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A RELATIONAL THEORY OF BIOLOGICAL SYSTEMS II

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The general Theory of Categories is applied to the study of the (M, R)-systems previously defined. A set of axioms is provided which characterize "abstract (M, R)-systems," defined in terms of the Theory of Categories. It is shown that the replication of the repair components of these systems may be accounted for in a natural way within this framework, thereby obviating the need for an *ad hoc* postulation of a replication mechanism.

A time-lag structure is introduced into these abstract (M, R)-systems. In order to apply this structure to a discussion of the "morphology" of these systems, it is necessary to make certain assumptions which relate the morphology to the time lags. By so doing, a system of abstract biology is in effect constructed. In particular, a formulation of a general Principle of Optimal Design is proposed for these systems. It is shown under what conditions the repair mechanism of the system will be localized into a spherical region, suggestive of the nuclear arrangements in cells. The possibility of placing an abstract (M, R)-system into optimal form in more than one way is then investigated, and a necessary and sufficient condition for this occurrence is obtained. Some further implications of the above assumptions are then discussed.

In a previous paper (Rosen, 1958b, hereinafter referred to as (II)) we outlined a possible approach to the theory of general systems, based on the notion of a *category*. The present work is an attempt to indicate how such an approach may, through the theory of (M, R)-systems introduced by us (Rosen, 1958a, hereafter referred to as (I)), be made to bear more directly on a number of important biological problems. We shall assume that the reader is familiar with the material contained in (I) and (II), and we shall use the notation and terminology of those papers without further comment.

I. *Abstract* (M, R)*-systems*. Our first task is to formalize the notion of (M, R)-system as described in (I), in terms of the more abstract theory which we developed in (II). We recall that an (**M**, **R**)-system consists essentially of a system **M**, together with certain auxiliary systems, designated by R_i, such that each component M_i of **M** has associated with it exactly one of the R_i, such that the output of R_i is a replica of the component M_i with which it is associated. The inputs to each R_i consist of a subset of the set of environmental outputs of **M**. We also imposed certain other structure on the (**M**, **R**)-systems in (I), in connection with the lags of the system; we shall deal with these properties in the next section. For the moment, however, we shall ignore the time lag structure, and consider an (**M**, **R**)-system as an assembly of components M_i with their associated systems R_i as described above.

It may be useful to have a concrete example before our eyes, on which we may explicitly illustrate the techniques which we shall develop below. To this end, let us consider the specific (**M**, **R**)-system which is labeled Figure 1 below (this is the same diagram as the one displayed on p. 254 of (I)); this figure represents the ordinary block diagram of the given system. By our discussion in the Introduction of (II), we know that this diagram is inherently ambiguous, in the sense that, when two arrows are shown as originating from a component (such as M_1 of the figure) we cannot tell whether these arrows represent different outputs of the component, or merely the same output being supplied to two different components. We shall henceforth assume that each component in this diagram which emits two arrows actually produces two distinct outputs.

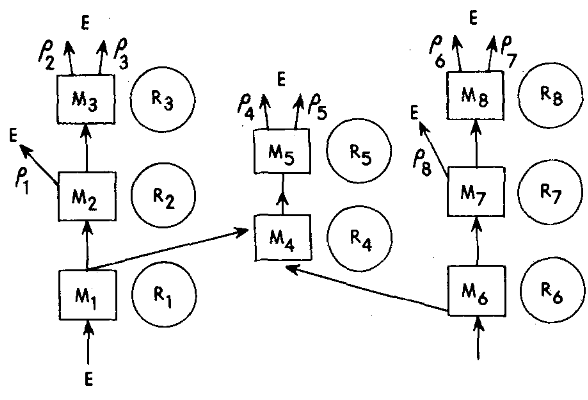


FIGURE 1

Let us now return to the general discussion. By Theorem 2 of (II) we know that we can find an *abstract* block diagram consisting of a collection of sets and mappings from a suitable subcategory of the category S of sets, which will represent a given system M. Further, we may suppose that this abstract block diagram is in the canonical form described by Theorem 4 of (II); in this case, each component of the system M under consideration is represented by a single mapping, the domain of which is contained in a cartesian product of sets of S. It will be found that this particular canonical representation will be most convenient both for the construction of our representation of abstract (M, R)-systems and for our subsequent discussion of the time-lag structure of these systems.

To see explicitly how to construct such canonical abstract block diagrams, we carry out this construction in detail for the special example mentioned above. First, we observe that, once the ambiguity concerning multiple outputs is resolved, we may apply Theorem 4 of (II) directly to ordinary block diagrams; we thereby obtain from a given ordinary block diagram a new ordinary block diagram in which each component emits a single output. Figure 2 below shows the diagram which results when this process is applied to the set of components M_i of Figure 1; the notation is chosen in the obvious manner. The emission of two arrows from a single component in this *canonical* diagram will now be seen to always have an unambiguous meaning; namely, the same output is to be provided to two components.

