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AN AXIOMATIC EXPLANATION OF COMPLETE SELF-REPRODUCTION

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A similarity between the concepts of reproduction and explanation is observed which implies a similarity between the less well understood concepts of complete self-reproduction and complete self-explanation. These latter concepts are shown to be independent from ordinary logical-mathematical-biological reasoning, and a special form of complete self-reproduction is shown to be axiomatizable. Involved is the question whether there exists a function that belongs to its own domain or range. Previously, Wittgenstein has argued, on intuitive grounds, that no function can be its own argument. Similarly, Rosen has argued that a paradox is implied by the notion of a function which is a member of its own range. Our result shows that such functions indeed are independent from ordinary logical-mathematical reasoning, but that they need not imply any inconsistencies. Instead such functions can be axiomatized, and in this sense they really do exist. Finally, the introduced notion of complete self-reproduction is compared with "self-reproduction" of ordinary biological language. It is pointed out that complete self-reproduction is primarily of interest in connection with formal theories of evolution.

1. Self-Reproduction and Self-Explanation. If an observer has examined and experimented with an object long enough that he can reproduce it or construct a model of it, he can be said to have understood it. Or, to be more precise, he has understood those properties of the object that he has modeled. There is, therefore, a definite sense in which an explanation can be identified with a reproduction.

If the object is concrete, for example a physical automaton, the explanation of it can be thought of as a description of how to reproduce the automaton from

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given physical parts. Since it is usually understood that the description can be effectively interpreted, it is in fact the whole reproduction that constitutes the explanation. The properties and the availability of the necessary parts are postulated, and not explained, in such a first-level explanation. A higher-level explanation would have to explain these latter concepts in other terms.

If the object is abstract, for example a theorem in a formal theory, the explanation of it involves a proof sequence whose last element is the theorem in question. Thus, the problem of explaining the theorem is the problem of finding a proof sequence that reproduces the theorem. In such a first-level explanation the axioms and rules of inference of the theory are taken for granted as well as the consistency of the theory. These concepts may, however, themselves be explained in an unexplained metatheory, and in this way a chain of higher-level explanations is conceivable.

Because of this similarity between explanation and reproduction, an explanation of reproduction can be thought of as an explanation of explanation, or as a reproduction of reproduction. Although such a reproduction of reproduction merely is a second-level concept, it is often referred to as "self-reproduction" (compare, for example, von Neumann, 1966; Penrose, 1958). This will not be done here, simply because such a second-level concept is incompletely self-referring in the sense that it requires externally reproduced objects (externally explained concepts). Instead we will use the name "self-reproduction" for a completely self-referring reproduction process.

Let π_2 be a function that reproduces (explains) an entity π_1:

$$\pi_2(\pi_1) = \langle \alpha, \beta, \cdots, \pi_1 \rangle. \tag{1}$$

When interpreted as a reproduction process, equation (1) may be read: π_2 is the behavior function of an automaton that reproduces the entity π_1. From the input π_1 the automaton thus produces a sequence of constructs $\alpha, \beta, \cdots$ that ends up in the final output π_1.

When interpreted as an explanation process, equation (1) may be read: π_2 is an explanation function that produces a proof sequence of the theorem π_1, that is, a sequence that begins with the axiom α of the reference theory and ends with the theorem π_1. To obtain a more general setting, we might consider π_2 not merely as an explanation function but as an explanation relation, thus admitting several proof sequences of one and the same theorem.

If we shift our attention from the entity π_1 towards π_2 and ask for its reproduction (explanation), we are led to a further relation π_3, such that: $\pi_3(\pi_2) = \langle \cdots, \pi_2 \rangle$. Continued enquiries produce a chain of higher-level explanations: $\pi_1 \in D\pi_2$; $\pi_2 \in D\pi_3$; $\pi_3 \in D\pi_4$; $\cdots$, such that π_i belongs to the domain, $D\pi_{i+1}$, of π_{i+1} and therefore is explained by π_{i+1}.

Our problem is to find out whether there are explicability chains that are complete in the sense that they are cyclic, as in the example: $\pi_1 \in D\pi_2$; $\pi_2 \in D\pi_1$.

Among such eventual complete explicability chains, that is, self-explanations, we shall distinguish between symbiotic and atomic self-explanations according to the following characterizations.

