_1\}$.

The *composition* of X_1 with X_2 is the digraph $X_1 \circ X_2$ given by $V(X_1 \circ X_2) = V_1 \times V_2$ and $E(X_1 \circ X_2) = \{(x, y) = [(x_1, x_2), (y_1, y_2)] \mid x_1, y_1 \in V_1, x_2, y_2 \in V_2,$ and either $(x_1, y_1) \in E_1$ or $x_1 = y_1$ and $(x_2, y_2) \in E_2\}$.

It is clear that the algebraic properties observed to hold for these operations in the undirected case hold for digraphs as well. Of course, they coincide with those defined previously (Mowshowitz, 1968, §4) when X_1 and X_2 are symmetric digraphs. The operations are illustrated in Figure 2.

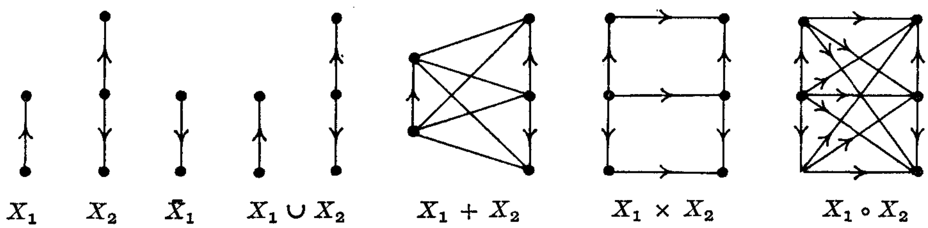


Figure 2. Operations on digraphs

Let X be a digraph. Then X can be expressed in the form

$$X = \bigcup_{i=1}^{r} \left(\bigcup_{j=1}^{k_i} X_{ij} \right)$$

where the X_{ij} are weakly connected components, and for each $i = 1, 2, \ldots, r$, $X_{ij} \cong X_i$ $(1 \leq j \leq k_i)$. As in the undirected case, it is well known that (*a*) $G(\bar{X}) = G(X)$ and (*b*) $G(X) = S_{k_1} \circ G(X_1) + \cdots + S_{k_r} \circ G(X_r)$, where S_{k_i} is the symmetric group of degree k_i, and $\circ$ and $+$ denote the composition and direct sum operations, respectively, defined on permutation groups.

Now, it is easily seen that Theorems 4.3 and 4.5 of the earlier paper (Mowshowitz, 1968) hold in the case of digraphs. We summarize in the following.

2.1 Theorem. Let X and X_i $(1 \leq i \leq n)$ be digraphs, and suppose $X_i \cong X$ $(1 \leq i \leq n)$ and $V(X_i) \cap V(X_j) = \emptyset$ for $i \neq j$. Then (*a*) $I_g(\bar{X}) = I_g(X)$, (*b*) $I_g(X_1 \cup X_2 \cup \cdots \cup X_n) = I_g(X)$, (*c*) $I_g(X_1 + X_2 + \cdots + X_n) = I_g(X)$.

Proof. The proofs for (*a*) and (*b*) are exactly analogous to those of Theorems 4.3 and 4.5 (*a*), respectively of the earlier paper (Mowshowitz, 1968). To prove (*c*), we need only note that as in the case of undirected graphs $X_i + X_j = \overline{\bar{X}_i \cup \bar{X}_j}$, and then appeal to the proof of 4.5 (*b*) of the earlier paper.

Next we will show that Theorem 4.7 (Mowshowitz, 1968) also holds in the directed case. Let X be a digraph. X is said to be *prime* with respect to the cartesian product if, whenever $X \cong X_1 \times X_2$, either X_1 or X_2 is the identity digraph (or graph) K_1. The result (Harary, 1959, 33) used in proving Theorem 4.7 hinges on a theorem of Sabidussi (1960, 456) which states that if $X_1, X_2, \ldots, X_n$ are connected prime (undirected) graphs with $V(X_i) \cap V(X_j) = \emptyset$ $(i \neq j)$, then $G(X_1 \cup X_2 \cup \cdots \cup X_n) \cong G(X_1 \times X_2 \times \cdots \times X_n)$. The proof given by Sabidussi is easily seen to carry over to the directed case if we can show that any digraph X is isomorphic to the cartesian product of prime digraphs which are unique up to isomorphism. Thus, we prove the following

2.2 Lemma. Let $Y = Y_1 \times Y_2 \times \cdots \times Y_n$ be a digraph where each Y_i $(1 \leq i \leq n)$ is prime with respect to the cartesian product. Let $F_i = E(Y_i)$ and

$$F^{(i)} = \{(x, y) = [(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)] \in E(Y) \mid (x_i, y_i) \in E(Y_i)\}.$$

Then the collection $\{F^{(i)}\}_{i=1}^n$ is a partition of $E(Y)$, that is, $\{F^{(i)}\}_{i=1}^n$ induces an equivalence relation on $E(Y)$.