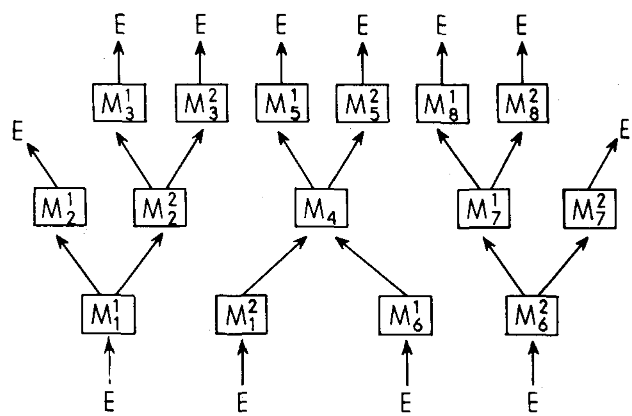


FIGURE 2

In Figure 3 below, we show the abstract block diagram obtained from the diagram of Figure 2 above by the procedure developed in (II). The notation is chosen so that the mapping f_i^k corresponds to the component M_i^k of the ordinary diagram; the set $A_{m,n}$ represents the output of the component M_m which serves as an input to the component M_n (regardless of the superscripts of these components). Sets with only a single subscript represent environmental inputs and outputs. It will be seen that this abstract diagram is already in the canonical form of Theorem 4 of (II).

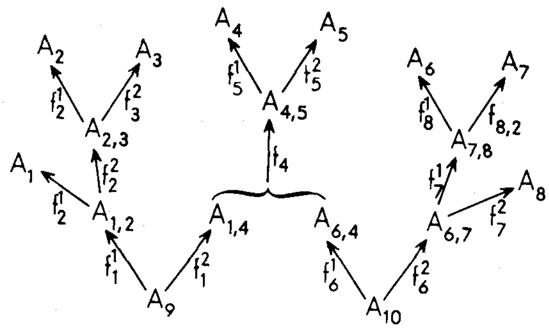


FIGURE 3

We must now find a means of representing the systems R_i in terms of the Theory of Categories. From our previous discussions in (I) and (II), it follows that the R_i must be represented by mappings of some type. An inspection of the definition of $(\mathbf{M}, \mathbf{R})$-system shows that the domain of these mappings must be represented by a cartesian product of sets which are elements of the set Θ of environmental outputs of the system $\mathbf{M}$ (it will be shown below that Θ is a set of sets), and the ranges of these mappings must contain the mappings which represent the related component M_i.

In greater detail, let us suppose that $\mathbf{M}$ is represented by an abstract block diagram in the manner previously described. We observe at the outset that certain sets in the abstract block diagram of $\mathbf{M}$ are expressible as cartesian products of other sets in the diagram; such sets will be referred to as *decomposable*. Sets of the diagram which cannot be expressed as a cartesian product of other sets in the diagram will be called *indecomposable*. For example, the set which is the domain of the mapping f_4 in Figure 3 above is decomposable; all other sets in this diagram are indecomposable. In particular, the canonical form which we have imposed on the diagram of **M** implies that the range of every mapping of the diagram is indecomposable (although the domains of these mappings are in general decomposable). In fact, it is not difficult to prove that this last is a necessary and sufficient condition for an abstract block diagram to be in the canonical form under discussion.

Let the indecomposable sets in the abstract block diagram of **M** be enumerated in some fashion; say as $\bar{A}_1, \bar{A}_2, \ldots, \bar{A}_n$. Then every other set in the diagram will, according to Postulate *A.B.D.* 2 of (II), necessarily be of the form $\prod_{k \epsilon K} \bar{A}_{i_k}$, where k runs through some finite index set K, and $\bar{A}_{i_k}$ is indecomposable for each $k \epsilon K$. If now f is a mapping in the abstract block diagram of **M**, the domain and range of which we shall denote respectively as $d(f)$ and $r(f)$, then by Postulate *A.B.D.* 1 of (II) there exists a set $A = \prod_{k \epsilon K} \bar{A}_{i_k}$ such that $d(f) \subset A$, and also an indecomposable set A_{i_o} such that $r(f) \subset A_{i_o}$. We shall find it convenient in the sequel to regard f as a mapping in the set of mappings $H(A, A_{i_o})$ (cf. (II), p. 321), even though f is not defined on all of A in general; we may accomplish this in a rigorous manner if we adopt the following artifice (which will be more thoroughly discussed in the next section): let f' be a mapping in $H(A, A_{i_o})$ with the property that on the set $d(f)$, $f = f'$. We shall require that for each such mapping f' the following condition shall be fulfilled: if x is any element of the set $A - d(f)$, then $f'(x)$ has an *infinite operation lag*. The usefulness of this artifice will become apparent as we proceed with our representation.

