BULLETIN OF
MATHEMATICAL BIOPHYSICS
VOLUME 30, 1968

PERISTALTIC FLOW IN TUBES

■ CHARLES BARTON AND SEVERIN RAYNOR
Department of Mechanical Engineering and Astronautical Sciences,
Northwestern University,
Evanston, Illinois

The study is concerned with the analysis of two flow domains of peristaltic motion in tubes. In the first analysis the wall disturbance wavelength is much larger than the average tube radius. There is a simple algebraic relation between the average flow rate and pressure differential across a wavelength. In the second analysis the disturbance wavelength may be as small as the average radius. A numerical technique may be used to determine the relation between average flow rate and pressure differential across a wavelength.

1. *Introduction.* The object of this study is to investigate analytically Newtonian flow induced by the peristaltic action of a flexible tube. The tube walls are excited by traveling sinusoidal waves that cause points on the wall to move only transversely to the net average fluid flow. Harmonic waves are chosen both because they are readily dealt with analytically and because the intestines might, as a first approximation, be thought of as tubes whose walls are moving in this manner.

An axisymmetric model is used with the stipulations that the fluid be Newtonian and incompressible. The analysis is valid for small Reynolds numbers.

2. *Previous Work.* The investigations involving flow in flexible but passive tubes include those by Morgan and Kiely (1954), who considered propagation of small disturbances with long wavelength in a flexible tube free to move both axially and radially. Whirlow and Rouleau (1965) analyzed viscous flow in a

thick-walled viscoelastic tube under specified flow and pressure conditions. Olsen and Shapiro (1967) investigated large amplitude motion of a viscous fluid in an elastic tube under specified flow and pressure conditions. Most analyses involving passive tube walls are related to blood circulation in the arteries. Rudinger (1966) summarized the analytical literature on blood flow and has a comprehensive bibliography.

Analytical studies involving active tube flow with externally excited walls are mainly concerned with various peristaltic phenomena. Latham (1966) investigated analytically and experimentally the behavior of a two-dimensional channel peristaltic pump. Shapiro (1967) considered retrograde diffusion in a two-dimensional peristaltic pump. Burns and Parkes (1967) used a perturbation technique to study channel and tube peristaltic motion.

Finally, there is a paper by Taylor (1951) concerned with swimming of flagellated organisms, utilizing Taylor series developments and perturbation techniques. Though this paper does not deal specifically with the flow of fluid in tubes, it is important to the present study because similar analyses are common to both.

3. *Long Wavelength Approximation to Peristaltic Flow in a Tube.* The method of analysis here is similar to that employed by Latham (1966) for the case of two-dimensional channel flow.

Consider the flow of a viscous fluid in a tube whose walls are executing traveling ring waves so that the tube surface is defined by

$$h(Z, t) = a + b \sin \frac{2\pi}{\lambda} (Z - ct). \tag{1}$$

The tube geometry is shown in Figure 1, which exhibits average radius a, wave

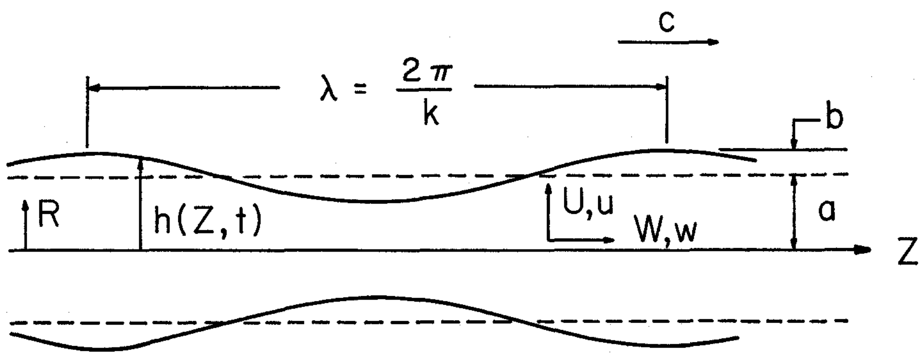

Figure 1. Tube geometry

amplitude b, and wavelength λ. The wave propagation velocity is c and the kinematic viscosity of the fluid filling the tube is ν. The fluid is assumed to be Newtonian and incompressible.

The R, Z coordinates are fixed in space while the r, z coordinates travel with the ring waves to the right at a velocity c. The variables z and r are defined by

$$z = Z - ct; \quad r = R. \tag{2}$$

The axial velocity measured in the moving coordinates is w, while the corresponding radial velocity is u. Corresponding velocities in the stationary R, Z coordinates are U and W.