Characterization 1. A *symbiotically self-reproducing* (*symbiotically self-explaining*) pair of distinct entities π_1 and π_2 must have a complete explicability chain of length two (or longer if more than a pair of entities), such that (in the pair-case):

$$\left.\begin{aligned} \pi_1(\pi_2) &= \langle \cdots, \pi_2 \rangle \\ \pi_2(\pi_1) &= \langle \cdots, \pi_1 \rangle \end{aligned}\right\}. \tag{2}$$

Characterization 2. An *atomically self-reproducing* (*atomically self-explaining*) entity π must constitute a unit-length complete explicability chain, such that:

$$\pi(\pi) = \langle \pi \rangle. \tag{3}$$

Atomic self-reproduction (self-explanation) shall result from symbiotic self-reproduction when all the distinct entities of the symbiotic case coalesce.

As is seen from equation (3), the existence of an atomic π implies the existence of a function that belongs to its own domain and range. Previously, Wittgenstein (1921; see in particular aphorism no 3.333) has argued, on intuitive grounds, that no function can be its own argument. Similarly, Rosen (1959, 1962) has argued that a paradox is implied by the notion of a function which is a member of its own range. Indeed, we shall see (Section 3) that such functions really cannot be derived from ordinary logical-mathematical reasoning. They need not, however, as we shall see in Section 4, imply any inconsistencies. Instead, an atomically self-reproducing entity can be axiomatized, and in this sense it really does exist.

2. Models of Sequences. The above independence and consistency questions will be studied in relation to axiomatic set theories. We need, therefore, a set-theoretic model of the sequences that appear in equation (1). This equation contains, beside the sequence of its right member, also sequences in its left member. The here occurring π-relations will namely, as usual, be represented with sets of ordered pairs (sequences of length two).

An ordered pair is by definition an entity $\langle x, y \rangle$ such that:

$$(\forall\, x, y, u, v) \colon [\langle x, y \rangle = \langle u, v \rangle] \Leftrightarrow [(x = u)\ \&\ (y = v)]. \tag{4}$$

Sequences of arbitrary length are recursively defined in terms of ordered pairs as follows (see for example Mendelson, 1964):

$$\langle x \rangle = x, \qquad \langle x_1, x_2, \cdots, x_{n+1} \rangle = \langle \langle x_1, x_2, \cdots, x_n \rangle, x_{n+1} \rangle. \tag{5}$$

The first set-theoretic model of the ordered pair was introduced by Wiener (1914), namely:

$$\langle x, y \rangle = \{\{\{x\}, \varnothing\}, \{\{y\}\}\}. \tag{6}$$

The dissymmetry between the two elements of $\langle x, y \rangle$, which makes this entity satisfy equation (4), is introduced by using the empty set, $\varnothing$. A similar model was at that time also suggested by Hausdorff (1914). A few years later Kuratowski (1921) offered a model free from any auxiliary sets, namely:

$$\langle x, y \rangle = \{\{x\}, \{x, y\}\}. \tag{7}$$

Comparing the two models we see that Wiener's raises the logical type of the pair by three units over that of its elements, whereas Kuratowski's raises the type by only two units. Furthermore, Wiener's model continues to separate the two elements of the pair even when they are identical, whereas in Kuratowski's model the separation mechanism ceases to work when the elements become identical. In this respect Kuratowski's model is more in conformity with Leibniz' *Identitas Indiscernibilium* and will prove quite suitable for our purposes.

It should be mentioned that Quine (1945) has suggested a model in which the logical type of the pair is the same as that of its elements. Hence by equation (5), all sequences, independent of their length, will maintain the type of their elements. This is certainly an attractive feature although not necessary for our purposes. Even though Quine's model does not conform with Leibniz' principle in the above sense, it appears that it can be modified towards harmony with this principle. However, Quine's model is based on the assumption that all entities of our universe of discourse are classes. Even though this assumption is fulfilled in Quine's "New Foundations . . .", with the Quine axiom of individuals (Quine 1945, 1961), we want primarily to discuss the independence of complete self-reproduction in a less specialized axiomatic set theory, namely the von Neumann-Bernays-Gödel theory, which possesses a standard model. Therefore, we will base the following discussions upon the Kuratowski model of sequences (eqs. (7) and (5)) which also in more general contexts is by far the most widely used.