In (I), we denoted the collection of environmental outputs of the system **M** as Θ. According to the representation theory developed in (II), Θ will be represented by a collection of (indecomposable) sets in the abstract block diagram of **M**. Thus in Figure 3 above, Θ consists of the sets $A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8$. Let the sets which comprise Θ be enumerated in some fashion as $B_1, B_2, \ldots, B_m$. Then the totality of possible inputs to the mappings which are to represent the components R_i of an (**M**, **R**)-system consist of the elements of the sets of the form $\prod_{j \epsilon J} B_{i_j}$, where J is a suitable index set, and $B_{i_j} \epsilon \Theta$ for each $j \epsilon J$. To make this clear, let us consider once again the (**M**, **R**)-system shown in Figure 1. The input to the component R_i, $i = 1, 2, \ldots, 8$, of this system is there denoted by Θ_i; according to our representation theory, we find for example that the set $\Theta_8 = \{\rho_2, \rho_5, \rho_8\}$ is to be represented in the corresponding abstract block diagram by the cartesian product $A_2 \times A_5 \times A_8$ of Figure 3 above; likewise the other sets Θ_i are similarly represented by cartesian products in the more abstract situation.

We may now undertake the definition of the representatives of the R_i components. If M_i is a component of a system M such that, in the abstract block diagram of M the component M_i corresponds to a mapping f, where $f: A \longrightarrow A_{i_o}$, then we assign to the associated R_i-system a mapping

$$\Phi_f: \prod_{j \in J} B_{i_j} \longrightarrow H(A, A_{i_o}).$$

In Figure 3 above, for example, this means that corresponding to the mapping $f_{8,2}$ say, we have a mapping

$$\Phi_{f_{8,2}}: A_2 \times A_5 \times A_8 \longrightarrow H(A_{7,8}, A_7)$$

and corresponding to the mapping f_4 we have a mapping

$$\Phi_{f_4}: A_5 \longrightarrow H(A_{1,4} \times A_{6,4}, A_{4,5}).$$

The reader may find it helpful to construct explicitly the mappings Φ_f corresponding to the remaining mappings f of Figure 3 above, and even to construct similar examples of his own, in order to better familiarize himself with the mechanics of this construction.

We thus find that we may construct an abstract representation of an (M, R)-system in the following manner: we construct the abstract block diagram (in the sense of (II)) of the system M; this abstract block diagram is then placed in the canonical form described in Theorem (4) of (II). We then adjoin to this abstract block diagram a family of mappings Φ_f, such that the following conditions are satisfied:

M.R. I: The correspondence $f \longrightarrow \Phi_f$ between the mappings in the abstract block diagram of **M** and the members of the family $\{\Phi_f\}$ is 1-1 onto.

M.R. II: If, in the block diagram of **M**, we have $f \epsilon H(A, A_{i_o})$, then the associated mapping Φ_f is an element of the set $H\left(\prod_{j \epsilon J} B_{i_j}, H(A, A_{i_o})\right)$, where the sets B_{i_j} represent the appropriate sets in the family Θ of environmental outputs of **M**.

M.R. III: If Φ_f is the mapping corresponding to f, then $f \epsilon r(\Phi_f)$.

The first of these conditions merely states that there is a mapping Φ_f for every mapping f in the abstract block diagram of the system **M**. The second and third conditions assure that Φ_f actually behaves so as to produce a replica of f from the appropriate outputs of **M**.

We now proceed to discuss some of the general implications of the formalization developed above.

A mapping f such that $r(f)$ consists of a single element will be called a *constant map*. It is not hard to see that an arbitrary system **M**, all the mappings of which are constant maps, can display no fluctuation of outputs in a changing environment; i.e. are completely invariant in their behavior. Therefore, the more closely the mappings of a system **M** approximate constant maps, the more invariant will its behavior appear over a wide range of environments. It seems to be precisely the non-constancy of many of the mappings which occur in biological systems which accounts for their characteristic flexibility in dealing with environmental changes. Of particular interest in this connection is the possible non-constancy of the mappings which we have denoted by Φ_f, and which may be considered as representing a portion of the "genetic" material of the abstract biological system under consideration. A non-constant mapping Φ_f may under certain circumstances produce as output a mapping different from the mapping f with which it is associated, thereby perhaps altering the structure of the entire system. The implications of this type of situation, which is of obvious interest and importance, will be more thoroughly explored in another place.

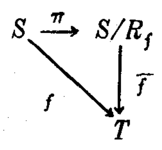
Thus far, our discussions of $(\mathbf{M}, \mathbf{R})$-systems have not included any mention of the possibility of the *replication* of the components we have labeled R_i in (I), and denoted by mappings Φ_f above (cf. however, Rashevsky, 1958, in this connection). Nevertheless, such replication is one of the fundamental characteristics of living systems, and must certainly be incorporated, in not too violently *ad hoc* a fashion, into any theory which makes a claim to general biological significance. The representation theory which we have outlined above enables us to make a number of remarks which may be pertinent to this problem.

Quite generally, let X, Y be arbitrary sets, and let $H(X, Y)$ denote (for the moment) the full set of mappings $f: X \longrightarrow Y$. If we now fix an element $x \epsilon X$, and allow f to vary through $H(X, Y)$, we find that we have an induced mapping

$$\psi_x : H(X, Y) \longrightarrow Y$$

defined by writing $\psi_x(f) = f(x)$ for all $f \epsilon H(X, Y)$. In this manner, we may make correspond to each $x \epsilon X$ a mapping $\psi_x \epsilon H[H(X, Y), Y]$. It is immediately verified that for any two elements $x_1, x_2 \epsilon X$, $\psi_{x_1} = \psi_{x_2}$ if and only if $x_1 = x_2$; thus in effect we have constructed an *embedding* of the arbitrary set X into the set of mappings $H[H(X, Y), Y]$. It may be noticed that this procedure is a generalization of the standard embedding of a vector space in its second conjugate (or dual) space.