The equations of motion and continuity governing fluid behavior in the tube are

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial R} + W\frac{\partial U}{\partial Z} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{U}{R^2} + \frac{\partial^2 U}{\partial Z^2}\right), \tag{3}$$

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial R} + W\frac{\partial W}{\partial Z} = -\frac{1}{\rho}\frac{\partial p}{\partial Z} + \nu\left(\frac{\partial^2 W}{\partial R^2} + \frac{1}{R}\frac{\partial W}{\partial R} + \frac{\partial^2 W}{\partial Z^2}\right), \tag{4}$$

and

$$\frac{\partial U}{\partial R} + \frac{U}{R} + \frac{\partial W}{\partial Z} = 0. \tag{5}$$

The continuity equation (5) may be written as

$$\frac{\partial(RU)}{\partial R} + \frac{\partial(RW)}{\partial Z} = 0. \tag{6}$$

Differentiation of (6) with respect to R results in

$$\frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R} - \frac{U}{R^2} = -\frac{\partial^2 W}{\partial R\, \partial Z}. \tag{7}$$

Expression (7) together with the identity

$$\frac{\partial^2 W}{\partial R^2} + \frac{1}{R}\frac{\partial W}{\partial R} = \frac{1}{R}\frac{\partial[R(\partial W/\partial R)]}{\partial R} \tag{8}$$

when introduced into the equations of motion (3) and (4) yield

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial R} + W\frac{\partial U}{\partial Z} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\frac{\partial^2 U}{\partial Z^2} - \frac{\partial^2 W}{\partial R\, \partial Z}\right), \tag{9}$$

and

$$\frac{\partial W}{\partial t} + U\frac{\partial W}{\partial R} + W\frac{\partial W}{\partial R} = -\frac{1}{\rho}\frac{\partial p}{\partial R} + \nu\left(\frac{1}{R}\frac{\partial[R(\partial W/\partial R)]}{\partial R} + \frac{\partial^2 W}{\partial Z^2}\right). \tag{10}$$

In order to linearize these equations, the order of magnitude of the various terms in the equations of motion will be considered. Let the symbol $\sim$ represent an order of magnitude relation.

At $R = h$ the radial velocity U is given by

$$U = -\frac{2\pi bc}{\lambda}\cos\frac{2\pi}{\lambda}(Z - ct). \tag{11}$$

From equation (11)

$$U \sim \frac{bc}{\lambda}. \tag{12}$$

The characteristic axial length is the wavelength λ. The characteristic time is the period λ/c. The characteristic radial dimension is the average radius a.

From the above characteristic dimensions

$$\frac{\partial[\;]}{\partial Z} \sim \frac{[\;]}{\lambda}, \quad \frac{\partial[\;]}{\partial t} \sim \frac{c[\;]}{\lambda}, \quad \text{and} \quad \frac{\partial[\;]}{\partial R} \sim \frac{[\;]}{a}. \tag{13}$$

This results in

$$\frac{\partial(RU)}{\partial R} \sim \frac{Rbc}{a\lambda} \tag{14}$$

and

$$\frac{\partial(RW)}{\partial Z} \sim \frac{RW}{\lambda}. \tag{15}$$

Then from the continuity relation

$$W \sim \frac{bc}{a}. \tag{16}$$

The order of magnitude of all terms in the equations of motion is

$$\begin{aligned}
&\frac{\partial U}{\partial t} \sim \frac{c^2b}{\lambda^2} && \frac{\partial W}{\partial t} \sim \frac{c^2b}{a\lambda} \\
&U\frac{\partial U}{\partial R} \sim \frac{c^2b^2}{a\lambda^2} && U\frac{\partial W}{\partial R} \sim \frac{c^2b^2}{a^2\lambda} \\
&W\frac{\partial U}{\partial Z} \sim \frac{c^2b^2}{a\lambda^2} && W\frac{\partial W}{\partial Z} \sim \frac{c^2b^2}{a^2\lambda} \\
&\frac{\partial^2 U}{\partial Z^2} \sim \frac{cb}{\lambda^3} && \frac{\partial^2 W}{\partial Z^2} \sim \frac{cb}{a\lambda^2} \\
&\frac{\partial^2 W}{\partial R\,\partial Z} \sim \frac{cb}{a^2\lambda} && \frac{1}{R}\frac{\partial[R(\partial W/\partial R)]}{\partial R} \sim \frac{cb}{a^3}.
\end{aligned} \tag{17}$$

Consider the case in which the wavelength is much larger than the average radius and the Reynolds number is small; $a/\lambda \ll 1$ and $ac/\nu \ll 1$.

$$\frac{\partial U}{\partial t} \Big/ \nu \frac{\partial^2 W}{\partial R\, \partial Z} \sim \frac{c^2 b}{\lambda^2} \Big/ \frac{\nu c b}{a^2 \lambda} = \frac{c a^2}{\nu \lambda} \ll 1 \tag{18}$$

$$U \frac{\partial U}{\partial R} \Big/ \nu \frac{\partial^2 W}{\partial R\, \partial Z}, \; W \frac{\partial U}{\partial Z} \Big/ \nu \frac{\partial^2 W}{\partial R\, \partial Z} \sim \frac{c^2 b^2}{a \lambda^2} \Big/ \frac{\nu c b}{a^2 \lambda} \ll 1 \tag{19}$$

$$\nu \frac{\partial^2 U}{\partial Z^2} \Big/ \nu \frac{\partial^2 W}{\partial R\, \partial Z} \sim \frac{\nu c b}{\lambda^3} \Big/ \frac{\nu c b}{a^2 \lambda} = \frac{a^2}{\lambda^2} \ll 1 \tag{20}$$

