BULLETIN OF
MATHEMATICAL BIOPHYSICS
VOLUME 30, 1968

ENTROPY AND THE COMPLEXITY OF GRAPHS: IV. ENTROPY MEASURES AND GRAPHICAL STRUCTURE

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The structural information content $I_g(X)$ of a graph X was treated in detail in three previous papers (Mowshowitz 1968a, 1968b, 1968c). Those investigations of I_g point up the desirability of defining and examining other entropy-like measures on graphs. To this end the chromatic information content $I_c(X)$ of a graph X is defined as the minimum entropy over all finite probability schemes constructed from chromatic decompositions having rank equal to the chromatic number of X. Graph-theoretic results concerning chromatic number are used to establish basic properties of I_c on arbitrary graphs. Moreover, the behavior of I_c on certain special classes of graphs is examined. The peculiar structural characteristics of a graph on which the respective behaviors of the entropy-like measures I_c and I_g depend are also discussed.

1. Introduction. In this paper we will discuss an entropy measure I_c defined with respect to a class of chromatic decompositions of a finite undirected graph. First, we will examine the behavior of this measure on arbitrary finite undirected graphs, and then specialize to particular cases. Second, we will compare I_c with I_g; finally, we will discuss the significance of the notion of graphical information content and summarize our results.

We begin with some definitions.* A *homomorphism* of a graph X into a graph Y is a mapping ϕ from $V(X)$ into $V(Y)$ such that whenever $[x, y] \in E(X)$, $[x, y]\phi = [x\phi, y\phi] \in E(Y)$. An equivalent way of defining this notion is to define an *elementary homomorphism* of a graph X as the identification of two non-adjacent points; then a homomorphism is just a sequence of elementary

* The definitions given here are largely those of Hedetniemi (1966).
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homomorphisms. ϕ is called a *full* homomorphism of X into Y if $[x\phi, y\phi] \in E(Y)$ implies that there exist points $u, v \in V(X)$ such that $x\phi = u\phi$, $y\phi = v\phi$ and $[u, v] \in E(X)$. The *image* of X under the homomorphism ϕ is the graph $X\phi$, with $V(X\phi) = \{x\phi \mid x \in V(X)\}$ and $E(X\phi) = \{[x\phi, y\phi] \mid [x, y] \in E(X)\}$. Clearly, $X\phi \subset Y$ if ϕ is a homomorphism of X into Y; moreover, $X\phi$ is a section subgraph of Y if ϕ is a full homomorphism. If ϕ maps $V(X)$ onto $V(Y)$, then ϕ is called a homomorphism of X *onto* Y. Note that if ϕ is a full homomorphism of X onto Y, then $E(X)\phi = E(Y)$; if, in addition, ϕ is one-one, then ϕ is an isomorphism. A homomorphism ϕ is said to be of *order* n if $n = |V(X\phi)|$, and is *complete of order* n if $X\phi \cong K_n$.

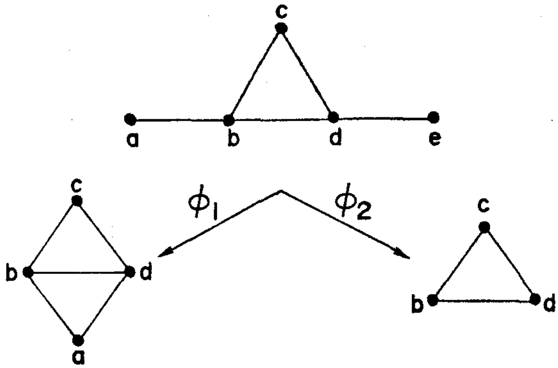
A coloring of a graph X is an assignment of colors to the points of X such that no two adjacent points have the same color. An *n-coloring* of X is a mapping f of $V(X)$ onto the set $\{1, 2, \ldots, n\}$ such that whenever $[x, y] \in E(X)$, $xf \neq yf$, that is a coloring of X which uses n colors. An n-coloring f is *complete* if for every i, j with $i \neq j$ there exist adjacent points such that $xf = i$ and $yf = j$. A decomposition $\{V_i\}_{i=1}^{k}$ of the set $V(X)$ of points of X is said to be a *chromatic decomposition* of X, if $x, y \in V_i$ imply that $[x, y] \notin E(X)$. Clearly, if f is an n-coloring of X, the sets $\{x \in V(X) \mid xf = i\}$ for $i = 1, 2, \ldots, n$ form a chromatic decomposition of X; conversely, a chromatic decomposition $\{V_i\}_{i=1}^{n}$ determines an n-coloring f. Thus, the sets V_i are called *color classes*. The *chromatic number* $\kappa(X)$ is the smallest number n for which X has an n-coloring, or, equivalently, the smallest n for which X has a chromatic decomposition with n color classes. Note that a graph X can have more than one n-coloring (or chromatic decomposition with n color classes). X is called n-chromatic if $\kappa(X) = n$.