Proof. (*a*) Let X be a symmetric digraph with $X = X_1 \times X_2 \times \cdots \times X_n$ where the X_i are prime. Then there exists a partition of $E(X)$ given by $\{E^{(i)}\}_{i=1}^n$ where $E^{(i)} = \{(x, y) \in E(X) \mid (x_i, y_i) \in E(X_i)\}$. Suppose $\hat{X} = \hat{X}_1 \times \hat{X}_2 \times \cdots \times \hat{X}_n$ is a digraph where each $\hat{X}_i$ is prime, $E(\hat{X}_i) \subset E(X_i)$ and $V(\hat{X}_i) = V(X_i)$ for $1 \leq i \leq n$ (i.e., each $\hat{X}_i$ is a subdigraph of X_i). Then, we claim that $\{\hat{F}^{(i)}\}_{i=1}^n$ with $\hat{F}^{(i)} = \{(x, y) \in E(\hat{X}) \mid (x_i, y_i) \in E(\hat{X}_i)\}$ is a partition of $E(\hat{X})$. For, suppose the contrary. Then there exists $(x, y) \in E(\hat{X}) \subset E(X)$

such that $(x_i, y_i) \in \hat{F}^{(i)} \subset E^{(i)}$ and $(x_i, y_i) \in \hat{F}^{(j)} \subset E^{(j)}$ for $i \neq j$. But this implies that $E^{(i)} \cap E^{(j)} \neq \emptyset$, which is impossible.

(*b*) Consider the symmetric digraph $X = K_{r_1} \times K_{r_2} \times \cdots \times K_{r_n}$ where $E(Y_i) \subset E(K_{r_i})$ and $V(Y_i) = V(K_{r_i})$. Clearly, the K_{r_i} are prime since the cartesian product of any two graphs cannot be complete. Moreover, $E^{(i)} \supset F^{(i)}$ for $1 \leq i \leq n$. Thus, according to (*a*), $\{F^{(i)}\}_{i=1}^{n}$ gives the desired partition of $E(Y)$.

The conclusion of Lemma 2.2 is equivalent to the existence of a unique (up to order and isomorphism) decomposition of a digraph into a cartesian product of prime factors. Thus, we define digraphs X and Y to be *relatively prime* with respect to the cartesian product if whenever $X \cong \hat{X} \times Z$ and $Y \cong \hat{Y} \times Z$, Z is the identity digraph K_1.

2.3 Theorem. Let digraphs X and Y be weakly connected and relatively prime with respect to the cartesian product. Then $I_g(X \times Y) = I_g(X) + I_g(Y)$.

Proof. Lemma 2.2 allows us to invoke the proof given by Sabidussi (1960, 456) to assert that $G(X \cup Y) \cong G(X \times Y)$, since X and Y are relatively prime. So, as shown by Harary (1959, 33) for undirected graphs, $G(X \times Y) = G(X) \times G(Y)$. (In general, we can now say that for any digraphs X and Y, $G(X) \times G(Y) = G(X \times Y)$ iff X and Y are relatively prime.) The theorem now follows from Lemma 4.6 (Mowshowitz, 1968).

The situation involving the composition of two digraphs is less tractable. The (necessary and sufficient) condition given by Sabidussi (1959, 694) for $G(X \circ Y)$ to be equal to $G(X) \circ G(Y)$ when X and Y are undirected graphs can be generalized as follows. Suppose X and Y are digraphs. Let the relations $\hat{R}$ and $\hat{S}$ on $V(X)$ be defined by

$$x \hat{R} y \quad \text{if } V_i(X; x) = V_i(X; y) \text{ and } V_o(X; x) = V_o(X; y),$$

$$x \hat{S} y \quad \text{if } V_i(X; x) \cup \{x\} = V_i(X; y) \cup \{y\} \text{ and } V_o(X; x) \cup \{x\} = V_o(X; y) \cup \{y\},$$

and let $\Delta = \{(x, x) \mid x \in V(X)\}$ (the trivial relation).

Now, the condition is

(*) Y is weakly connected whenever $\hat{R} \neq \Delta$ and $\overline{Y}$ is weakly connected whenever $\hat{S} \neq \Delta$.

It is easy to see that Sabidussi's proof of necessity also shows that the condition (*) is necessary for $G(X \circ Y)$ to be equal to $G(X) \circ G(Y)$. However, the sufficiency proof given by Sabidussi does not appear to carry over to the directed case. Moreover, since a counterexample is not forthcoming, we shall have to be satisfied with a weaker result in the directed case.