$$\frac{\partial W}{\partial t} \Big/ \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R} \sim \frac{b c^2}{a \lambda} \Big/ \frac{\nu c b}{a^3} = \frac{c a^2}{\nu \lambda} \ll 1 \tag{21}$$

$$U \frac{\partial W}{\partial R} \Big/ \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R}, \; W \frac{\partial W}{\partial Z} \Big/ \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R} \sim \frac{c^2 b^2}{a^2 \lambda} \Big/ \frac{\nu c b}{a^3} \ll 1 \tag{22}$$

$$\nu \frac{\partial^2 W}{\partial Z^2} \Big/ \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R} \sim \frac{\nu c b}{a \lambda^2} \Big/ \frac{\nu c b}{a^3} = \frac{a^2}{\lambda^2} \ll 1. \tag{23}$$

If the Reynolds number is small and the wavelength long when compared to the radius, it can be seen from expressions (18), (19), and (20) that

$$\frac{\partial U}{\partial t} \ll \nu \frac{\partial^2 W}{\partial R\, \partial Z},$$

$$U \frac{\partial U}{\partial R} \ll \nu \frac{\partial^2 W}{\partial R\, \partial Z},$$

$$W \frac{\partial U}{\partial R} \ll \nu \frac{\partial^2 W}{\partial R\, \partial Z},$$

and

$$\frac{\partial^2 U}{\partial Z^2} \ll \frac{\partial^2 W}{\partial R\, \partial Z}. \tag{24}$$

Similarly, for terms in the equation of motion in the axial direction it can be seen from expressions (21), (22), and (23) that

$$\frac{\partial W}{\partial t} \ll \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R} \qquad U \frac{\partial W}{\partial R} \ll \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R}$$

$$W \frac{\partial W}{\partial Z} \ll \frac{\nu}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R} \qquad \frac{\partial^2 W}{\partial Z^2} \ll \frac{1}{R} \frac{\partial [R(\partial W/\partial R)]}{\partial R}. \tag{25}$$

The smaller terms may be deleted from the equations of motion, resulting in the following linear equations of motion

$$\frac{1}{\rho}\frac{\partial p}{\partial R} = -\nu\frac{\partial^2 W}{\partial R\,\partial Z} \tag{26}$$

and

$$\frac{1}{\rho}\frac{\partial p}{\partial Z} = \frac{\nu}{R}\frac{\partial[R(\partial W/\partial R)]}{\partial R}. \tag{27}$$

Furthermore,

$$\frac{\partial p}{\partial R}\bigg/\frac{\partial p}{\partial Z} = -\nu\frac{\partial^2 W}{\partial R\,\partial Z}\,\frac{\nu}{R}\frac{\partial[R(\partial W/\partial R)]}{\partial R} \sim \frac{\nu cb}{a^2\lambda}\bigg/\frac{\nu cb}{a^3} = \frac{a}{\lambda} \ll 1 \tag{28}$$

or

$$\frac{\partial p}{\partial R} \ll \frac{\partial p}{\partial Z}. \tag{29}$$

Thus the pressure is approximately a function of the axial coordinate Z and time alone. It is only weakly dependent on R. Due to the small dependence of pressure on the radius, say

$$p = p(Z, t) = p(z) \tag{30}$$

since

$$z = Z - ct.$$

In the moving r, z coordinates the equations of motion and continuity reduce to

$$\frac{dp}{dz} = \mu\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) \tag{31}$$

and

$$\frac{\partial(ru)}{\partial r} + \frac{\partial(rw)}{\partial z} = 0. \tag{32}$$

The solutions to (31) and (32) must satisfy the following boundary conditions at $r = h$.

$$w = -c$$

$$u = \frac{\partial h}{\partial t}. \tag{33}$$

As $dp/dz = f(z)$ integrate (31) at constant z to obtain

$$w = -c - \frac{1}{4\mu}\frac{dp}{dz}(h^2 - r^2). \tag{34}$$

To an observer moving at the velocity c in the axial direction, the pressure and flow appear stationary; and therefore, the flow rate q measured in the moving coordinates is a constant that varies with neither time nor position along the tube axis.

$$q = 2\pi \int_0^h rw\, dr \tag{35}$$

where w is given by equation (34).

$$q = -c\pi h^2 - \frac{\pi h^4}{8\mu}\frac{dp}{dz}. \tag{36}$$

The pressure gradient dp/dz may then be expressed as

$$\frac{dp}{dz} = -\frac{8\mu q}{\pi h^4} - \frac{8\mu c}{h^2}. \tag{37}$$

Substituting equation (37) into the equation for w yields

$$w = -c + 2\left(\frac{q}{\pi h^4} + \frac{c}{h^2}\right)(h^2 - r^2). \tag{38}$$

The transverse velocity u may be found by integrating the continuity equation at constant z. Noting that $u = 0$ at $r = 0$,

$$ru = -\int_0^r r\frac{\partial w}{\partial z}\, dr. \tag{39}$$

Since $u(0, z) = 0$, this requires that

$$u = -\frac{dh}{dz}\left[\frac{cr^3}{h^3} - \frac{2qr}{\pi h^3} + \frac{2qr^3}{\pi h^5}\right]. \tag{40}$$