The following remarks concerning the relationship between homomorphisms and n-colorings (illustrated in Fig. 1) are necessary for the sequel. It is easy to show (Hedetniemi, 1966, 10) that a graph X has a complete n-coloring f if and only if there exists a complete homomorphism ϕ of X onto the complete graph K_n. From this it follows that if $\kappa(X) = n$, then X has a complete homomorphism of order n; and that the smallest order of all homomorphisms of a graph X is just the chromatic number $\kappa(X)$. Thus, it is clear that to each chromatic decomposition $\{V_i\}_{i=1}^{n}$ of an n-chromatic graph X, there corresponds a homomorphism ϕ of X onto K_n such that each V_i is of the form

$$\{x\phi = u \mid x \in V(X)\}$$

for some $u \in V(K_n)$.

2. *The chromatic information content of a graph.* Since the automorphism group of a graph X gives rise to a unique decomposition of $V(X)$, we were able



$\phi_1: a, e \to a$
$b \to b$
$c \to c$
$d \to d$

$\phi_2: a, d \to d$
$b, e \to b$
$c \to c$

4-coloring:
$f_1: a, e \to 1$
$b \to 2$
$c \to 3$
$d \to 4$

3-coloring:
$f_2: a, d \to 1$
$b, e \to 2$
$c \to 3$

Figure 1. Homomorphisms and n-colorings

to define $I_g(X)$ (Mowshowitz, 1968a) as the entropy of a unique finite probability scheme associated with X. As we indicated above, there is, in general, no unique chromatic decomposition of a graph. So, in order to define a unique information measure which reflects the chromatic structure of a graph, we shall have to consider maximizing or minimizing the entropy function over a certain class of chromatic decompositions. For reasons which will become clear presently, we choose to minimize.

2.1. Definition. Let X be a graph with n points, and let

$$\hat{V} = \{V_i\}_{i=1}^{h} (|V_i| = n_i(\hat{V}))$$

be an arbitrary chromatic decomposition of X where $h = \kappa(X)$. Then the *chromatic information content* $I_c(X)$ of X is given by

$$I_c(X) = \min_{\hat{V}} \left\{ - \sum_{i=1}^{h} \frac{n_i(\hat{V})}{n} \log \frac{n_i(\hat{V})}{n} \right\}.$$

The principal reason for restricting the class of chromatic decompositions, over which to minimize, is convenience, since a great deal is known about the chromatic number of a graph. Moreover, as illustrated by the graph of Figure 2, $I_c(X)$ does not necessarily give the minimum over all chromatic decompositions. However, the following theorem shows that in certain cases, $I_c(X)$ does give the minimum over all chromatic decompositions.

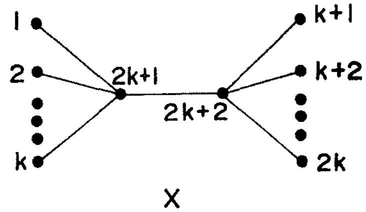


Figure 2.