3. The Independence of Complete Self-Reproduction. We want to demonstrate the independence of complete self-reproduction with respect to an ordinary set theory, namely the von Neumann-Bernays-Gödel theory (NBG). This will be done by showing that symbiotic, as well as atomic, self-reproduction implies the negation of the axiom of restriction, a negation which is known to be independent from NBG.

The *axiom of restriction*, axiom R:

$$\forall S\colon (S \neq \varnothing) \Rightarrow \exists\, u\colon [(u \in S)\ \&\ \forall\, v\colon (v \in u) \Rightarrow (v \notin S)] \tag{8}$$

says that with each nonempty set S there exists an element u of S, such that no element of u is also an element of S. Axiom R is known to be both consistent and independent, that is completely independent, with respect to ordinary set theory in the form of NBG (see Mendelson, 1964; Mendelson is one of the contributors to this result). This means that also $\neg R$ is independent from NBG. We can in fact consistently add either R or its negation, $\neg R$, as an axiom to NBG (provided, of course, that this set theory is consistent).

Let us first see that both symbiotic and atomic self-reproduction imply epsilon cycles.

Theorem 1. Symbiotic self-reproduction, as characterized by equation (2), implies the presence of at least one epsilon-cycle.

Proof. Equation (2) is obviously equivalent to the following statement: "$\langle \pi_2, \langle \cdots, \pi_2 \rangle\rangle \in \pi_1$ and $\langle \pi_1, \langle \cdots, \pi_1 \rangle\rangle \in \pi_2$." If each of the sequences $\langle \cdots, \pi_1 \rangle$ and $\langle \cdots, \pi_2 \rangle$ is of length two or larger, the Kuratowski sequence model (eqs. (7) and (5)) transforms the above version of equation (2) into:

$$\text{"}\{\{\pi_2\}, \{\pi_2, \{\{\cdot\}, \{\cdot, \pi_2\}\}\}\} \in \pi_1 \quad \text{and} \quad \{\{\pi_1\}, \{\pi_1, \{\{\cdot\}, \{\cdot, \pi_1\}\}\}\} \in \pi_2.\text{"}$$

Here the presence of epsilon cycles is transparent. One of these is for example: $\pi_1 \in \{\pi_1\} \in \{\{\pi_1\}, \{\,\}\} \in \pi_2 \in \{\pi_2\} \in \{\{\pi_2\}, \{\,\}\} \in \pi_1$, that is, a cycle of length six. All epsilon cycles are illustrated with the graph of Figure 1. There are two independent sources of these cycles. One is the part of Characterization 1 that reads: "$\pi_1 \in D\pi_2$ and $\pi_2 \in D\pi_1$." This part alone produces the innermost cycles of Figure 1, each of length six. (In the atomic case this part corresponds to the existence of a function that is a member of its own domain.) The other cycle source is the part of Characterization 1 that reads: "$\langle \cdots, \pi_1 \rangle \in R\pi_2$ and $\langle \cdots, \pi_2 \rangle \in R\pi_1$." This part alone produces the cycle of length ten in Figure 1. (In the atomic case it corresponds to the existence of a function that belongs to its own range.) Finally, if instead the sequences $\langle \cdots, \pi_1 \rangle$ and $\langle \cdots, \pi_2 \rangle$ are of unit length, that is, equal to $\langle \pi_1 \rangle$ $(= \pi_1)$ and $\langle \pi_2 \rangle$ $(= \pi_2)$ respectively, the epsilon cycles reduce to the innermost cycle of length six in Figure 1. This completes the proof of Theorem 1.