2.4 Theorem. Let X and Y be digraphs. Then

$$I_g(X \circ Y) \leq I_g(X) + I_g(Y).$$

Proof. This follows immediately from Theorem 5.1 (Mowshowitz, 1968) since $G(X) \circ G(Y) \leq G(X \circ Y)$, as in the undirected case.

We will conclude the section by establishing a connection between the information content of undirected and directed graphs.

2.5 Lemma. Let G_1 and G_2 be permutation groups with the same object set V satisfying $G_1 \leq G_2$; and let P_1 and P_2 be the finite probability schemes constructed from the orbits of G_1 and G_2, respectively. Then $H(P_2) \leq H(P_1)$, where H is the entropy function.

Proof. Let $p_i = n_i/n$ $(1 \leq i \leq h)$, where $n = |V|$ and n_i is the number of elements in the ith orbit of G_1. Since G_1 and G_2 have the same object set, and G_1 is a subgroup of G_2, each orbit of G_2 will be a union of orbits of G_1. So, suppose O' is an orbit of G_2 with $O' = O_i \cup O_j$ $(i \neq j)$, and that every other orbit of G_2 is equal to some orbit O_k $(k \neq i,\ k \neq j)$ of G_1. Then

$$\begin{aligned} H(P_1) - H(P_2) &= -p_i \log p_i - p_j \log p_j + (p_i + p_j) \log (p_i + p_j), \\ &= p_i[\log (p_i + p_j) - \log p_i] + p_j[\log (p_i + p_j) - \log p_j], \\ &= p_i \log [1 + (p_j/p_i)] + p_j \log [1 + (p_i/p_j)] > 0. \end{aligned}$$

Now, suppose that the orbits of G_1 are O_{ij} $(1 \leq i \leq h',\ 1 \leq j \leq r_i)$, and the orbits of G_2 are given by $O_i' = \bigcup_{j=1}^{r_i} O_{ij}$. Moreover, let $p_{ij} = |O_{ij}|/n$, $\sum_{j=1}^{r_i} p_{ij} = p_i$. Clearly, we must have $\sum_{i=1}^{h'} r_i = h$, the number of orbits of G_1, and $\sum_{i=1}^{h'} p_i = 1$. It is obvious from the above that

$$H(P_1) - H(P_2) = \sum_{j=1}^{h'} \sum_{j=1}^{r_i} p_{ij} \left[\log \left(\frac{\sum_{j=1}^{r_i} p_{ij}}{p_{ij}}\right)\right];$$

and since

$$\frac{\sum_{j=1}^{r_i} p_{ij}}{p_{ij}} > 1, \qquad H(P_1) - H(P_2) > 0.$$

Since $H(P_1) = H(P_2)$ iff G_1 and G_2 have the same orbits, the Lemma is proved.

2.6 Lemma. Let X be a digraph and X' its associated graph. Then $G(X) \leq G(X')$.

Proof. By the definition of X', $(x, y) \in E(X')$ iff $(y, x) \in E(X')$. Suppose $\phi \in G(X)$. Then (x, y) and (y, x) are in $E(X')$ iff (x, y) or (y, x) is in $E(X)$. But this is true iff $(x, y)\phi$ or $(y, x)\phi$ is in $E(X)$ which, in turn, can hold iff $(x, y)\phi$ and $(y, x)\phi$ are in $E(X')$.

An immediate consequence of Lemmas 2.5 and 2.6 is the following.

2.7 Theorem. Let X and X' be as in 2.6. Then $I_g(X') \leq I_g(X)$.

3. The construction of digraphs with given information content. As noted earlier, an undirected graph is just a symmetric digraph, from the standpoint of the automorphism group. Thus, it is natural to expect digraphs to exhibit a greater variety of group structures than undirected graphs. For example, the smallest undirected graph (excluding K_1) whose group consists of the identity alone has six points. In the case of digraphs, as we shall show presently, for every $n \geq 1$ there exists a digraph X with n points for which $G(X) = \{e\}$. This of course means that for every $n \geq 1$ there exists a digraph X with n points such that $I_g(X) = \log n$. Much more can be said, however. We will now give a construction showing that for every $n \geq 1$, there exists a weakly connected digraph X such that

$$I_g(X) = -\sum_{i=1}^{h} \frac{n_i}{n} \log \frac{n_i}{n}$$

where n_i $(1 \leq i \leq h)$ are any positive integers satisfying $\sum_{i=1}^{h} n_i = n$.