2.2. Theorem. Let X be a graph with n points and

$$\hat{V} = \{V_i\}_{i=1}^{k}(|V_i| = n_i(\hat{V}, k))$$

be an arbitrary chromatic decomposition of X. Suppose X does not have a complete k-coloring for $k > h = \kappa(X)$. Then

$$I_c(X) = \min_{\hat{V}, k}\left\{-\sum_{i=1}^{k} \frac{n_i(\hat{V}, k)}{n} \log \frac{n_i(\hat{V}, k)}{n}\right\}.$$

Proof. Let f be the k-coloring corresponding to a chromatic decomposition $\{V_i\}_{i=1}^{k}$ for $k > h$. Since f is not complete, there exist i and j ($i \neq j$) such that $[x, y] \notin E(X)$ for every $x \in V_i$ and $y \in V_j$. So, without loss of generality, let $i = k - 1$ and $j = k$, and let $\{U_i\}_{i=1}^{k-1}$ be the chromatic decomposition given by $U_i = V_i$ for $1 \leq i \leq k - 2$ and $U_{k-1} = V_{k-1} \cup V_k$. Then the respective entropies of the finite probability schemes P_k and P_{k-1} associated with $\{V_i\}_{i=1}^{k}$ and $\{U_i\}_{i=1}^{k-1}$ are given by

$$H(P_k) = -\sum_{i=1}^{k} \frac{n_i}{n} \log \frac{n_i}{n} \text{ and}$$

$$H(P_{k-1}) = -\sum_{i=1}^{k-2} \frac{n_i}{n} \log \frac{n_i}{n} - \frac{n_{k-1} + n_k}{n} \log \frac{n_{k-1} + n_k}{n}.$$

Hence,

$$H(P_k) - H(P_{k-1})$$

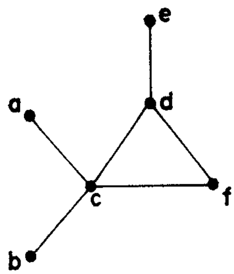
$$= \frac{1}{n}\left[n_{k-1} \log \frac{n}{n_{k-1}} + n_k \log \frac{n}{n_k} - (n_{k-1} + n_k) \log \frac{n}{n_{k-1} + n_k}\right]$$

$$= \frac{1}{n}[-n_{k-1} \log n_{k-1} - n_k \log n_k + (n_{k-1} + n_k) \log (n_{k-1} + n_k)]$$

$$= \frac{1}{n}\left\{n_{k-1}\left[\log\left(1+\frac{n_k}{n_{k-1}}\right)\right] + n_k\left[\log\left(1+\frac{n_{k-1}}{n_k}\right)\right]\right\} \geq 0.$$

So, for each chromatic decomposition of X with $k(>\kappa(X))$ color classes having corresponding probability scheme P_k, there exists a chromatic decomposition with $h(=\kappa(X))$ color classes having probability scheme P_h such that $H(P_h) \leq H(P_k)$. This means that a chromatic decomposition of X, which gives rise to a scheme with minimal entropy, must have $\kappa(X)$ color classes, which proves the theorem.

Figure 3 illustrates the fact that a graph can have different chromatic decompositions with a fixed number of color classes.



$\hat{V}_1$: $\{a, b, e, f\}$, $\{c\}$, $\{d\}$ $\hat{V}_2$: $\{a, d\}$ $\{c, e\}$, $\{b, f\}$

Figure 3. Chromatic decompositions of a graph

Since the minimal order of all homomorphisms of a graph X is $\kappa(X)$, $I_c(X)$ can be interpreted as the amount of information needed to construct a complete homomorphism of minimal order, or, equivalently, as the amount of information needed to construct a $\kappa(X)$-coloring. If X is taken to be a planar graph which corresponds to a map (that is the points of X represent territories and two points x and y are adjacent if and only if their respective territories have a common frontier), then $I_c(X)$ can be viewed as the amount of information needed to color the map (using as few colors as possible) so that no two bordering territories have the same color.

Another closely related interpretation of I_c is in terms of the notion of independence. A set S of points of a graph X is called *independent* if no two points of S are adjacent in X. (Note that the color classes of a chromatic decomposition are independent sets.) The number of points of X in a maximally independent set is the *independence number* of X. Intuitively speaking, $I_c(X)$ is inversely related to the number of points in a maximally independent set. This is clear for the complete graph and the null graph, since $I_c(K_n) = \log n$

($\kappa(K_n) = n$) and $I_c(\bar{K}_n) = 0$ ($\kappa(\bar{K}_n) = 1$); and the independence numbers of K_n and $\bar{K}_n$ are 1 and n, respectively. Of course, the relationship is only suggestive, since two graphs with the same independence number can have different chromatic information content.