Once again quite generally, let S, T be arbitrary sets, and let $f: S \longrightarrow T$ be a mapping in $H(S, T)$. The mapping f then induces an equivalence relation R_f in S, defined by writing $x_1 R_f x_2$ in S if and only if $f(x_1) = f(x_2)$. If we define the *quotient space* S/R_f of S under f to be the set of equivalence classes of S obtained in this manner, we can form the diagram of mappings



where f is the given mapping, π is the natural mapping which maps each element of S on its equivalence class in S/R_f, and $\bar{f}$ is the (uniquely determined) mapping which makes the diagram commutative. It follows from our construction that $\bar{f}$ is a 1–1 mapping, and hence it has an inverse, $\bar{f}^{-1}: T \longrightarrow S/R_f$. In particular, if f was originally 1–1, then $S/R_f = S$ and $f = \bar{f}$. Likewise, if the equivalence class of an element $x_1 \epsilon S$ consists of x_1 alone, then we may write $\bar{f}^{-1}(f(x_1)) = x_1$ in S itself.

Let us now apply these general considerations to the abstract block diagrams we have defined above. Suppose that as before we enumerate the environmental outputs of the abstract block diagram of a system **M** as $B_1, B_2, \ldots, B_m$. As we pointed out above, the domains of the mappings Φ_f consist of the various cartesian products $\prod B_{i_j}$. Suppose that $f: A \to B$ is a mapping in the abstract block diagram of **M**, and that $\prod_{j \in J} B_{i_j}$ is the domain of the associated mapping Φ_f. If we now write $X = \prod_{j \in J} B_{i_j}$, and $Y = H(A, B)$, then from the above general considerations, we obtain an embedding of $\prod_{j \in J} B_{i_j}$ into the set of mappings

$$\mathbf{H} = H\left\{H\left[\prod_{j \in J} B_{i_j}, H(A, B)\right], H(A, B)\right\}$$

Now writing $S = H\left(\prod_{j \in J} B_{i_j}, H(A, B)\right)$, $T = H(A, B)$, we see that for each mapping $F \in (\mathbf{H})$, we obtain an induced mapping $\bar{F}^{-1}$, which maps the elements of $H(A, B)$ into equivalence classes of mappings in the set $H\left(\prod_{j \in J} B_{i_j}, H(A, B)\right)/R_F$. In case the equivalence class of a particular mapping consists of a single element, we have observed that the image of such a mapping under $\bar{F}^{-1}$ may be considered as an element of $H\left(\prod_{j \in J} B_{i_j}, H(A, B)\right)$.

Let us now recall the significance of the various constructions we have performed above. Our embedding of the set $\prod_{j \in J} B_{i_j}$ into the set of mappings (1) shows that a particular family of outputs of a system **M** $\left(\text{i.e. an element in the set } \prod_{j \in J} B_{i_j}\right)$ may sometimes itself be considered as a mapping. The domain of this mapping is the set $H\left(\prod_{j \in J} B_{i_j}, H(A, B)\right)$, which consists of mappings of the type we have designated as Φ_f; the range of this mapping is the set $H(A, B)$, (which in particular contains the mapping $f: A \to B$ which represents a component of **M**). This mapping, called F in the above discussion, then induces a mapping $\bar{F}^{-1}$, which maps each element of $H(A, B)$ onto an equivalence class of mappings of the form Φ_f. The above discussion shows when this class of mappings is reduced to a single element.

We have shown that, in the presence of suitable outputs of the system M, the mappings f of an abstract (M, R)-system may act in such a manner as to generate their associated mappings Φ_f. Thus we see that a means of replication of the components R_i of an (M, R)-system, very similar in nature to the one postulated separately by N. Rashevsky (1958) is already contained within the formalism we have developed above, and requires the introduction of no new assumptions. It should be emphasized, however, that the above discussion of replication was introduced primarily to demonstrate the scope of our abstract formulation, and not necessarily as a definitive presentation of the ideas outlined therein.

II. *The Time Lags.* We recall from our earlier discussions in (I) and (II) that every system inherently possesses two different types of time lags; namely, the *operation lags*, caused by the actual functioning of the components, and the *transport lags*, which arise from the delays occasioned by the moving of materials or other stimuli from the neighborhood of the components which produce them to the neighborhood of the components to which they serve as inputs. We have already noticed in (I) that both types of lag depend on both the structure of the given system and the nature of the environment in which the system is operating. The formalization we have developed above makes it possible for us to define these lags (in an (M, R)-system) in a more precise fashion, and facilitates our investigation of the possible relation of the time lag structure to the behavior of actual biological systems.