First, some preliminaries. We will call a digraph X a *path of length* n (≥ 0) if $V(X) = \{x_0, x_1, \ldots, x_n\}$ and $E(X) = \{(x_1, x_2), \ldots, (x_{n-1}, x_n)\}$ (if $n = 0$, $V(X) = \{x\}$, $E(X) = \emptyset$); X will be called a (directed) *cycle* of length n (≥ 1), if

$V(X) = \{x_1, x_2, \ldots, x_n\}$ and

$$E(X) = \{(x_1, x_2), (x_2, x_3), \ldots, (x_{n-1}, x_n), (x_n, x_1)\}$$

(if $n = 1$, $V(X) = \{x\}$, $E(X) = \emptyset$).

3.1 Lemma. Let X_n be a (directed) path of length n (≥ 0) and Y_n a (directed) cycle of length n (≥ 1). Then (a) The orbits of $G(X_n)$ consist of the individual points of X, so $I_g(X_n) = \log(n + 1)$; (b) $G(Y_n)$ has exactly one orbit $V(X)$, so $I_g(Y_n) = 0$.

Proof. (a) Let $V(X) = \{0, 1, \ldots, n\}$ and $E(X) = \{(0, 1), (1, 2), \ldots, (n - 1, n)\}$ as in Figure 3; and let X_n' be the associated symmetric digraph. From Lemma 2.6 we have that $G(X_n) \leq G(X_n')$, so that each orbit of $G(X_n')$ is a subset of an orbit of $G(X_n')$. Now, the orbits of X_n' are

$$\begin{cases} \{0, n\}, \{1, n-1\}, \ldots, \left\{\dfrac{n+1}{2}, \dfrac{n+1}{2} + 1\right\} & \text{if } n + 1 \text{ is even} \\[2ex] \{0, n\}, \{1, n-1\}, \ldots, \left\{\dfrac{n+2}{2} - 1, \dfrac{n+2}{2} + 1\right\}, \left\{\dfrac{n+2}{2}\right\} & \text{if } n + 1 \text{ is odd.} \end{cases}$$

Since $od(0) = 1$, $od(n) = 0$, 0 and n must be in different orbits, so $\{0\}$, $\{n\}$ are orbits of $G(X_n)$. Hence, any automorphism of X_n must be an automorphism of the subgraph X_n^- consisting of points $1, 2, \ldots, n-1$. But X_n^- is isomorphic to X_{n-2}. Thus, by repeating the argument, we see that $\{0\}, \{1\}, \ldots, \{n\}$ must be the orbits of $G(X)$.

(*b*) Let

$$V(Y_n) = \{1, 2, \ldots, n\} \quad \text{and} \quad E(Y_n) = \{(1, 2), (2, 3), \ldots, (n-1, n), (n, 1)\}$$

as in Figure 3. It is obvious that the cycle $(12\cdots n)$ is an automorphism of Y_n, so $G(Y_n)$ has but one orbit.