Consider the flow rate Q measured in the stationary R, Z coordinate system. The axial velocity W measured in the stationary coordinates are

$$W = w + c. \tag{41}$$

Hence

$$Q = 2\pi \int_0^h WR\, dR, \tag{42}$$

or

$$Q = q + \pi ch^2. \tag{43}$$

Now compute the time average flow $\bar{Q}$. If T is a complete period for h, then $T = \lambda/c$ and

$$\bar{Q} = \frac{1}{T}\int_0^T Q\,dt \tag{44}$$

or

$$\bar{Q} = q + \pi c(a^2 + b^2/2). \tag{45}$$

The flow rate q depends on the pressure gradient dp/dz as shown in equations (36) and (37).

In the investigation that follows for a tube with wall disturbance wavelengths as short as the average radius, the case of no pressure drop across a wavelength will be considered. Therefore, the present investigation is limited to the same case. The pressure change over one wavelength, Δp_λ, is the same whether measured in the moving or stationary coordinates. For simplicity of integration, evaluate it in the moving coordinates:

$$\Delta p_\lambda = \int_0^\lambda \frac{dp}{dz}\,dz. \tag{46}$$

For the case of a sinusoidal traveling wave, the value $h = a + b\sin[(2\pi/\lambda)z]$ is introduced into equations (37) and (46). Δp_λ is evaluated in terms of a new variable τ, which is defined by

$$\tau = \frac{2\pi z}{\lambda}. \tag{47}$$

$$\Delta p_\lambda = -\frac{4\mu\lambda}{\pi^2}\int_0^{2\pi}\left[\frac{q}{(a + b\sin\tau)^4} + \frac{\pi c}{(a + b\sin\tau)^2}\right]d\tau \tag{48}$$

$$\Delta p_\lambda = -\frac{8\mu\lambda}{\pi(a^2 + b^2)}\left[\frac{q}{(a^2 + b^2)}P_3\left(\frac{a}{\sqrt{a^2 - b^2}}\right) + \pi c P_1\left(\frac{a}{\sqrt{a^2 - b^2}}\right)\right]. \tag{49}$$

For the special case where $\Delta p_\lambda = 0$,

$$\bar{Q} = \frac{\pi c b^2}{2}\left[\frac{16a^2 - b^2}{2a^2 + 3b^2}\right]. \tag{50}$$

From equation (40) for the case of a sinusoidal traveling wave the radial velocity is given by

$$U = \frac{-2\pi b c r\cos(2\pi/\lambda)z}{\lambda h^3}\left[r^2 + \frac{4(a^2 - b^2)^2}{2a^2 + 3b^2}\left(1 - \frac{r^2}{h^2}\right)\right] \tag{51}$$

utilizing the result of equation (50).

Inserting the expression (50) for q into the expression (37) for the axial velocity yields an expression for axial velocity in a tube with sinusoidal traveling ring waves:

$$W = -c\left\{1 + \left[\frac{4(a^2 - b^2)^2}{(2a^2 + 3b^2)h^4} - \frac{1}{h^2}\right](h^2 - r^2)\right\}. \tag{52}$$

The expression (50) for $\bar{Q}$ may be checked by an independent method for the completely occluded case in which b is equal to a.

For the completely occluded case

$$\bar{Q}_c = \frac{c\pi}{\lambda}\int_0^\lambda h^2\, dZ \tag{53}$$

or

$$\bar{Q}_c = \frac{3\pi a^2 c}{2}, \tag{54}$$

which is the same given by equation (50) for the completely occluded case.

If $b/a \ll 1$, equation (50) leads to

$$\bar{Q} = 4\pi c b^2. \tag{55}$$

Approximate Computation of Peristaltic Flow in a Tube with Square Cross-section. Latham (1966) shows that for a two-dimensional channel with a long wavelength disturbance on the walls and no pressure change over an integral number of wavelengths, the total flow rate per unit width is

$$Q_T = \frac{3cb^2 a}{(a^2 + b^2/2)}; \tag{56}$$

a is the average channel half-thickness and b is the disturbance height.

In a square channel with an average transverse dimension $2a$ for both width and depth and small amplitude $b \ll a$, the average flow rate Q_c would be approximately

$$Q_c \simeq 12cb^2. \tag{57}$$

The expression for Q_c includes the assumption that having a square channel with equal disturbances on all four walls exactly doubles the average flow velocity over that of a two-dimensional channel of the same average width. This speculation is only justified intuitively.

Comparing equation (57) and equation (55) shows that there is a five per cent difference in flow through a square channel and a tube of equal cross-sectional areas when the wall disturbance heights are equal and much smaller than the average channel or tube thickness.

4. *Short Wavelength Approximation to Peristaltic Flow in a Tube.* Consider the flow of a viscous incompressible fluid in a long tube whose walls are located at

$$R = h = a + b \sin k(Z - ct). \tag{58}$$

The tube geometry is shown in Figure 1.