3. *Properties of the measure* I_c

In this section, we will prove some general results concerning chromatic information content and then look at the behavior of I_c on uniquely n-colorable graphs, Kronecker product graphs, and trees.

3.1. Theorem. $I_c(X) \leq \log \kappa(X)$ for any finite undirected graph X.

Proof. This follows immediately from Definition 2.1 using the same argument as in Lemma 4.1 of a previous paper (Mowshowitz, 1968b).

Now, we give some results which are simple consequences of Theorem 3.1 and of known facts about the chromatic number of a graph.

3.1a. Corollary. For a graph X and complete homomorphism ϕ, $I_c(X) \leq I_c(X\phi)$.

Proof. $\kappa(X) \leq \kappa(X\phi)$, and $I_c(X\phi) = \log \kappa(X\phi)$, since $X\phi$ is complete.

Various bounds for the chromatic number of a graph are known. It is obvious that upper bounds are directly applicable in the present context. We use one such bound in the following:

3.1b. Corollary. Let X be a graph with $d_0 = \max_x d(x)$ where $d(x)$ denotes the degree of x. Then $I_c(X) \leq \log (d_0 + 1)$.

Proof. This follows from the fact that $\kappa(X) \leq d_0 + 1$.

If X and Y are graphs with the same set of vertices, $\kappa(X \cup Y) \leq \kappa(X) \cdot \kappa(Y)$; if $V(X) \cap V(Y) = \varnothing$, then $\kappa(X \cup Y) = \max \{\kappa(X), \kappa(Y)\}$. So, we have

3.1c. Corollary. Let X and Y be graphs. Then

(i) $I_c(X \cup Y) \leq \log \kappa(X) + \log \kappa(Y)$ if $V(X) = V(Y)$
(ii) $I_c(X \cup Y) \leq \log (\max \{\kappa(X), \kappa(Y)\})$ if $V(X) \cap V(Y) = \varnothing$.

The following relates the chromatic information content of a graph to that of its complement.

3.1d. Corollary. Let X be a graph with n points. Then

(i) $I_c(\bar{X}) \leq \log[n - \kappa(X) + 1]$

(ii) $I_c(\bar{X}) \leq \log\left[\left(\frac{n+1}{2}\right)^2\right] - \log\kappa(X)$.

Proof. (i) and (ii) follow from the inequalities

$$2n^{\frac{1}{2}} \leq \kappa(X) + \kappa(\bar{X}) \leq n + 1$$

and

$$n \leq \kappa(X) \cdot \kappa(\bar{X}) \leq \left[\left(\frac{n+1}{2}\right)^2\right], \quad \text{respectively.}$$

3.1e. Corollary. For graphs X and Y,

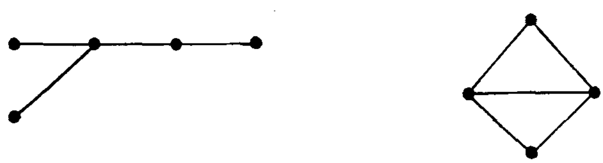
$$I_c(X + Y) \leq \log[\kappa(X) + \kappa(Y)]$$

Proof. The result follows from the fact that $\kappa(X + Y) = \kappa(X) + \kappa(Y)$.

Hedetniemi (1966, 14) calls a graph X *uniquely k-colorable* if it has only one homomorphism of order k, that is, if for any two homomorphisms ϕ_1 and ϕ_2 of X onto K_n, and any two points x, y, $x\phi_1 = y\phi_1$ if and only if $x\phi_2 = y\phi_2$. Uniquely k-colorable graphs are particularly well-suited for our purposes, since if X is uniquely k-colorable, where $k < n = |V(X)|$, then $\kappa(X) = k$ and X has a unique chromatic decomposition with k color classes. Thus, if $\{V_i\}_{i=1}^{k}$ with $|V_i| = n_i$ is the (unique) chromatic decomposition of a uniquely k-colorable graph X,

$$I_c(X) = -\sum_{i=1}^{k} \frac{n_i}{n} \log \frac{n_i}{n}.$$

Figure 4 shows some uniquely k-colorable graphs.