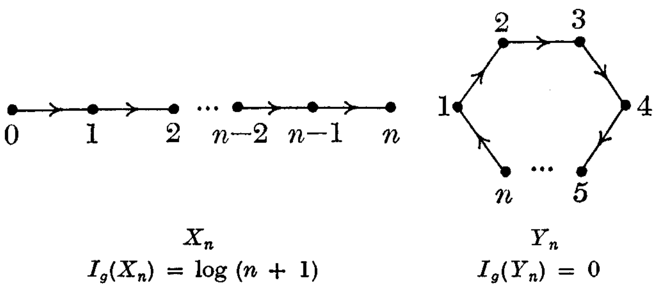


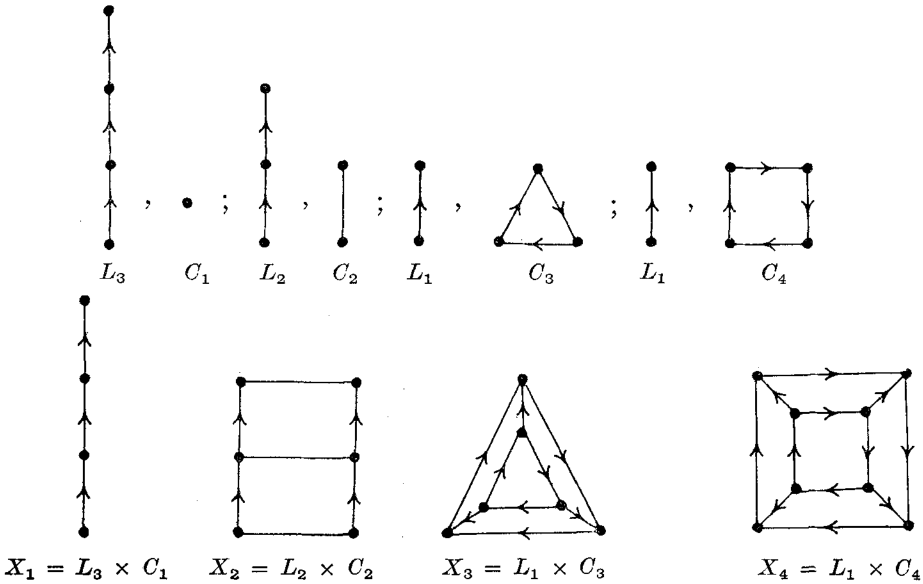
Figure 3. Path and cycle digraphs

3.2 Theorem. Let n be any positive integer, and suppose $P = \{n_{ij}\}$ is a partition of n where $n_{ij} = n_i$ $(1 \leq j \leq r_i)$, $n_{i_1} \neq n_{i_2}$ $(i_1 \neq i_2)$, and $i = 1, 2, \ldots, k$. Then there exists a weakly connected digraph X with n points such that $G(X)$ has exactly $r = \sum_{i=1}^{k} r_i$ orbits, and for each n_{ij} there is an orbit A with $|A| = n_{ij}$; and, hence,

$$I_g(X) = H(P) = -\sum_{i=1}^{k} r_i \left(\frac{n_i}{n} \log \frac{n_i}{n}\right).$$

Proof. If $n = 1$, the proof is trivial. So let $n > 1$. For each $i = 1, 2, \ldots, k$ consider the digraph $X_i = L_{r_i - 1} \times C_{n_i}$ where L_{r_i-1} is a path of length $r_i - 1$, and C_{n_i} is a cycle of length n_i. Since L_{r_i-1} and C_{n_i} are relatively prime with respect to the cartesian product, the set of orbits of $G(X_i)$ is just the cartesian product of the respective orbits of $G(L_{r_i-1})$ and $G(C_{n_i})$. Hence, $G(X_i)$ has exactly r_i orbits each consisting of n_i elements, since by Lemma 2.8 $G(L_{r_i-1})$ has r_i orbits each consisting of one element, and $G(C_{n_i})$ has one orbit with n_i elements. Now consider the digraph $X = X_1 + X_2 + \cdots + X_k$. Since $X_i \not\cong X_j$ for $i \neq j$, it is clear from the discussion preceding Theorem 2.1 that $G(X) = G(X_1) + G(X_2) + \cdots + G(X_k)$, that is, $G(X)$ is the direct sum of $G(X_1), \ldots, G(X_k)$. Hence, the orbits of $G(X)$ are just the union of the orbits of the $G(X_i)$, that is, $G(X)$ has r orbits, r_i of which each contain n_i elements for $i = 1, 2, \ldots, k$. Moreover, since each $L_{r_i - 1}$ and C_{n_i} are weakly connected, X_i is weakly connected; and, thus, X is weakly connected, as required.

Figure 4 illustrates the theorem for $n = 24$, $P = \{1^4, 2^3, 3^2, 4^2\}$.



$$X = X_1 + X_2 + X_3 + X_4$$

$$I_g(X) = -\left[4\left(\frac{1}{24}\log\frac{1}{24}\right) + 3\left(\frac{2}{24}\log\frac{2}{24}\right) + 2\left(\frac{3}{24}\log\frac{3}{24}\right) + 2\left(\frac{4}{24}\log\frac{4}{24}\right)\right]$$
$$= H(\{1^4, 2^3, 3^2, 4^2\})$$

Figure 4. The construction of a 24-point digraph with given information content

4. *An extension of the information measure to infinite graphs.* In an earlier paper (Mowshowitz, 1968) we defined a graph‡ as an irreflexive symmetric binary relation on a *finite* set. By dropping the finiteness restriction, one obtains an *arbitrary graph* which may have infinitely many points and lines. The definitions given in Section 2 of the previous paper are the same in the general case with the following modifications: Let X be an arbitrary graph. A sequence

$$S = (\ldots, l_{-n}, \ldots, l_{-1}, l_0, l_1 \ldots, l_n \ldots)$$

of lines $l_i = [x_i, x_{i+1}] \in E(X)$ may be finite or infinite. However, as in the

‡ To simplify the discussion, we will consider only undirected graphs, and use the notation of the earlier paper (1968).

finite case, two points x and y are connected if there is a finite sequence S with initial point x and endpoint y. As before, the set of automorphisms of X forms a group $G(X)$. An orbit (which may contain infinitely many points of X) of $G(X)$ is given by $\{xg \mid g \in G(X)\}$ for some $x \in V(X)$. Again, the collection of orbits of $G(X)$ is a decomposition of $V(X)$ into disjoint subsets whose union is $V(X)$.

An arbitrary graph X is said to be *countable* if $|V(X) \cup E(X)|$ is countable; X is *locally finite* if $d(x)$ (the degree of x) is finite for every $x \in V(X)$. In what follows we will be dealing exclusively with countable graphs which may or may not be locally finite.