The wavelength may be of the order of a. For sufficiently low wall velocities and high fluid viscosities the fluid motion may be described by the linearized Navier-Stokes equations (59) and (60) and by the continuity equation for incompressible fluids (61). The equations of motion differ from those used in the previous section for a long wavelength traveling wave disturbance by their inclusion of the second partial derivatives with respect to Z and by the inclusion of the equation of motion in the radial direction.

Equations (20) and (23) show that the second partial derivatives with respect to Z are of the same order as the second radial derivatives if the wall disturbance wavelength is of the order of the mean tube radius a.

In addition, from equation (28) the radial pressure derivative may be seen to be of the order of the axial pressure derivative if the wall disturbance wavelength is of the order of the mean tube radius. Therefore, the equation of motion in the radial direction (59) must be included.

The pertinent equations for this particular fluid flow are then

$$\frac{1}{\mu}\frac{\partial p}{\partial R} = \nabla^2 u - \frac{u}{R^2}\cdot \tag{59}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial Z} = \nabla^2 w, \tag{60}$$

and

$$\frac{\partial u}{\partial R} + \frac{u}{R} + \frac{\partial w}{\partial Z} = 0. \tag{61}$$

The notation used in this analysis will differ slightly from that used for the case of the long wavelength disturbance in Section 3. Here u and w will be used to denote velocity components in the radial and axial directions respectively. A non-dimensional radial variable r is defined as

$$r = kR$$

where k is the wave number $k = 2\pi/\lambda$. And a non-dimensional axial parameter s is introduced as $s = k(Z - ct)$.

From equations (59), (60), and (61) it follows that

$$\nabla^2 p = 0. \tag{62}$$

At the boundary $R = h$, the axial and radial velocities are

$$\left.\begin{aligned} u &= -bkc \cos s \\ w &= 0. \end{aligned}\right\} \quad \text{at} \quad R = h. \tag{63}$$

The solution for p is assumed to be separable and of the form

$$\frac{p}{\mu k} = p_0(s) + p_1(r) + \sum_{n=1}^{\infty} f_n(r) F_n(s). \tag{64}$$

Since equation (62) must be valid for all s and all r it will be satisfied if

$$\frac{d^2 p_0(s)}{ds^2} = 0, \tag{65}$$

$$\frac{d^2 p_1(r)}{dr^2} + \frac{1}{r} \frac{dp_1(r)}{dr} = 0, \tag{66}$$

and

$$\sum_{n=1}^{\infty} \left[F_n(s) \frac{d^2 f_n(r)}{dr^2} + \frac{F_n(s)}{r} \frac{df_n(r)}{dr} + f_n(r) \frac{d^2 F_n(s)}{ds^2} \right] = 0. \tag{67}$$

First consider the homogeneous equation (65) for $p_0(s)$ which has for its solution

$$p_0(s) = A' + B's. \tag{68}$$

Since the case to be considered is one in which there is no pressure change over a complete wavelength, the linear $B's$ term is inadmissible in this analysis. If there were a pressure change over a complete wavelength, it could be incorporated in the analysis by the $B's$ term. The constant term has no effect on fluid motion and may thus be omitted with no change in velocities subsequently obtained by use of the equations of motion.

Now examine $p_1(r)$, which is defined by the homogeneous differential equation (66) whose solution is

$$p_1(r) = A'' + B'' \ln r. \tag{69}$$

The constant A'' term in $p_1(r)$ may be combined with the constant term in $p_1(s)$ and may be omitted for the same reason. This means that p is in reality the overpressure caused by the wall disturbance, rather than the actual total pressure. The coefficient B'' of the logarithmic term in equation (69) must be zero because the pressure should be finite at $r = 0$.

This brings the form of the pressure expression to

$$\frac{p}{\mu k} = \sum_{n=1}^{\infty} f_n(r) F_n(s). \tag{70}$$

Satisfying the homogeneous partial differential equation (67) term by term yields

$$F_n(s)\frac{d^2f_n(r)}{dr^2} + \frac{F_n(s)}{r}\frac{df_n(r)}{dr} + \frac{d^2F_n(s)}{ds^2}f_n(r) = 0. \tag{71}$$

Since the pressure should be finite for $r = 0$ and periodic over $s = 2\pi$, it follows that

$$\frac{p}{\mu k} = \sum_{n=1}^{\infty} I_0(nr)[A_n \cos ns + C_n \sin ns]. \tag{72}$$

With the stipulations that the velocities be periodic with the fundamental period $s = 2\pi$ and be finite, the solutions of the differential equations (59), (60), and (61) are then

$$\begin{aligned} w = V_0 &- \sum_{n=1}^{\infty}\left\{A_n\left[\frac{1}{n}I_0(nr) + \frac{r}{2}I_1(nr)\right] + B_nI_0(nr)\right\}\sin ns \\ &+ \sum_{n=1}^{\infty}\left\{C_n\left[\frac{1}{n}I_0(nr) + \frac{r}{2}I_1(nr)\right] + D_nI_0(nr)\right\}\cos ns \end{aligned} \tag{73}$$

and

$$u = \sum_{n=1}^{\infty}\left[\frac{A_nr}{2}I_0(nr) + B_nI_1(nr)\right]\cos ns + \sum_{n=1}^{\infty}\left[\frac{C_nr}{2}I_0(nr) + D_nI_1(nr)\right]\sin ns. \tag{74}$$

equation (73) may be integrated over $0 \le R \le h$ to yield the instantaneous flow rate at any location on the tube axis.

