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ON CHAINS OF RELATED SETS

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If **K** is a partition of a set K which is partially ordered by the relation R and **R** is a collection of pairs of sets of **K** such that the sets of each pair are related by R in the sense of Rashevsky, then **R** is a relation which partially orders **K**. Necessary and sufficient conditions that **K** be a chain are obtained, and if **K** is a chain under these conditions, it is shown that **K** is unique.

Certain propositions on relations between sets were presented by N. Rashevsky (1961) and M. Sommerfield (1963). In this paper K denotes a finite set which is partially ordered by a relation denoted by R, i.e. $R \subset K \times K$ which is reflexive, antisymmetric and transitive. We investigate collections of subsets of K such that if A, B is a pair of subsets of K, then A and B are related by R in the sense of Rashevsky, and state conditions under which such a collection forms a chain.

For convenience we restate the definitions (Rashevsky, 1961) of relations between sets. The statement that the set A is *strongly related* by R to the set B, symbolized by ARB, means that $A \times B \subset R$. The statement that A is *weakly related* by R to B, symbolized by $AR'B$, means that $A \times B \not\subset R$ and if $R_{AB} = A \times B \cap R$, then R_{AB} is from A onto B. We use the symbol $A\mathbf{R}B$ to signify ARB or $AR'B$.

The following theorem is a slight extension of Theorem 13 (Sommerfield, 1963).

Theorem 1: If each of A, B and C is a set of a collection of subsets of K, $A\mathbf{R}B$ and $B\mathbf{R}C$, then $A\mathbf{R}C$. Furthermore, if ARB or BRC, then ARC.

Proof: If $x \epsilon A$, there is a $y \epsilon B$ such that $(x, y) \epsilon R$, and if $y \epsilon B$, there is a $z \epsilon C$ such that $(y, z) \epsilon R$. Hence $(x, z) \epsilon R$. Similarly, if $v \epsilon C$, there is a $u \epsilon A$ such that $(u, v) \epsilon R$. Hence $A\mathbf{R}C$. Suppose ARB or BRC. Then if $v \epsilon C$ and if $u \epsilon A$, $(u, v) \epsilon R$ so that ARC.

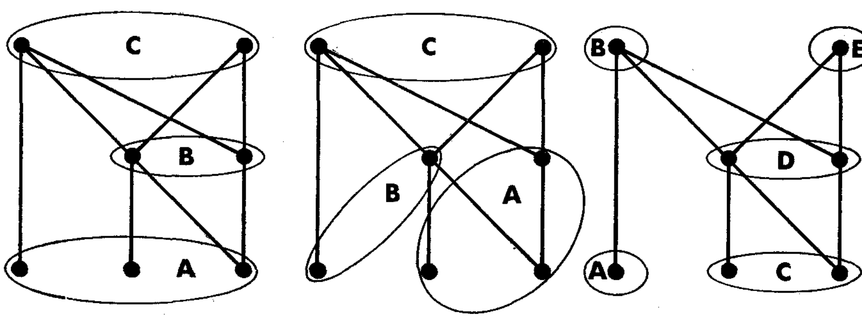
We observe that ARB implies that not more than two elements of K are common to A and B. Hence ARB does not imply ARA. Since the remainder of this paper is concerned with collections of subsets of K no two of which contain a common element, it may happen, for example, that if A, B and C are subsets of K, then ARB, $AR'C$, $AR'A$ and BRB.

The statement that $\mathbf{K}$ is a *partition* of K means that $\mathbf{K}$ is a collection of subsets of K no two of which intersect such that $\cup_{A \epsilon \mathbf{K}} A = K$. With $\mathbf{R}$ denoting the collection of pairs (A, B) of sets of $\mathbf{K}$ such that $A\mathbf{R}B$, the following theorem shows that the relation R induces a partial ordering of $\mathbf{K}$ by $\mathbf{R}$.

Theorem 2: If $\mathbf{K}$ is a partition of K, then $\mathbf{K}$ is partially ordered by $\mathbf{R}$.