Ideally, any extension $\hat{I}_g$ of the measure I_g to the (countably) infinite case should satisfy: (*a*) $\hat{I}_g(X) = I_g(X)$ for all finite graphs X; (*b*) $\hat{I}_g(X)$ is defined and unique for all countable graphs X. Unfortunately, the particular extension we will examine satisfies (*a*) but not (*b*). However, the reason for its failure to satisfy (*b*) turns out to be interesting in itself, as will be seen presently.

Let X be a countable graph. A sequence $\{X_n\}_{n=1}^{\infty}$ of finite graphs X_n with $V_n = V(X_n)$ and $E_n = E(X_n)$ is said to *converge* to X as a *limit* (written $\lim_{n\to\infty} X_n = X$) if $\lim_{n\to\infty} V_n = V(X)$ and $\lim_{n\to\infty} E_n = E(X)$ where the latter two limits are simply limits of a sequence of sets. Note that in general, if $\{A_n\}_{n=1}^{\infty}$ is a sequence of sets A_n,

$$\liminf_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} A_n \right)$$

and

$$\limsup_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} \left(\bigcup_{n=k}^{\infty} A_n \right).$$

$\{A_n\}_{n=1}^{\infty}$ is said to converge to A (written $\lim_{n\to\infty} A_n = A$) if $\limsup A_n = \liminf A_n = A$. Moreover, if $\{A_n\}_{n=1}^{\infty}$ converges to A, then any subsequence $\{A_{n_k}\}_{k=1}^{\infty}$ also converges to A.

4.1 Lemma. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of finite graphs, and let $h(n)$ be the number of orbits of $G_n = G(X_n)$. If $\sup_n h(n) \leq h_0$, then $\sup_n I_g(X_n) \leq \log h_0$.

Proof. For each n, let $k_i(n)$ be the number of elements in the ith orbit of G_n, $f(n) = |V(X_n)|$, and let $\hat{k}_i(n) = [h(n)/f(n)]k_i(n)$. Clearly,

$$\sum_{i=1}^{h(n)} \frac{\hat{k}_i(n)}{h(n)} = 1,$$

and

$$-\sum_{i=1}^{h(n)} \frac{\hat{k}_i(n)}{h(n)} \log \frac{\hat{k}_i(n)}{h(n)} \leq \log h(n),$$

for each n. But

$$I_g(X_n) = -\sum_{i=1}^{h(n)} \frac{k_i(n)}{f(n)} \log \frac{k_i(n)}{f(n)} = -\sum_{i=1}^{h(n)} \frac{[h(n)/f(n)]k_i(n)}{h(n)} \log \frac{[h(n)/f(n)]k_i(n)}{h(n)}$$

$$= -\sum_{i=1}^{h(n)} \frac{\hat{k}_i(n)}{h(n)} \log \frac{\hat{k}_i(n)}{h(n)}.$$

So, $h(n) \leq h_0$ for all n gives $\sup_n I_g(X_n) \leq \log h_0$.

A sequence $\{X_n\}_{n=1}^{\infty}$ of finite graphs X_n will be called a *defining sequence* for a countable graph X if $X_n \subset X_{n+1}$ for every n, and $\lim_{n\to\infty} X_n = X$. Note that for any sequence $\{A_n\}_{n=1}^{\infty}$ of sets A_n with $A_n \subset A_{n+1}$ for all n, the limit $\lim_{n\to\infty} A_n$ always exists and is equal to $\bigcup_{n=1}^{\infty} A_n$; so, $\lim_{n\to\infty} X_n = \bigcup_{n=1}^{\infty} X_n$. It is clear that every countable graph X has a defining sequence. For, if X is a countable graph, we can take $V(X) = \{x_1, x_2, \ldots\}$ and define a sequence $\{X_n\}_{n=1}^{\infty}$ by the relations

$$V(X_1) = \{x_1\} \text{ and } E(X_1) = \emptyset;$$

$$V(X_{n+1}) = V(X_n) \cup \{x_{n+1}\} \text{ and}$$

$$E(X_{n+1}) = E(X_n) \cup \{[x_{n+1}, y] \in E(X) \mid y \in V(X_n)\}.$$

It is trivial to verify that $X_n \subset X_{n+1}$ for all n, and

$$\lim_{n\to\infty} X_n = \bigcup_{n=1}^{\infty} X_n = X.$$

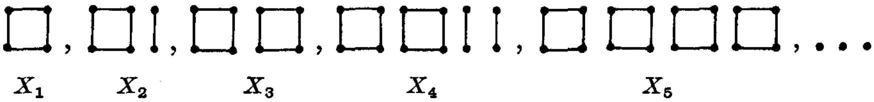
Now we are in a position to define an extension of I_g.