We shall be concerned below with abstract (M, R)-systems satisfying the conditions laid down in Section I above. Let us suppose that f represents a mapping in a certain abstract (M, R)-system; we have observed that the time lag of the component represented by f depends on the particular input on which f is acting. That is, each object in the domain of f (i.e. each possible input to the component which f represents) will be associated with some positive real number, which corresponds to the operation lag of the component in question when the given object is the input to that component. In more formal terminology, we see that the time lag associated with the operation of a component is a mapping, the domain of which coincides with the domain of the mapping f which represents the component, and the range of which is contained in the set R^+ of non-negative real numbers. If we denote this mapping by τ_f, we have

$$\tau_f : d(f) \longrightarrow R^+.$$

Naturally, similar considerations apply to the operation lags of the systems R_i, and represented by the mappings Φ_f. We shall allow the value ∞ as an admissible operation lag; as mentioned in Section I, a useful artifice for formally extending the domain of a mapping f to a larger set $A \supset d(f)$ is to replace f by a mapping f' such that $d(f') = A$, with the property that $f(x) = f'(x)$ for $x \epsilon d(f)$, and to write $\tau_f(x) = \tau_{f'}(x)$ for $x \epsilon d(f)$, $\tau_{f'}(x) = \infty$ otherwise.

The transport lags are somewhat more complicated to represent, owing to the fact that, speaking roughly, they depend in general both on the component of origin and the component of destination, as well as on the particular material or stimulus involved. The most natural way of dealing with this problem seems to be the following: let $\mathbf{F}$ be the cartesian product of the set of mappings of the system $\mathbf{M}$ of an abstract $(\mathbf{M}, \mathbf{R})$-system with itself. For each indecomposable set $A \epsilon \mathbf{M}$, we consider the cartesian product $\mathbf{F} \times A$ (i.e. the set of all triples of the form (f, g, a), where $f, g \epsilon \mathbf{M}$, $a \epsilon A$). We define the transport lag $\sigma_A^{f,g}$ of the elements of A between the components of $\mathbf{M}$ represented by the mappings f, g respectively to be a mapping

$$\sigma_A^{f,g} : \mathbf{F} \times A \longrightarrow R^+$$

subject to the following conditions:

1. $\sigma_A^{f,g} \equiv \infty$ if A is not a factor of $d(g)$.
2. $\sigma_A^{f,g} \equiv \infty$ if A is not contained in $r(f)$.
3. $\sigma_A^{f,f} \equiv 0$ if (1) or (2) does not hold.

The first two conditions express the fact that if the elements of the set $A \epsilon \mathbf{M}$ fail to be either inputs of the component represented by g or outputs of the component represented by f, then the transport lag between f and g of the elements of A is not defined (i.e. is infinite, for formal purposes); the third condition expresses the fact that the transport lag of the output of a component to the component itself is zero. With these conventions, we obtain a general definition of the transport lag structure of abstract systems which is consonant with physical intuition, and which will be flexible enough for later applications.

Let us now attempt to investigate in what manner the study of the time lag structure of an abstract (**M**, **R**)-system may throw light on other aspects of the system's behavior. In particular, we desire to extend our discussion of the relations between the time lag structure and the "morphology" of abstract systems which was undertaken in (I). In order to do this, we must first make some assumption which serves to connect the two aspects of the structure of these systems. There is, naturally, a wide variety of possible assumptions which can be made in this context, each one corresponding to what N. Rashevsky (1956) has called a "system of abstract biology." Hence it should be recognized that the tentative assumptions we put forward below are meant primarily to illustrate the type of results which may be obtained from these lines of thought, and therefore should not be considered as final.

We make the following assumptions:

*L*1: The transport lag $\sigma_A^{f,g}$ is directly proportional to the physical distance between the components represented by the mappings f and g; the constant of proportionality may vary with the particular $a \epsilon A$ involved.

In symbols, if we write $\rho(f, g)$ for the distance between the components represented by f and g, this hypothesis becomes

$$\sigma_A^{f,g}(a) = k(a) \cdot \rho(f, g).$$

Of course, this hypothesis is meaningful only if the transport lag $\sigma_A^{f,g}$ is defined (i.e. finite).

*L*2: If σ_A^{f,g_1} and σ_A^{f,g_2} are both finite, then the elements of A (which are outputs of the component represented by f) always go to the component, the representative mapping of which (either g_1 or g_2) gives the smaller of σ_A^{f,g_1}, σ_A^{f,g_2}.

Physically, the assumption (*L*2) may be expressed as follows: the finiteness of σ_A^{f,g_1} and σ_A^{f,g_2} means, by definition, that the set A serves as a factor of both $d(g_1)$ and $d(g_2)$. By Assumption (*L*1), these lags are measures of the distances $\rho(f, g_1)$, $\rho(f, g_2)$, since in this case the same output a is being delivered by f to both g_1 and g_2. Therefore, Assumption (*L*2) in its widest form requires that the output of a component of an abstract system is delivered *in* *full* to the nearest component which is capable of accepting it as an input.