2-colorable 3-colorable

Figure 4. Uniquely k-colorable graphs

Since the task of computing $I_c(X)$ is considerably simplified in the case of uniquely k-colorable graphs, it is worthwhile trying to determine such graphs.

Hedetniemi (1966, 16) mentions that K_n and $K_n - x$ (the graph formed by removing a point from K_n) are uniquely n- and $(n-1)$-colorable; $\bar{K}_n$ is uniquely 1-colorable, and all connected bipartite (2-chromatic) graphs are uniquely 2-colorable. The same author also shows that $X = X_1 + X_2$ is uniquely k-colorable if and only if X_1 and X_2 are, respectively, uniquely k_1- and k_2-colorable where $k = k_1 + k_2$.

3.2. Theorem. Let X_1 and X_2 be uniquely k_1- and k_2-colorable graphs with m and n points, respectively. Then

$$I_c(X_1 + X_2) = $$
$$= \frac{m}{m+n} I_c(X_1) + \frac{n}{m+n} I_c(X_2) + \left[\log(m+n) - \frac{m\log m + n\log n}{m+n}\right].$$

Proof. Let

$$I_c(X_1) = -\sum_{i=1}^{k_1} \frac{m_i}{m}\log\frac{m_i}{m} \quad\text{and}\quad I_c(X_2) = -\sum_{i=1}^{k_2} \frac{n_i}{n}\log\frac{n_i}{n}.$$

Then it is clear that

$$\begin{aligned} I_c(X_1 + X_2) &= -\sum_{i=1}^{k_1} \frac{m_i}{m+n}\log\frac{m_i}{m+n} - \sum_{i=1}^{k_2} \frac{n_i}{m+n}\log\frac{n_i}{m+n} \\ &= -\frac{1}{m+n}\left[\sum_{i=1}^{k_1} m_i\log m_i + \sum_{i=1}^{k_2} n_i\log n_i - (m+n)\log(m+n)\right] \\ &= \log(m+n) - \frac{1}{m+m}\left[\sum_{i=1}^{k_1} m_i\log m_i + \sum_{i=1}^{k_2} n_i\log n_i\right] \\ &= \log(m+n) \\ &\qquad - \frac{1}{m+n}(m\log m + n\log n - m\, I_c(X_1) - n\, I_c(X_2)), \end{aligned}$$

from which the result follows.

In particular, if $X_1 \cong X_2$, we have

3.2a. Corollary. $I_c(X_1 + X_2) = I_c(X) + 1$ where X_1 and X_2 are uniquely k-colorable graphs isomorphic to X.

Proof. It is obvious that any two isomorphic graphs have the same chromatic information content. So, setting $I_c(X_1) = I_c(X_2) = I_c(X)$ and $m = n$ in the theorem, we obtain the desired result.

As an example, consider the path L_2 of length two. $I_c(L_2) = \log 3 - \frac{2}{3}$ and $I_c(L_2 + L_2) = \log 3 + \frac{1}{3} = I_c(L_2) + 1$.

As an additional basis for comparison with I_g, we consider the behavior of I_c on the Kronecker product of two graphs. Hedetniemi (1966, 26) points out that if $X = X_1 \otimes X_2$, then either there exist homomorphisms ϕ_1 and ϕ_2 of X such that $X\phi_1 = X_1$ and $X\phi_2 = X_2$, or X_1 or X_2 is totally disconnected. Using this fact, he proves some results concerning $\kappa(X_1 \otimes X_2)$ which we summarize as follows.

3.3. Lemma. Let X and Y be graphs. Then

(i) $\kappa(X \otimes Y) \leq \min\{\kappa(X), \kappa(Y)\}$
(ii) $\kappa(X \otimes X) = \kappa(X)$
(iii) If $X \otimes Y \supset Y' \cong Y$, then $\kappa(X \otimes Y) = \kappa(Y)$, and, hence $\kappa(Y) \leq \kappa(X)$.