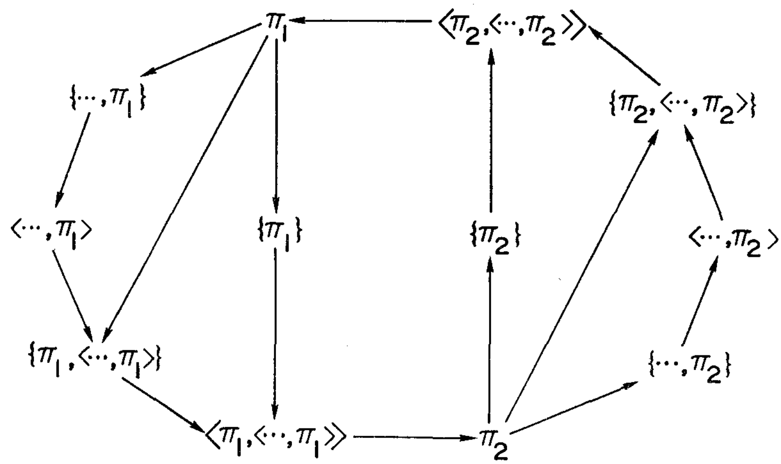

Figure 1. Epsilon graph of symbiotic self-reproduction as characterized by equation (2). An arrow from label x to label y means that $x \in y$

Theorem 2. Atomic self-reproduction, as characterized by equation (3), implies the presence of an epsilon cycle.

Proof. Equation (3) is equivalent with $\langle \pi, \pi \rangle \in \pi$, or, according to equation (7), $\{\{\pi\}\} \in \pi$. There is therefore an epsilon cycle, $\pi \in \{\pi\} \in \{\{\pi\}\} \in \pi$, which is of length three if π, $\{\pi\}$ and $\{\{\pi\}\}$ are all distinct, but shorter if they are not distinct. (Later, in Section 4, we shall actually narrow the characterization of atomic self-reproduction towards a definition such that both the epsilon cycle and the explicability cycle will have unit length.)

Next, we want to demonstrate that the existence of an epsilon cycle implies $\neg R$, the negation of the axiom of restriction.

Theorem 3. The presence of an epsilon cycle implies $\neg R$.

Proof. Let us carry out the proof for one of the displayed epsilon cycles, for example $\pi \in \{\pi\} \in \{\{\pi\}\} \in \pi$, the cycle of atomic self-reproduction. It is obvious how the proof generalizes to a general case. We first define a nonempty set $S = \{\pi, \{\pi\}, \{\{\pi\}\}\}$, whose elements are the elements of the cycle. Since $\{\{\pi\}\} \in \pi$, obviously each element u of S satisfies $\exists\, v$: $(v \in u)$ & $(v \in S)$. If, for example, $u = \pi$, the element $\{\{\pi\}\}$ of u is also an element of S. Hence we have demonstrated that $\exists\, S: S \neq \varnothing$ & $\forall\, u: [(u \in S) \Rightarrow \exists\, v: (v \in u)$ & $(v \in S)]$, which is precisely $\neg R$ (compare eq. (8)). This completes the proof.

Theorem 3 implies, via Theorem 1 and 2 that complete self-reproduction is independent from NBG. For if it were not independent, it could be derived in

NBG. We could then demonstrate the existence of epsilon cycles in NBG, that is, we could derive $\neg R$ in NBG, which contradicts the independence of $\neg R$ relative to NBG. Thus we have reached:

Corollary 1. Complete self-reproduction is independent from ordinary set theory in the form of NBG in the following sense. The existence of a symbiotically self-reproducing entity, as characterized by equation (2), cannot be derived in NBG. Nor can the existence of an atomically self-reproducing entity, as characterized by equation (3), be derived in NBG.

The consistency of complete self-reproduction with respect to NBG will not be dealt with here.† Instead we want to demonstrate the consistency of atomic self-reproduction relative to another axiomatic set theory, namely Quine's New Foundations.

4. The Axiomatizability of Atomic Self-Reproduction. In order to be able to deal with the consistency of atomic self-reproduction, we will have to specify all relevant properties of this concept. This means that we must sharpen Characterization 2 towards a definition of atomic self-reproduction.

What further characteristics can naturally be imposed? First of all we should require of an atomic entity π that it be a unit class. Next, with the automata-interpretation of self-reproduction in mind, it is natural to consider not only a single reproduction step as the normal activity of the reproducing automaton, but also a repetition of such steps. This means that not only the production sequence $\langle\pi\rangle$ shall be associated with $\pi(\pi)$ but also the production sequences $\langle\pi, \pi\rangle$, $\langle\pi, \pi, \pi\rangle$, etc. Again, with the proof-sequence interpretation in mind, it is natural to accept a sequence like $\langle\pi, \pi\rangle$ as a proof of π. It begins with the axiom π, ends with the explained entity π, and appears not to hide any unexplained rules of inference. We are thus led to the following definition of atomic self-reproduction (which obviously obeys Characterization 2; compare eq. (3)).