The flow rate is defined as

$$Q = 2\pi\int_0^h Rw(R)\,dR. \tag{75}$$

Substituting equation (73) for the axial velocity into equation (75) yields

$$\begin{aligned} Q = \pi\Bigg\{V_0h^2 &- \frac{2}{k^2}\sum_{n=1}^{\infty}\left[\frac{A_n}{n}\left(\frac{kh}{n}I_1(nkh) + \frac{k^2h^2}{2}I_2(nkh)\right) + \frac{B_nkh}{n}I_1(nkh)\right]\sin ns \\ &+ \frac{2}{k^2}\sum_{n=1}^{\infty}\left[\frac{C_n}{n}\left(\frac{kh}{n}I_1(nkh) + \frac{k^2h^2}{2}I_2(nkh)\right) + \frac{D_nkh}{n}I_1(nkh)\right]\cos ns\Bigg\}. \end{aligned} \tag{76}$$

It now remains to evaluate the constants V_0, A_n, B_n, C_n, and D_n in the equations for radial velocity, axial velocity, and over-pressure by use of the boundary conditions (63).

As the radial boundary defined by equation (58) does not coincide with a constant R surface, the standard boundary value problem technique of satisfying equations (73) and (74) term by term is not an acceptable method for evaluating the constants in the series for the fluid velocities.

The technique used to evaluate the constants was a computer-aided process. The velocity series equations (73) and (74) were truncated at some $n = N$, and the included constants evaluated so as to satisfy boundary conditions at a selected number of points along a wavelength. This technique gave values to all the constants in the truncated series. The number of terms included in the series could be changed to study the effect on the solution. And the program gave velocity, instantaneous flow, and average flow numerical results at selected stations over a wavelength as well.

5. *Results.* The instantaneous flow rate Q as well as the non-dimensional velocities u/c and w/c were calculated for a wide range of tube geometries with the aid of a digital computer.

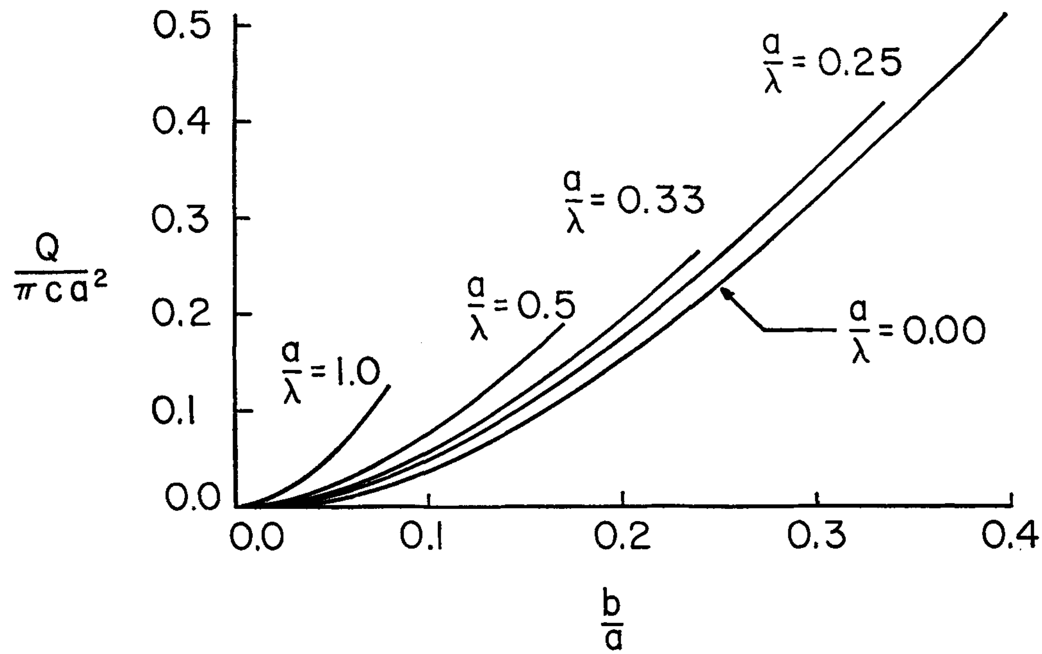

Figure 2. Non-dimensional flow rate for tube with short wavelength wall disturbance

The average non-dimensional average flow rate $Q/\pi ca^2$ is displayed in Figures 2 and 3 for a range of the two parameters b/a and a/λ. A similar curve is plotted on the same figure for the tube analysis of Section 3.