Proof: If $A \epsilon \mathbf{K}$ and $x \epsilon A$, then $(x, x) \epsilon R_{AA}$, so that $(A, A) \epsilon \mathbf{R}$. Suppose $(A, B) \epsilon \mathbf{R}$ and $B \neq A$. If $x \epsilon A$, there is a $y \epsilon B$ such that $(x, y) \epsilon R$. Suppose if $y \epsilon B$, there is a $z \epsilon A$ such that $(y, z) \epsilon R$. Then if $x \epsilon A$, there is a $z \epsilon A$, $z \neq x$, such that $(x, z) \epsilon R$. Since A is finite, this leads to the contradiction of a pair u, v of elements of A such that $(u, v) \epsilon R$ and $(v, u) \epsilon R$. Hence there is a $y \epsilon B$ such that if $x \epsilon A$, then $(y, x) \notin R$. Thus $(B, A) \notin \mathbf{R}$. It follows from Theorem 1 that $\mathbf{R}$ is transitive.

Examples of finite partially ordered sets are conveniently given by means of diagrams. If M is partially ordered by S, the statement that an element q of M *covers* the element p of M means that $(p, q) \epsilon S$ and if $(p, z) \epsilon S$, $z \neq p$ and $z \neq q$, then $(z, q) \notin S$. A figure obtained by representing elements of M by dots so that if q covers p, then the dot for q is above the dot for p, and connecting the dots for p and q with a line segment is called a *diagram* of M. According to a theorem of Birkhoff (1935) every finite partially ordered set is representable by a diagram. In Figure 1 are represented three partitions of a 7-element set K.



Figures 1a, b, c.

Figure 1a shows $AR'C$ and BRC, while Figure 1b depicts ARC and $BR'C$. In neither case are A and B related. Figure 1c represents ARB, $CR'D$, DRB, DRE, CRB and CRE.

The following theorem is a direct result of the definitions.

Theorem 3: If $\mathbf{K}$ is a partition of K, then $\cup_{(A,B)\epsilon\mathbf{R}} R_{AB} \subset R$.

A partition $\mathbf{K}$ of K is said to be *R-complete* provided $\cup_{(A,B)\epsilon\mathbf{R}} R_{AB} = R$. The partition represented in Figure 1c is R-complete; those shown in Figures 1a and 1b are not. A set M which is partially ordered by the relation S such that if $(p, q)\epsilon M \times M$, then $(p, q)\epsilon S$ or $(q, p)\epsilon S$ is called a *chain*. If a subset C of K is a chain, then C is said to be a *subchain* of K. The statement that a partition $\mathbf{K}$ of K is a *stratification* of K means that if C is a subchain of K, no two elements of C belong to one set of $\mathbf{K}$. The statement that a partition $\mathbf{H}$ is a *refinement* of a partition $\mathbf{K}$ means that each set of $\mathbf{H}$ is a subset of a set of $\mathbf{K}$. The statement that $\mathbf{K}$ is a *maximal stratification* of K means that the only stratification of K of which $\mathbf{K}$ is a refinement is $\mathbf{K}$. Figure 1c represents a stratification and Figure 1a depicts a maximal stratification, whereas the partition represented in Figure 1b is not a stratification. We are now ready to state a condition necessary and sufficient that a stratification of K be a chain.

Theorem 4: If $\mathbf{K}$ is a stratification of K, each of the following two statements implies the other:

(i) $\mathbf{K}$ is maximal and R-complete;
(ii) $\mathbf{K}$ is a chain with respect to $\mathbf{R}$.

Proof: suppose $\mathbf{K}$ is not a chain. Then there are sets A and B of $\mathbf{K}$ such that $(A, B)\notin\mathbf{R}$. Suppose $\mathbf{K}$ is R-complete. Then if $x\epsilon A$ and $y\epsilon B$, $(x, y)\notin R$. Hence there is a stratification of K containing $A \cup B$ and $\mathbf{K}$ is not maximal. It follows that (i) implies (ii).

Suppose $\mathbf{K}$ is not R-complete. Then there is a pair $(x, y)\epsilon R$ and sets A and B of $\mathbf{K}$ such that $x\epsilon A$, $y\epsilon B$ and $(A, B)\notin\mathbf{R}$. Hence $\mathbf{K}$ is not a chain. Suppose $\mathbf{K}$ is not maximal. Then there are sets A and B of $\mathbf{K}$ and a stratification of K which contains $A \cup B$, so that if $x\epsilon A$ and $y\epsilon B$, then $(x, y)\notin R$. Thus $(A, B)\notin\mathbf{R}$ and $\mathbf{K}$ is not a chain. Therefore (ii) implies (i).