An immediate consequence of the lemma is

3.4. Theorem. Let X and Y be graphs. Then

(i) $I_c(X \otimes Y) \leq \log[\min\{\kappa(X), \kappa(Y)\}]$
(ii) $I_c(X \otimes X) \leq \log \kappa(X)$
(iii) If $X \otimes Y \supset Y' \cong Y$, then $I_c(X \otimes Y) \leq \log \kappa(Y)$.

We conclude the section with a discussion of the chromatic information content of some particular kinds of trees. First, note that all trees are 2-chromatic and, thus, as indicated earlier, all trees are uniquely 2-colorable. Moreover, $I_c(T) \leq \log 2 = 1$ for any tree T.

3.5. Theorem. Let L_{n-1} be a path of length $n - 1$ for $n \geqslant 2$. Then

$$I_c(L_{n-1}) = \begin{cases} \log 2n - \dfrac{1}{2n}[(n+1)\log(n+1) + (n-1)\log(n-1), & \text{if } n \text{ is odd} \\ 1, & \text{if } n \text{ is even} \end{cases}$$

Proof. Let $V(L_{n-1}) = \{x_1, x_2, \ldots, x_n\}$ and $E(L_{n-1}) = \{[x_1, x_2], \ldots, [x_{n-1}, x_n]\}$. Since L_{n-1} is uniquely 2-colorable, we need only produce a single 2-coloring to determine the desired chromatic decomposition. Let f be a coloring defined by

$$x_1 f = 1, \quad x_2 f = 2 \quad \text{and} \quad x_{i+1} f = \begin{cases} 1 & \text{if } x_i f = 2 \\ 2 & \text{if } x_i f = 1 \end{cases}.$$

A simple induction shows that f is always a 2-coloring, and it is obvious that $|\{x \in V(L_{n-1}) \mid xf = i\}|$ equals $(n+1)/2$ for $i = 1$ and $(n+1)/2 - 1$ for $i = 2$ when n is odd, and $n/2$ for both $i = 1$ and $i = 2$ when n is even. So, when n is odd,

$$I_c(L_{n-1}) = \frac{n+1}{2n} \log \frac{2n}{n+1} + \frac{n-1}{2n} \log \frac{2n}{n-1},$$

and when n is even $I_c(L_{n-1}) = 2(n/2n \log 2n/n) = \log 2 = 1$, as required.

A *star* S_n is an n-point tree with a point of degree $n - 1$; all other points are of degree 1. Clearly,

$$I_c(S_n) = \frac{1}{n} \log n + \frac{n-1}{n} \log \frac{n}{n-1}$$

since we can assign color 1 to the (distinguished) point of degree $n - 1$ and 2 to each of the other points. We consider a generalization of the star, for which we need to define the notion of distance. If x and y are connected points of a graph X, the *distance* $\rho(x, y)$ between x and y is the minimal number of edges in a path with initial point x and endpoint y.

3.6. Theorem. Let $S_t(\subset X)$ be a star with distinguished point x_0, and let X be homeomorphic to S_t. Let $n_i = |\{x \in V(X) \mid \rho(x_0, x) = i\}|$ for

$$i = 0, 1, 2, \ldots, k \text{ where } k = \max_x \rho(x_0, x).$$

If

$$r = \sum_{\substack{0 \le i \le k \\ i \text{ even}}} n_i, \qquad s = \sum_{\substack{0 \le i \le k \\ i \text{ odd}}} n_i,$$

then

$$I_c(X) = \frac{r}{n} \log \frac{n}{r} + \frac{s}{n} \log \frac{n}{s} \quad \text{where } n = |V(X)|.$$