4.2 Definition. Let $\{X_n\}_{n=1}^{\infty}$ be a defining sequence for a countable graph X. Then the *structural information content* $\hat{I}_g(X; X_n)$ *of* X *with respect to the sequence* $\{X_n\}_{n=1}^{\infty}$ is given by $\hat{I}_g(X; X_n) = \lim_{n\to\infty} I_g(X_n)$, if the limit exists. If $\lim_{n\to\infty} I_g(X_n)$ diverges, we will write $\hat{I}_g(X; X_n) = \infty$. Consider the countable graph X (shown in Fig. 5) defined by the sequence $\{X_n\}_{n=1}^{\infty}$ where $X_1 = C_4$ (the cycle of length four), $X_2 = C_4 \cup K_2$, and $X_{n+1} = 2X_{n-1}$ for $n \geq 2$.

Now, it is easy to see that

$$I_g(X_n) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \log 3 - \frac{2}{3} & \text{if } n \text{ is even,} \end{cases}$$

for all n. Hence, $\liminf_{n\to\infty} I_g(X_n) = 0$ and $\limsup_{n\to\infty} I_g(X_n) = \log 3 - \frac{2}{3}$, so $\hat{I}_g$ is not defined for the sequence $\{X_n\}_{n=1}^{\infty}$. However, the subsequences $\{Y_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ with $Y_n = X_{2n-1}$ and $Z_n = X_{2n}$ for $n = 1, 2, \ldots$ are both defining sequences for X, and $\hat{I}_g(X; Y_n) = 0$ and $\hat{I}_g(X; Z_n) = \log 3 - \frac{2}{3}$.



$$X = \lim_{n\to\infty} X_n = \bigcup_{n=1}^{\infty} X_n$$

Figure 5. A countable graph with no unique information content

What this example points out is that the information content (whenever it exists) of a countable graph depends on the way the graph is defined (or constructed). From one point of view, this is a shortcoming of the definition. However, from the standpoint of measuring the complexity of a countable graph, it might be desirable to have a measure which is a function of the way in which the graph is constructed. For example, such a measure might be useful for characterizing the relative complexity of an algorithm. For, if the computation at each step can be associated with a finite graph (or, perhaps, a digraph), the algorithm can be represented as a sequence of finite graphs. In any case, it is intuitively plausible that in certain cases the complexity of an object is not an intrinsic property of some structural feature, but rather depends on the way the object is constructed.

Now we shall examine some of the properties of the extension $\hat{I}_g$.

4.3 Theorem. Let X be a countable graph with defining sequence $\{X_n\}_{n=1}^{\infty}$. Then (*a*) If X is finite, $\hat{I}_g(X; X_n)$ exists and is equal to $I_g(X)$. (*b*) There exists a subsequence $\{Y_n\}_{n=1}^{\infty}$ such that $\hat{I}_g(X; Y_n)$ exists.

Proof. (*a*) Since X is assumed finite, it is clear that there exists a positive integer N such that $X_n = X_m$ for all $n, m \geq N$. Hence

$$\lim_{n\to\infty} I_g(X_n) = I_g(X).$$

(*b*) If the sequence $\{I_g(X_n)\}_{n=1}^{\infty}$ is bounded, there is a convergent subsequence $\{I_g(X_{n_k})\}_{k=1}^{\infty}$; if not, there is a subsequence whose limit is infinite. In either case, the subsequence has a limit, so we can choose a subsequence $\{Y_n\}_{n=1}^{\infty}$ of the defining sequence such that $\hat{I}_g(X; Y_n)$ exists.

Theorem 4.3 shows that there always exists a defining sequence for a countable graph for which $\hat{I}_g$ is defined. The variation in information content given by different sequences, however, can be infinite. Consider the countable graph X (shown in Fig. 6) with defining sequence $\{X_n\}_{n=1}^{\infty}$ given by $X_1 = K_2$, $X_2 =$

$2K_1 \cup K_2$, and for $n \geq 2$

$$X_{n+1} = \begin{cases} \bigcup_{\{i,r \mid ir=k\}} rK_i & \text{if } X_n = K_k. \\ K_m & \text{if } X_n = \bigcup_{\{i,r \mid ir=k\}} rK_i \text{ and } m = \sum_{\{i,r \mid ir=k\}} ir. \end{cases}$$

Clearly, $I_g(X_{2n-1}) = 0$ for all $n = 1, 2, \ldots$; and $I_g(X_{2n})$ is of the form $k(r/kr \log kr/r) = \log k$ where $|V(X_{2n})| = kr$. In the latter case, it is obvious that $\lim_{n\to\infty} I_g(X_{2n}) = \infty$. Hence, there are defining sequences for X given by $Y_n = X_{2n-1}$ and $Z_n = X_{2n}$ such that $\hat{I}_g(X; Y_n) = 0$ and $\hat{I}_g(X; Z_n) = \infty$.