*L*3: ("Principle of Optimal Design") If f, g_1, g_2, ..., g_n are mappings of an abstract system such that the sets $r(g_i)$ are contained in the factors of $d(f)$, and each factor of $d(f)$ contains one of the $r(g_i)$, then the components represented by these mappings are spatially distributed in such a fashion that

$$\sigma_{r(g_1)}^{g_1, f} = \sigma_{r(g_2)}^{g_2, f} = \ldots = \sigma_{r(g_n)}^{g_n, f}.$$

This principle specifies that if the mappings of a system are noncontractible (as we always assume; see (I)), then there are no extraneous "waiting lags" caused by the failure of inputs to arrive "on time."

Assumption (*L*3) is a precise formulation of the notion of optimality which was discussed in a more intuitive fashion in (I). Moreover, it seems to be a natural extension to cellular systems of the ideas of optimality developed by N. Rashevsky ("Principle of Maximum Simplicity") in his discussions of morphology (1948), and considerably extended by D. L. Cohn (1954, 1955).

The assumptions (*L*2) and (*L*3) are no doubt excessively restrictive, but we remind the reader that they have been introduced primarily for illustrative purposes. Thus, for example, a more realistic assumption along the lines of (*L*2) would be to assume that, if a component represented by a mapping f has the property that $r(f)$ is a factor of the domains $d(g_1)$ and $d(g_2)$, where g_1, g_2 represent other components of the system, then there is a distribution of the output of f proportionately between g_1 and g_2 depending on their relative distances from f. Likewise, a less stringent condition of type (*L*3) would require, instead of equality between the transport lags $\sigma_{r(g_i)}^{g_i, f}$, that arbitrary differences of these lags should be smaller in absolute value than some preassigned positive number, which depends in general on the mapping f; i.e.

$$\left| \sigma_{r(g_i)}^{g_i, f} - \sigma_{r(g_j)}^{g_j, f} \right| < \epsilon(f)$$

for each i and j. For the present discussion, however, we shall retain (*L*2) and (*L*3) in their present form.

Let us now investigate in what manner the assumptions ($L1$)-($L3$) affect the structure of an abstract ($\boldsymbol{M}, \boldsymbol{R}$)-system. Let us therefore suppose that a definite such system, which we may denote by Λ, has been given. Then according to our previous discussions, we can find an abstract block diagram for Λ, which we shall denote by the same symbol Λ, and which consists of a certain number n of mappings of type f, and hence also n associated mappings of type Φ_f. Since the range of each mapping (of either type) is an indecomposable set of the category (see (II)) in which our representation of Λ is carried out, there must be at least n indecomposable sets in the category (excluding the indecomposable sets which represent the purely environmental inputs to the system, and which are assumed to have zero transport lag).

We observe that if there exists an indecomposable set $A_{i_o} \supseteq r(f)$ (for some $f \epsilon \Lambda$), such that A_{i_o} is a factor of the domain of two distinct mappings g_1, $g_2 \epsilon \Lambda$, then ($L1$) and ($L2$) imply that, for the system to function properly, we must have

$$\rho(f, g_1) = \rho(f, g_2).$$

Otherwise, the total output of f will go to the "nearer" of g_1, g_2; thus the "farther" of g_1, g_2 will by non-contractibility produce no output, and ultimately the entire dependent set of this mapping (see (I)) will fail to be produced.

As a simple example of the type of conclusions which may be drawn from this last argument, let us suppose that A_{i_o} represents an environmental output of $\boldsymbol{M}$ such that A_{i_o} is a factor of the domain of each mapping Φ_f, $f \epsilon \boldsymbol{M}$. Suppose that $A_{i_o} \supseteq r(\bar{f})$, say. Then by the above, we must have

$$\rho(\bar{f}, \Phi_f) = \text{constant}$$

for each $f \epsilon \boldsymbol{M}$. Recalling the physical significance of the objects involved, this condition signifies, roughly speaking, that the components corresponding to the mappings Φ_f(i.e. to the "synthetic" or "repair" structure of the system Λ) is spatially distributed in a spherical fashion about the component represented by the mapping $\bar{f}$. If we were to attempt to regard a free-living single cell as being, to a first approximation, a system of ($\boldsymbol{M}, \boldsymbol{R}$)-type, then it would be natural to associate the mappings $f \epsilon \boldsymbol{M}$ with biological functions carried out by what is normally called the cytoplasm, while the mappings Φ_f would be associated with nuclear biological functions. If the cellular system also obeyed the laws $(L1)$-$(L3)$, then the structural implication for the cell would be that the nuclear material (as observed cytologically) is distributed spherically around what would ordinarily be regarded as a purely cytoplasmic component. Hence this model implies an intimate relation between what we habitually call "nucleus" and "cytoplasm"; in terms of relative biological activity, our model suggests the presence of certain biological activities which would usually be associated with cytoplasm, lying within the cytologically observed region assigned to the nucleus. It is in fact well known (cf. for example Allfrey, Mirsky & Osawa, 1957) that isolated nuclei do carry out such cytoplasmic activities as the synthesis of proteins and the generation of high-energy phosphate bonds.