We next show that a stratification of K which forms a chain is unique. If M is a chain with respect to S, we denote by $\underset{\cdot}{m}$ the *least*, and by $\dot{m}$ the *greatest*, element of M, i.e. if $z\epsilon M$, then $(\underset{\cdot}{m}, z)\epsilon S$ and $(z, \dot{m})\epsilon S$. The statement that M is a *chain from p to q* means that $p = \underset{\cdot}{m}$ and $q = \dot{m}$. If M is a proper subset of no chain from p to q, then M is said to be a *maximal chain* from p to q.

Theorem 5: Not more than one stratification of K is a chain with respect to $\mathbf{R}$.

Proof: If a stratification $\mathbf{K}$ of K is a chain, C is a maximal subchain of K, $A\epsilon\mathbf{K}$ and $c\epsilon A$, then if $B\epsilon\mathbf{K}$, $(B, A)\notin\mathbf{R}$. Hence if $\underset{\cdot}{C} = \{\underset{\cdot}{c}\}$, then $\underset{\cdot}{C}$ is the least element of **K**. Suppose a stratification **H** of K is a chain and $\mathbf{H} \neq \mathbf{K}$. Then there is a $D \epsilon \mathbf{K} \cap \mathbf{H}$ and a maximal subchain **C** of **K** from C to D which is a maximal subchain of **H**, and there is an $A \epsilon \mathbf{K}$ and a $P \epsilon \mathbf{H}$, $A \neq P$, such that each of A and P covers D. In one of the sets A and P there is an element which is not in the other. Suppose $u \epsilon A$, $u \notin P$. Since u is not in a set of **C**, there is a $Q \epsilon \mathbf{H}$ such that $u \epsilon Q$ and $(P, Q) \epsilon \mathbf{R}$. If $x \epsilon A \cap P$, then $(x, u) \notin R$. Hence there is a $t \epsilon P$ such that $t \notin A$ and $(t, u) \epsilon R$. Then there is a $B \epsilon \mathbf{K}$ such that $(A, B) \epsilon \mathbf{R}$ and $t \epsilon B$, so that $(t, u) \epsilon B \times A$ and $\cup_{(A,B) \epsilon \mathbf{R}} R_{AB} \neq R$. But from Theorem 4 $\cup_{(A,B) \epsilon \mathbf{R}} R_{AB} = R$. This contradiction shows that **H** is not a chain.

No stratification of the set shown in Figure 1 is a chain. Figure 2 represents a set K and the stratification of K which produces a chain.

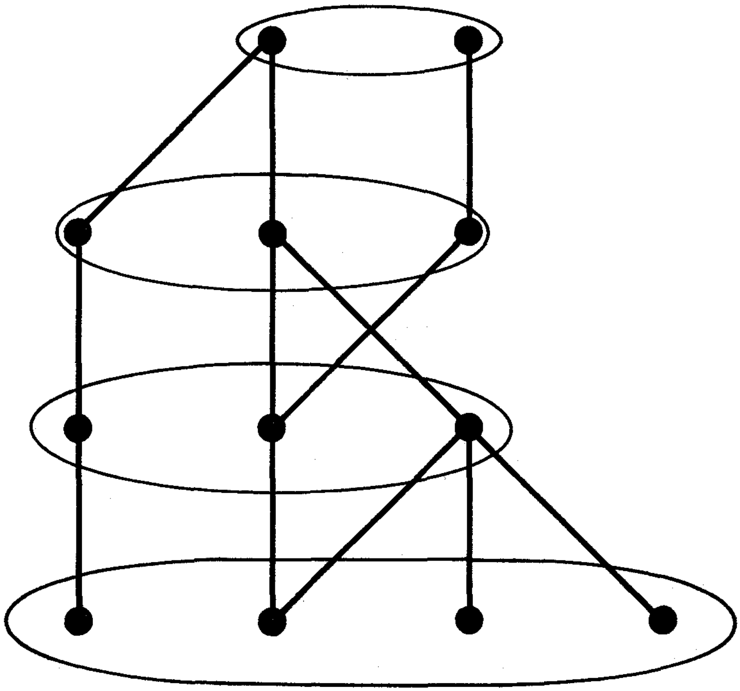


Figure 2.

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