Definition 1. An *atomically self-reproducing* (*atomically self-explaining*) entity is a unit-class relation π such that: $\pi\pi\langle\pi\rangle$ (π stands in the relation π to $\langle\pi\rangle$), $\pi\pi\langle\pi, \pi\rangle$, $\pi\pi\langle\pi, \pi, \pi\rangle$, etc.

† Both atomic and various forms of symbiotic self-reproduction are consistent with respect to NBG. This is easily established with a modeling technique due to Ernst Specker 1957, and further elaborated upon by Petr Hájek 1965. Independently, Dana Scott suggested the same modeling technique in the unpublished manuscript "A Different Kind of Model for Set Theory" which was read at the 1960 International Congress on Logic, Methodology and Philosophy of Science at Stanford. We are grateful to Professor Scott for pointing out these works as a solution to the above consistency problem.

For another formulation of atomic self-reproduction, we need the concept of a Quine individual (compare Quine, 1961), defined as follows:

Definition 2. S is a *Quine individual* if and only if:

$$\forall\, u\colon (u \in S) \Leftrightarrow (u = S). \tag{9}$$

Although of remote origins, the two definitions actually do specify one and the same type of entity.

Theorem 4. An entity π is atomically self-reproducing (atomically self-explaining) if and only if π is a Quine individual.

Proof. We first prove that $\pi = \{\pi\}$ is equivalent to the statement that π is atomically self-reproducing. Let π be an atomically self-reproducing entity according to Definition 1. Since π is a unit-class relation, it is a set of a single ordered pair which means that π is a function. Therefore, since each of the sequences $\langle\pi\rangle$, $\langle\pi, \pi\rangle$ etc. stands in the relation π to π, we must have $\langle\pi\rangle = \langle\pi, \pi\rangle = \langle\pi, \pi, \pi\rangle = \cdots$. This condition is equivalent with $\pi = \langle\pi, \pi\rangle$ (compare eq. (5)). On the other hand, $\langle\pi, \pi\rangle \in \pi$ because $\pi\pi\langle\pi\rangle$. Hence $\pi \in \pi$, and since π is a unit class we must have $\pi = \{\pi\}$. Let us, for the proof of the converse, assume that π is an entity such that $\pi = \{\pi\}$. This means that $\pi = \{\pi\} = \{\{\pi\}\} = \{\{\{\pi\}\}\} = \{\{\{\pi\}, \{\pi\}\}\} = \{\{\{\pi\}, \{\pi, \pi\}\}\} = \{\langle\pi, \pi\rangle\}$ (compare eq. (7)). Hence $\pi\pi\pi$, that is, $\pi\pi\langle\pi\rangle$. We also have $\pi = \{\pi\} = \{\{\pi\}\} = \langle\pi, \pi\rangle$, that is, $\pi = \langle\pi\rangle = \langle\pi, \pi\rangle = \langle\pi, \pi, \pi\rangle = \cdots$. This means that π is a unit-class relation such that $\pi\pi\langle\pi\rangle$, $\pi\pi\langle\pi, \pi\rangle$, $\pi\pi\langle\pi, \pi, \pi\rangle$, etc., that is, π is an atomically self-reproducing entity (Definition 1). This completes the proof of the equivalence between $\pi = \{\pi\}$ and the statement that π is an atomically self-reproducing entity. Next, we shall prove the equivalence between $\pi = \{\pi\}$ and the statement that π is a Quine individual. If $\pi = \{\pi\}$, we obviously have $\forall\, u\colon (u \in \pi) \Leftrightarrow (u = \pi)$, that is, π is according to Definition 2 a Quine individual. On the other hand, if π is a Quine individual, the identity $\pi = \pi$ implies that $\pi \in \pi$. Furthermore, since a Quine individual can have but one element, we must have $\pi = \{\pi\}$. This completes the proof of the last stated equivalence and thus also the proof of the whole Theorem 4.

By a work of Scott (1962), we know that the statement "there exists a Quine individual" is completely independent (that is, both independent and consistent) with respect to set theory in the form of Quine's New Foundations (Quine, 1961). We have thus reached the following corollary.

Corollary 2. The existence of an atomically self-reproducing (atomically self-explaining) entity (Definition 1) can be consistently added as a new and independent axiom to set theory in the form of Quine's New Foundations.