For $b/a \leq 0.90$ and $0.02 \leq a/\lambda \leq 0.05$ the analyses of Sections 3 and 4 give

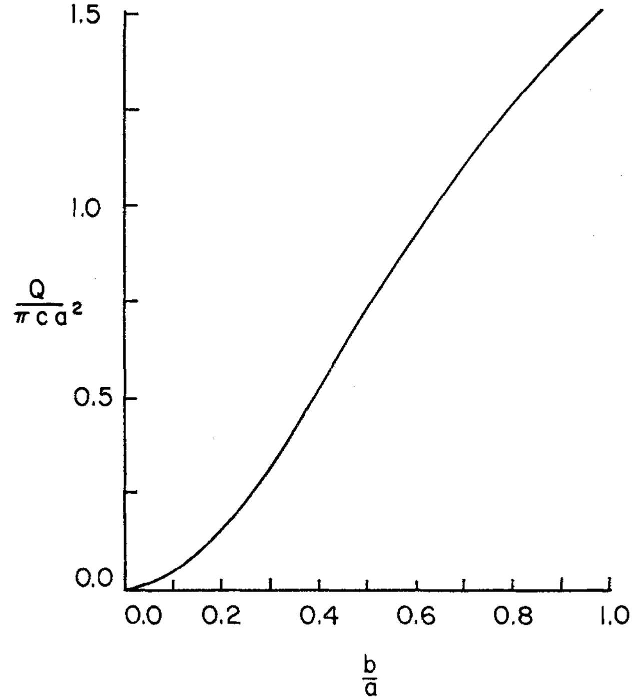

Figure 3. Non-dimensional flow rate for tube with long wavelength wall disturbance

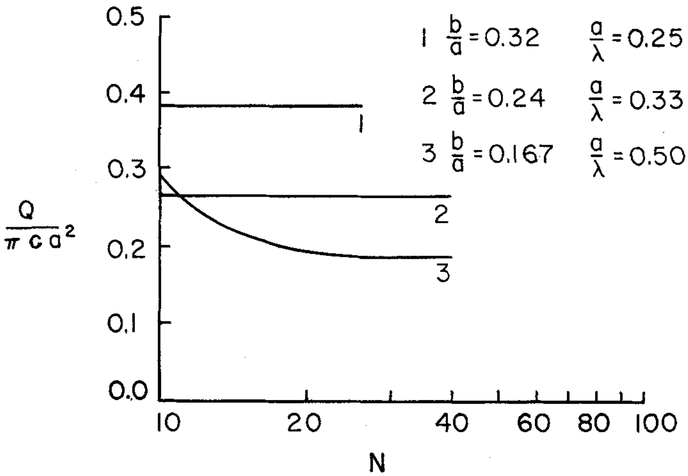

Figure 4. Non-dimensional flow rate versus number of terms included in the series (76) for flow in a tube

the same flow rate. Therefore, the long wavelength approximation is valid in this flow regime.

However, there is less than ten per cent difference in flow predicted by the long and short wave analyses for $0.05 \leq a/\lambda \leq 0.25$ and $b/a \leq 0.33$.

Convergence of the series (76) for flow in a tube was examined to determine the region of validity. The number of terms retained was varied and the effect on the flow rate noted. A graph of computed flow rate versus the number of included terms is shown in Figure 4.

From the flow rate variations it may be said that the series (76) when terminated after the twentieth term gives valid results over the region shown in Figure 5. The series (72), (73), and (74) for fluid velocities and overpressure are valid over the same region.

Computer calculations were performed over a geometry region larger than that shown in Figure 5 in order to determine the domain of validity.

The table shown below reports data relative to the transport of chyme through the small intestine.

Quantity	Value	Reference
Average length; in male of small intestine	$22\frac{1}{2}$ feet	Piersol
Average maximum inflated diameter	25 mm	Piersol
Peristaltic wavespeed	2 cm/min	Evans
Wavelength ratio	$\frac{a}{\lambda} = 0.156$	Houssay, *et al.*
Amplitude ratio	$\frac{b}{a} = 0.60$	Houssay, *et al.*
Time for chyme to pass through small intestine	$4\frac{1}{2}$ hours	Fulton

The long wavelength analysis formula for flow rate yields a non-dimensional flow rate for the amplitude ratio $b/a = 0.60$ of $Q/\pi ca^2 = 0.915$. This may be roughly considered as an average non-dimensional axial velocity $\overline{W}/c$. From the intestinal length and the time required for chyme to traverse the small intestine, the average chyme velocity may be seen to be 2.54 cm/min. From the long wavelength analysis, the average chyme velocity for this intestinal model is predicted to be 0.915×2 cm/min or 1.83 cm/min. The latter velocity