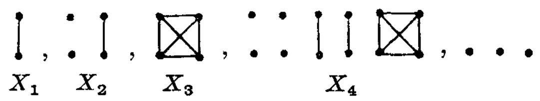


Figure 6. A countable graph showing an infinite variation in information content

4.4 Theorem. Let X be a countable graph with defining sequence $\{X_n\}_{n=1}^{\infty}$, and let $h(n)$ be the number of orbits of $G(X_n)$. If $\sup h(n) = h_0 < \infty$, there exists a defining sequence $\{Y_n\}_{n=1}^{\infty}$ such that $\hat{I}_g(X; Y_n) \leq \log h_0$.

Proof. By Lemma 4.1,

$$\sup_n I_g(X_n) \leq \log h_0,$$

so

$$\hat{I}_g(X; Y_n) \leq \log h_0$$

whenever it exists. Theorem 4.3 assures the existence of the appropriate defining sequence.

When the condition of the theorem is satisfied, it is easy to see that $G(X)$ has finitely many orbits. However, the graph given in Figure 6 shows that if $G(X)$ has only a finite number of orbits (namely, one) it is still possible for $\hat{I}_g(X; X_n)$ to be infinite.

A less pathological case is characterized in the following.

4.5 Theorem. If X is a countable graph and $|E(X)| < \infty$, $\hat{I}_g(X; X_n) = 0$ for all defining sequences $\{X_n\}_{n=1}^{\infty}$.

Proof. Let Y be the (finite) subgraph of X defined by $E(Y) = E(X)$ and $V(Y) = \{x \in V(X) \mid d(x) > 0\}$; let A_i with $|A_i| = r_i$ $(1 \leq i \leq h)$ be the orbits of $G(Y)$. Since $|E(X)| < \infty$, it is clear that there exists a positive integer N such that $E(X_n) = E(X)$ for all $n \geq N$ where $\{X_n\}_{n=1}^{\infty}$ is an arbitrary defining sequence for X; and $|V(X_n)| \geq |V(Y)|$. Let $|V(Y)| + k_n = |V(X_n)|$. Then

$$I_g(X_n) = \sum_{i=1}^{h} \frac{r_i}{r + k_n} \log \frac{r + k_n}{r_i} + \frac{k_n}{r + k_n} \log \frac{r + k_n}{k_n}$$

where

$$|V(Y)| = \sum_{i=1}^{h} r_i = r.$$

Hence,

$$\lim_{n\to\infty} I_g(X_n) = \sum_{i=1}^{h} \lim_{n\to\infty} \left[\frac{r_i}{r + k_n} \log \frac{r + k_n}{r_i}\right] + \lim_{n\to\infty} \left[\frac{k_n}{r + k_n} \log \frac{r + k_n}{k_n}\right] = 0,$$

since $\{k_n\}$ is a monotonically increasing sequence and $\lim_{x\to\infty} (\log x)/x = 0$.

Since the notion of a defining sequence allows for infinite variation in the information content of the same countable graph, it is appropriate to consider various restriction on such sequences. In particular, it seems reasonable to require that the groups of the respective graphs in a defining sequence for a countable graph X reflect the orbit structure of $G(X)$. So, let us call a defining sequence $\{X_n\}_{n=1}^{\infty}$ for a countable graph X a *G-defining sequence* if $G(X)$ has a finite (infinite) number of orbits when and only when $\{h_n\}_{n=1}^{\infty}$ is a bounded (unbounded) monotonically increasing sequence, where h_n is the number of orbits of $G(X_n)$. An immediate consequence of this definition and Theorem 4.4 is

4.6 Theorem. Let $\{X_n\}_{n=1}^{\infty}$ be a G-defining sequence for a countable graph X. If $G(X)$ has finitely many orbits, there is a subsequence $\{Y_n\}_{n=1}^{\infty}$ such that $\hat{I}_g(X; Y_n)$ exists and is finite.

Some simple examples of G-defining sequences are given in Figure 7.

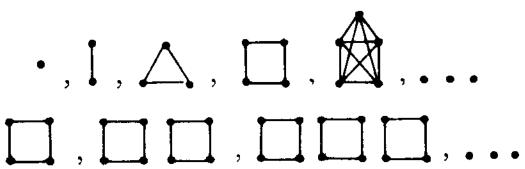


Figure 7. G-defining sequences

Many other special types of defining sequences could no doubt be invented which would yield results more profound than Theorem 4.6. However, it seems likely that such an investigation would be more fruitful if one were interested in particular applications of an information measure on infinite graphs.

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