Let us now carry the above construction one step further. Suppose as above that A_{i_o} represents an environmental output of **M** which factors the domain of each mapping Φ_f of the system Λ. Suppose in addition that there exist environmental outputs A_{i_1}, $A_{i_2}, \ldots, A_{i_r}$ of **M** such that, for each mapping $\Phi_f \epsilon \Lambda$, exactly one of the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$ factors $d(\Phi_f)$. This process in effect decomposes the set of mappings $\{\Phi_f\}_{f \epsilon M}$ into r classes, the mappings in the k^{th} class being such that A_{i_k} is a factor of their domains. From what we have already said above, it follows that the mappings in the k^{th} class must lie on a sphere S_k, such that the component which produces the output of **M** represented by the set A_{i_k} lies at the center of S_k. But we have seen that every mapping Φ_f, regardless of its class relative to the sets $A_{i_1}, A_{i_2}, \ldots, A_{i_r}$, must already lie on the sphere S_o the center of which is the component which produces the output A_{i_o} (since by hypothesis the set A_{i_o} factors the domain of every mapping Φ_f). Hence the mappings Φ_f in the k^{th} class are restricted to lie on the *intersection* of the sphere S_k with the sphere S_o. This intersection is a circle; the important thing to notice, however, is that the circle is a *one-dimensional* manifold. The fact that this model requires the "genetic" material of the system Λ to lie along one-dimensional manifolds is suggestive of the linear arrangement of the hereditary material found uniformly in all types of organisms. Our model already incorporates the feature that the mappings Φ_f are not randomly distributed along the sphere S_o, but are organized linearly according to their inputs and outputs according to the rules $(L1)$-

($L3$); this structure may be related to a conjecture of M. Demerec and Z. E. Demerec (1956). These authors suggest that the linear arrangement of the genetic material of cells may be connected with the existence of a sequence of coupled chemical reactions, each of which is mediated by an enzyme controlled in turn by a particular gene in the linear array. The reaction sequence is arranged in such a manner that the i^{th} reaction of the sequence is mediated by the enzyme controlled by the i^{th} gene in the linear array. We shall not, however, pursue this point any farther at this time.

It should finally be noticed that the basic characteristics of the above model are not greatly affected by a weakening of conditions ($L2$) and ($L3$) along the lines mentioned previously on p. 121. The reader may readily convince himself that the principal effect of such a weakening will be to introduce a "spread" about the intersections of the various spheres which we have designated by S_{i_o}, S_{i_k} above; within reasonable limits, this spreading of circles into tori does not affect the conclusions we have drawn from the model.

These simple results already indicate the directions in which a theory of the type developed above may lead. As remarked in (I), the segregation of the synthetic material into nuclei by living cells is one of the most commonplace observations of biology, and yet is one for which no theoretical justification of any kind has been provided. We emphasize once more that the above very simple considerations should not be taken too seriously in this regard; they were presented primarily for their heuristic value, in pointing out the possible usefulness of some of the methods developed above. However, the fact that our results are for the most part invariant under a loosening of some of our hypotheses may indicate that by imposing more realistic conditions on the time-lag structure of (M,R)-systems, we may obtain results in closer accord with observation.

Let us now turn our attention to certain other aspects of the physical structure of (M,R)-systems, which may be inferred from the hypotheses ($L1$)-($L3$). We propose to investigate the relations between the abstract block diagram of the (M,R)-system Λ and the number of its possible different optimal realizations. To begin with, let us consider a single mapping of type $\Phi_f \epsilon \Lambda$ and let us suppose that

$$d(\Phi_f) = A_{i_1} \times A_{i_2} \times \dots \times A_{i_s}$$

and $A_{i_k} \supseteq r(f_{i_k})$ for $k = 1, \ldots, s$, where the f_{i_k} are terminal mappings of M. If the entire system Λ begins to operate at time $t = 0$, we shall denote the time at which the first element of the set A_{i_k} produced by the system appears as T_{i_k} (thus T_{i_k} is, roughly speaking, the time elapsed between environment and output). Each T_{i_k} is readily seen to be a sum of operation and transport lags in M. The non-contractibility of the mapping Φ_f implies that the component which corresponds to Φ_f cannot begin to operate (to produce a replica of the component corresponding to the mapping f, or some related structure) until the time

$$\max\left(T_{i_1} + \sigma_{A_{i_1}}^{f_{i_1}, \Phi_f}, \quad T_{i_2} + \sigma_{A_{i_2}}^{f_{i_2}, \Phi_f}, \ldots, T_{i_s} + \sigma_{A_{i_s}}^{f_{i_s}, \Phi_f}\right).$$

If we now assume that the system Λ is optimal (i.e. that the condition $(L3)$ is satisfied), we must have

$$T_{i_1} + \sigma_{A_{i_1}}^{f_{i_1}, \Phi_f} = T_{i_2} + \sigma_{A_{i_2}}^{f_{i_2}, \Phi_f} = \ldots = T_{i_s} + \sigma_{A_{i_s}}^{f_{i_s}, \Phi_f}.$$

This is a set of $s - 1$ conditions on the operation and transport lags of Λ.