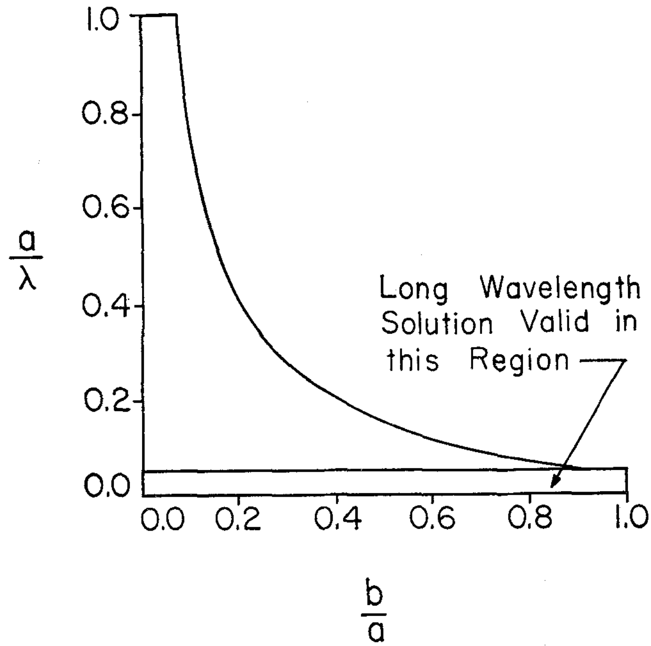

Figure 5. Region of validity for short wavelength tube solution truncated at $N = 20$

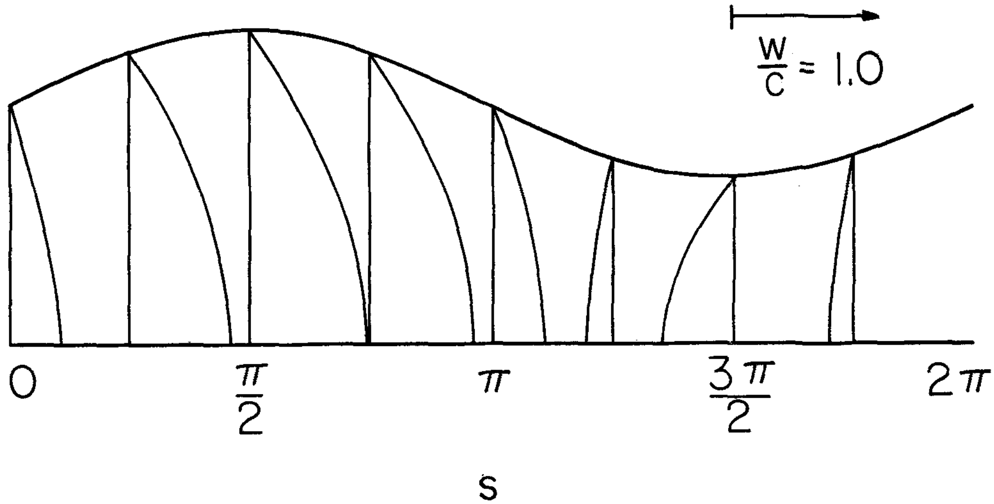

Figure 6. Non-dimensional axial velocity in tube

$$\frac{a}{\lambda} = 0.25 \quad \frac{b}{a} = 0.24$$

is predicted on the basis of no average pressure gradient in the intestine and steady peristaltic waves throughout the process. The calculated velocity differs from the observed by 28 per cent. A slight driving pressure gradient would increase the average calculated axial velocity and thus reduce the error.

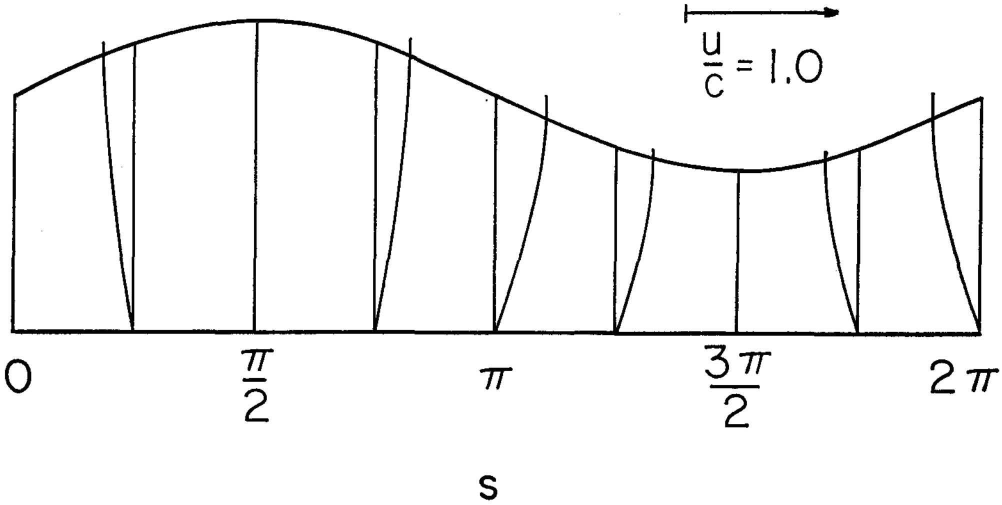

Figure 7. Non-dimensional radial velocity in tube

$$\frac{a}{\lambda} = 0.25 \quad \frac{b}{a} = 0.24$$

The effect of pressure gradient on the average flow rate was evaluated analytically but since no experimental data could be found on this subject the analysis was not reported in this paper.

Thanks are due to the Bio-Medical Engineering Center at Northwestern University for awarding one of the authors, Charles Barton, a training stipend under N.I.H. Training Grant 5T16M874-04 which permitted the completion of this paper.

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RECEIVED 5-7-58