Proof. By induction on k. The cases $k = 0$ and $k = 1$ follow trivially from the preceding remarks. So, suppose the theorem holds for $k = m(>1)$ and the mapping f with $xf \in \{1, 2\}$ is the (unique) 2-coloring of X. Then if $k = m + 1$, it is clear that the mapping f' given by $xf' = xf$ for

$$x \in \{y \in V(X) \mid \rho(y, x_0) \le m\}$$

and

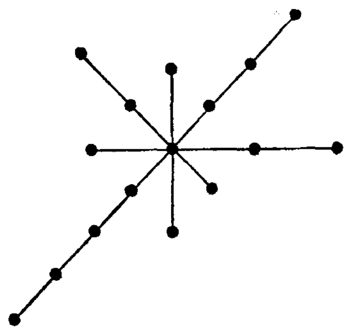
$$xf' = \begin{cases} 1 \text{ if } yf' = 2 & \text{and} \quad \rho(y, x_0) = m \\ 2 \text{ if } yf' = 1 & \text{and} \quad \rho(y, x_0) = m \end{cases} \text{for } \rho(x, x_0) = m + 1$$

is the desired 2-coloring for the case $k = m + 1$. The conclusion of the theorem follows immediately.

Figure 5 illustrates the theorem.

4. Comparison of the two measures of graphical information

The principal reason for introducing the measure I_c in addition to the one studied in earlier papers (Mowshowitz 1968a, 1968b, 1968c) is to point out the



X

$I_c(X) = \frac{6}{16} \log \frac{16}{6} + \frac{10}{16} \log \frac{16}{10}$

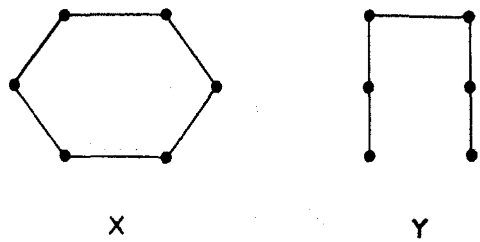
Figure 5. Chromatic information content of a graph homeomorphic to a star

dependence of such measures on certain peculiar structural features of a graph. In one sense, we are only belaboring the obvious, since it is clear from the way an entropy measure on a graph is defined that it cannot characterize the graph's structure completely. However, the juxtaposition of these two different measures serves a useful purpose in the sense that it prevents one from making outrageous claims about how adequately the information content of a graph reflects the complexity of a graph; and, on the positive side, it suggests the possibility of studying classes of entropy measures defined on graphs.

It is fairly obvious from the preceding section that there is relatively little agreement between I_g and I_c. This, of course, follows from the fact that the symmetries (automorphisms) of a graph X are, at best, tenuously related to the various colorings of X. For example, the automorphism group of a cycle of length n is transitive for any n; whereas, the chromatic number of such a graph is two or three, depending on whether n is even or odd. A more serious source of divergence is provided by the class of trees. The chromatic number of a tree is two, so $I_c(T) \leq 1$ for any tree T. However, for every $n \geq 7$, there exists a tree T with n points whose automorphism group consists of the identity alone, so that $I_g(T) = \log n$. Figure 6 presents graphs X and Y for which $I_g(X) < I_c(X)$ and $I_g(Y) > I_c(Y)$.

As one would expect, it is rather difficult to determine classes of graphs on which I_g and I_c agree. Two simple examples are furnished by null graphs and stars. Figure 7 exhibits two graphs on which the two measures agree.

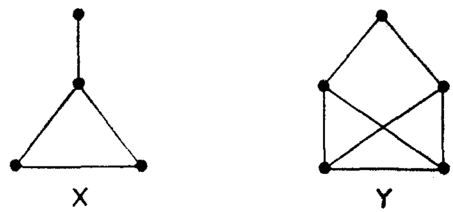
From our brief survey of the properties of I_c, it would appear that this measure is much less well-behaved with respect to graph operations than I_g.