Thus we see that, using the hypothesis $(L3)$, the abstract structure of the block diagram of an (M, R)-system, we impose conditions upon the time lag structure of the system. Let φ represent an arbitrary mapping of an (M, R)-system (i.e. of type f or of type Φ_f). We shall denote by $q(\varphi)$ the number of factors of the domain of φ, which *do not represent purely environmental inputs* (i.e. factors which contain the ranges of other mappings of the system; these are the only types of set which impose conditions on the time lag structure). Then we observe immediately that precisely the same argument as given above implies that for each mapping $f \epsilon M$, the mapping Φ_f implicitly imposes a total of $q(\Phi_f) - 1$ conditions on Λ, and hence there are $\sum_{f \epsilon M} (q(\Phi_f) - 1) = \sum_{f \epsilon M} q(\Phi_f) - n$ conditions on the lags of Λ, due to the mappings Φ_f alone, (where n is the number of mappings in M). To enumerate the number of conditions placed upon Λ by the mappings $f \epsilon M$, we observe that for the mappings f, as for the mappings Φ_f, $q(f)$ non-environmental factors in

$d(f)$ implies a total of $q(f) - 1$ conditions on the lags of M. Hence the assumption of hypothesis ($L3$) imposes a total of

$$\sum_{f \in M} (q(\Phi_f) - 1) + \sum_{f \in M} (q(f) - 1) = \sum_{f \in M} (q(\Phi_f) + q(f)) - 2n$$

conditions.

In a somewhat similar fashion, the hypothesis ($L2$) also imposes a number of further conditions on Λ. More precisely, let A be a set in Λ which represents a non-environmental input to some component (i.e. $A \supseteq r(\psi)$ for some mapping $\psi \in \Lambda$). Let us denote by $p(A)$ the number of mappings $\varphi \in \Lambda$ such that A is a factor of $d(\varphi)$. Then the same line of reasoning which we have employed above shows that there must be a total of $p(A) - 1$ conditions imposed on Λ by each such set A, and hence there will be

$$\sum_{A \in \Lambda} (p(A) - 1)$$

conditions on the lags of Λ by the hypothesis ($L2$). Since there must be n such sets, we find that this number can be written

$$\sum_{A \in \Lambda} p(A) - n.$$

Finally, there are n further conditions to be imposed on the lags of Λ if we make our usual assumption concerning the finite life-time of the components of Λ (see (I) for the details). Combining all of these conditions, we find that we have a total of

$$\sum_{f \in \Lambda} [q(f) + q(\Phi_f)] + \sum_{A \in \Lambda} p(A) - 2n$$

separate conditions on the time lags of Λ.

It is clear that if n is the number of mappings in the system M of Λ, then there are a total of $3n$ lags in Λ, of which n are transport lags, and $2n$ are operation lags due to the action of the components represented by the mappings f and Φ_f. Combining this information with what we have previously obtained, we find that we are confronted with two possibilities:

$$1.\ \sum_{f\in\Lambda}[q(f)+q(\Phi_f)]+\sum_{A\in\Lambda}p(A)-2n<3n;$$

$$2.\ \sum_{f\in\Lambda}[q(f)+q(\Phi_f)]+\sum_{A\in\Lambda}p(A)-2n\geq 3n.$$

If the first possibility is realized, then we have too few conditions to completely determine the lag structure, and hence there are many (in fact, *infinitely many*) different optimal forms which the abstract (M,R)-system Λ can assume. Furthermore, since the given conditions only tell us *how many* of the lags we may solve for, but not which ones, it is entirely arbitrary which lags we choose to solve for, and which ones we choose to select at random. Hence we discover a great latitude in constructing different optimal forms of the same abstract system; we may choose to completely fix the operation lags of the system (if we have a sufficient number of conditions), thereby obtaining systems which differ in the relative distances between components, or we may choose to leave certain operation lags undetermined, thereby opening up the possibility of choosing different physical realizations for the same component (cf. our discussion of "amplifier" in (I), p. 247). Thus, a study of what we may call the "morphology" of abstract (M,R)-systems of this type is of a degree of complication approaching what we have become accustomed to in the study of actual biological systems.

If the second possibility holds, we face the existence of two further sub-possibilities; namely, either all the conditions imposed on the system are all independent, or else they are not independent. If the conditions are independent and the equality sign holds, then there is exactly one optimal form for the system in question. If the conditions are independent and the inequality holds, then the system in question *cannot be put into an optimal form*. If the conditions are not independent, then the total number of independent conditions among them must total less than $3n$ in order for optimal systems to exist in great numbers. In systems of the type discussed above, it is immaterial whether the imposed conditions are independent or not, since the totality of conditions is already less than the number of lags of the system.

Thus we see how the principles which we have advanced above may enable us to make inferences concerning the structure of physical realizations of abstract systems, using the abstract structure alone. As we have emphasized above, it is not the specific form of the principles which we have labeled $(L1)$–$(L3)$ above which is of primary interest; it is the fact that such principles can lead to useful results that is of significance. We emphasize again that the above discussion is presented primarily as an example, to illustrate in what manner similar, but more realistic, principles can be used to obtain results of a theoretical nature which may have general biological importance.

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