5. General Comments on Complete Self-Reproduction and its Relation to Theories of Biological Evolution. Corollary 2 shows that no inconsistency is implied by the concept of atomic self-reproduction, that is, inconsistency with respect to a set theory like Quine's New Foundations. Let us in this connection examine an interesting suggestion of Rosen (1959, 1962), namely that a self-reproducing automaton has a behavior function which is a member of its own range and that "the very notion of such automata appears to contain an internal inconsistency." It is not clear what is meant with such an unqualified "internal inconsistency." If instead the statement had been formulated "the existence of such automata is inconsistent with respect to a theory which contains the axiom of restriction," it would have been clear and correct. However, if the alleged inconsistency had been referred to a theory like Quine's New Foundations (with or without Quine individuals) or to NBG, it would have been incorrect.

The point is that our ordinary logical-mathematical-biological reasoning is carried out in a language that does not commit itself on questions like the axiom of restriction, R. It does not contain statements equivalent to R or to $\neg R$ but is open for a further axiomatic development along either alternative. This complete independence of R, which is reflected in an intuitive discomfort with the concept (compare Wittgenstein, 1921; Rosen 1959, 1962), should not be confused with an inconsistency, however.

In trying to resolve the alleged paradox of Rosen, Guttman (1966) claims that no "logical difficulty" is implied upon postulating an automaton Φ^* that produces another automaton Φ that is slightly different from Φ^* no matter how small this difference may be. Here Φ^* and Φ are the behavior functions of the two automata, and $\Phi^*(A) = \Phi$ is to have the meaning that an automaton with behavior Φ^* from the input A, produces an automaton with behavior Φ. Guttman's claim is incorrect in the sense that $\Phi^*(A) = \Phi$ & $\Phi^* \neq \Phi$ & $\Phi^* \subset \Phi$ implies an epsilon cycle just as well as $\Phi(A) = \Phi$ (a function that is a member of its own range). This is easily seen with the aid of Kuratowski's pair model:

$$[\Phi \neq \Phi^* \subset \Phi = \Phi^*(A)] \Rightarrow [\Phi(A) = \Phi] \Rightarrow [\langle A, \Phi\rangle \in \Phi]$$
$$\Rightarrow [\{\{A\}, \{A, \Phi\}\} \in \Phi] \Rightarrow [\Phi \in \{A, \Phi\} \in \{\{A\}, \{A, \Phi\}\} \in \Phi].$$

It would seem that many misunderstandings concerning the concept of self-reproduction are due to the different meanings which are commonly attached with it. Von Neumann (1966) and Penrose (1958), for example, use the word self-reproduction for a second-level reproduction, whereas we, following Rosen (1959), use self-reproduction in a complete sense.

In ordinary biological language the name self-reproduction is mostly used for second-level reproduction, for example, when the mechanisms of cell-division are used to explain the "self-reproducing" properties of the cell. That no logical difficulty arises in connection with this type of "self-reproduction" is well known (and can in fact also be demonstrated as an easy consequence of the Kleene recursion theorem).

There is, however, one area of biology where self-reproduction, in the complete sense of this paper, is of interest. This is the theory of biological evolution. If we ask not merely how a cell can reproduce in a suitable surrounding, but how this property has evolved, then we are faced with an explanation of reproduction in a complete sense. Our results show that such a theory of evolution cannot be derived with an ordinary logical-mathematical-biological reasoning, but that it instead will have to contain new and independent axioms. The question of how these should be formulated offers interesting problems. As has been demonstrated in the above, one possibility is to axiomatize atomic self-reproduction. Another and more interesting possibility may be to introduce axioms on a deeper level, such that atomic or similar forms of self-reproduction can be derived. The question whether an eventual axiomatization of symbiotic self-reproduction can serve this purpose is open.

Finally we want to emphasize that it is not *a priori* clear that a formal theory of biological evolution should exist. If such a theory could be formulated, however, it still remains to investigate whether it is recursively axiomatizable (that is, whether there are effective methods for the decision if a given well-formed expression is an axiom or not). Informal theories of evolution, on the other hand, have already proved to be influential both within and far beyond the biological domain. The problem to examine the formalizability of such theories seems to be a highly rewarding task.

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