$I_g(X) = 0$ $\quad$ $I_g(Y) = \log 3$

$I_c(X) = 1$ $\quad$ $I_c(Y) = 1$

Figure 6. Graphs on which I_g and I_c differ



$I_g(X) = I_c(X) = \frac{3}{2}$ $\quad$ $I_g(Y) = I_c(Y) = \log 5 - \frac{4}{5}$

Figure 7. Graphs on which I_g and I_c agree

However, it is quite possible that the operations considered are not appropriate for the measure I_c. That is to say, the operations we chose to consider in connection with I_g are of the following kind. To each graph operation $\circ$ (such as the cartesian product), there corresponds a group operation ∇ (such as the cartesian product for groups) such that $G(X) \nabla G(Y) \leq G(X \circ Y)$ for any graphs X and Y. When the orbits of $G(X) \nabla G(Y)$ are cartesian products of orbits of $G(X)$ and $G(Y)$, this condition insures that every orbit of $G(X \circ Y)$ is a union of orbits of $G(X) \nabla G(Y)$. This, in turn, insures that $I_g(X \circ Y) \leq I_g(X) + I_g(Y)$. In order to have a comparable result for the chromatic information content, we would have to find an operation $\circ$ on graphs which satisfies the following: if $\{W_m\}_{m=1}^{t}$ is a chromatic decomposition of $X \circ Y$ with $t = \kappa(X \circ Y)$, then each W_m is a union of sets of the form $U_i \times V_j$ where $\{U_k\}_{i=1}^{h}$ and $\{V_i\}_{i=1}^{k}$ are chromatic decompositions of X and Y, respectively, with $h = \kappa(X)$ and $k = \kappa(Y)$. The binary operations (cartesian product, composition and Kronecker product) studied earlier might be found to satisfy this condition for certain classes of graphs. In any case, specifying such operations and finding conditions which insure that $I_c(X \circ Y) = I_c(X) + I_c(Y)$ are certainly interesting problems for further study.

5. *Conclusion*

What we have attempted to accomplish in this and the preceding papers (Mowshowitz 1968a, 1968b, 1968c) is the development and exploration of some tools for measuring the structural complexity of a particular mathematical object. The basic idea in our approach to this problem consists in constructing a finite probability scheme from a decomposition of the set of points of the object in question. As we have seen in the case of the measure I_g defined relative to the automorphism group of a graph, the entropy of such a finite probability scheme provides a compact analytic device for a partial characterization of the relative complexity of an object.

The previous papers were devoted to a systematic study of the measure I_g in order to demonstrate the usefulness of the concept of structural information in classifying graphs according to their relative complexity. In the present paper we have discussed an entropy measure I_c defined relative to a class of chromatic decompositions of a graph and compared it with the previous measure.

Perhaps the most significant fact to emerge in this study is the (mathematical) feasibility of using the entropy function to characterize the structure of an object. More general objects than graphs can be studied from this point of view. For example, suppose an object X is defined as a system $X = \langle U; E_1, E_2, \ldots, E_k \rangle$ where U is a non-empty finite set and each E_i is a binary relation on U. If $X = \langle U; E_1, \ldots, E_k \rangle$ and $Y = \langle V; F_1, F_2, \ldots, F_k \rangle$ are two such objects, we can define a homomorphism of X into Y as a mapping ϕ of U into V such that for every $x, y \in U$, $(x, y) \in E_i$ implies that $(x, y)\phi = (x\phi, y\phi) \in F_i$ for each $i = 1, 2, \ldots, k$. A homomorphism ϕ which is a one-one mapping of U onto V can then be called an isomorphism of X onto Y; and if $U = V$, it is reasonable to call ϕ an automorphism of X. As in the case of graphs, it is easily verified that the set of all automorphisms of an object X forms a group under the usual composition of mappings.

If ϕ is a homomorphism of X onto Y, it is clear that the sets given by $\{x \in U \mid x\phi = y\}$ for some $y \in V$ form a decomposition of the set U. Moreover, the orbits of the group of the object X form a decomposition of U. Thus, just as we did for graphs, we can define an information measure on such an object by taking the entropy of a finite probability scheme constructed from the decomposition given by its group; or we can define a measure relative to a class of decompositions corresponding to a class of homomorphisms. In any case, it is feasible to develop the notion of structural information content for a variety of mathematical objects.

The author is indebted to Professors A. Rapoport, M. Kochen, N. Rashevsky, and to Dr. S. Hedetniemi for advice and criticism.

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RECEIVED 6-16-67