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Gibbs' Phenomenon for Sampling Series and What To Do About It

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ABSTRACT. Gibbs' phenomenon occurs for most orthogonal wavelet expansions. It is also shown to occur with many wavelet interpolating series, and a characterization is given. By introducing modifications in such a series, it can be avoided. However, some series that exhibit Gibbs' phenomenon for orthogonal series do not for the associated sampling series.

1. Introduction

Gibbs' phenomenon, which involves overshoot of the partial sums of a series approximation to a function with a jump discontinuity, has been recognized for about a century. It was originally formulated by Gibbs for Fourier series [6], but also exists for other orthogonal expansions as well as for integral transforms [3, 12]. It was shown by Kelly [9] to occur under certain conditions for orthogonal wavelet approximations. Shim and Volkmer [14] then showed that these conditions for Gibbs' phenomenon to exist are satisfied for all reasonable wavelets.

When we turn to sampling (interpolating) series rather than orthogonal series, few results are known. Recently, Helmsberg [8] has shown Gibbs' phenomenon occurs for Fourier interpolation. One of us [13] has shown it also exists for interpolating series in some wavelet subspaces for functions continuous on the right. Also, Gomes and Cortina [7] have some results related to convolution integrals. But to our knowledge no other results involving interpolating series are known.

In this work we extend these results to other wavelet interpolating series. We shall show that it occurs for many of the standard wavelets, but not for all. We shall characterize it by a condition for interpolating series similar to that in [9] for orthogonal series.

We calculate an approximation to the amount of overshoot in certain cases. We then show that Gibbs' phenomenon can be avoided by using an alternate interpolating series. For certain cases, notably for Franklin wavelets and Daubechies wavelets with four taps, it does not occur for interpolating series even though it does for orthogonal series.

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2. Background

The prototype of an interpolating or sampling series is the Shannon series

$$f_0(t) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(t-n)}{\pi(t-n)} .$$

It interpolates exactly a function $f \in L^2(\mathbb{R})$ which is π bandlimited. Such functions belong to the Paley–Wiener space B_π of entire functions of exponential type $\leq \pi$. The function $\phi(t) = \sin \pi t/\pi t$ may also be considered the scaling function for a wavelet system, i.e.,

(i) $\{\phi(t-n)\}$ is an orthonormal sequence

(ii) $\phi(t) = \sum_{k=-\infty}^{\infty} c_k \phi(2t-k)$ for some $c_k \in l^2$

(iii) the closed linear span of $\{\phi(2^m t - n)\}_{n,m\in\mathbb{Z}}$ is $L^2(\mathbb{R})$

Most wavelet systems lead to similar sampling series [15]. However, in such cases a distinction must be made between the sampling function $S(t)$ and the scaling function $\phi(t)$. In the Shannon case above, they are the same. Each wavelet system has an associate "multiresolution analysis" consisting of a nested sequence $\{V_m\}$ of subspaces of $L^2(\mathbb{R})$ where the space V_m is the closed linear span of $\{\phi(2^m t - n)\}_{n\in\mathbb{Z}}$. A continuous function in $L^2(\mathbb{R})$ may be approximated by its projection onto V_m or by its sampling (i.e., interpolating) series in V_m. These are not the same even for the Shannon system. The former may exhibit Gibbs' phenomenon while the other may not. We shall be concerned only with the latter, whose properties are not, however, so well known.

We shall assume throughout that $\phi(t)$ is a continuous orthonormal scaling function such that

$$\begin{aligned} (i)\quad \phi(t) &= 0\left(|t|^{-\beta}\right) \quad as \quad t \longrightarrow \pm\infty, \quad \beta > 5/2, \\ (ii)\quad \widehat{\phi}^*(\omega) &= \sum_n \phi(n)e^{-iwn} \neq 0, \quad w \in \mathbb{R} . \end{aligned} \tag{2.1}$$

Then it was shown in [15] that there is a sampling function $S(t) \epsilon V_0$ such that for each $f \epsilon V_0$

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)S(t-n), \quad t \epsilon R .$$

These sampling series can be used to obtain an approximation in V_m for each continuous $f \epsilon L^2(\mathbb{R})$. It is given by

$$f_m(t) = \sum_{n=-\infty}^{\infty} f\left(2^{-m}n\right) S\left(2^m t - n\right) . \qquad (S)$$

It was shown in [13] that $f_m(t) \to f(t)$ uniformly for $f \epsilon H^\alpha$ (the Sobolev space) for $\alpha > \frac{1}{2}$. This required an additional hypotheses on ϕ, that it belong to Z_λ, which can be shown to be true for ϕ which satisfies (i) and (ii) above [15].

In order to study the Gibbs' phenomenon, we require that f be piecewise continuous and in $L^2(\mathbb{R})$. We shall also suppose that a jump discontinuity be at a dyadic rational number, so that by translation we can take it to zero, which we do. The spaces V_m are not translation invariant for irrational translations in general. We shall also assume the jump is in the positive direction, i.e., that $f(0^+) > f(0^-)$. If there is a sequence $t_m \downarrow 0$ such that

$$f_m\left(t_m\right) \to \gamma^+ > f\left(0^+\right) \tag{2.2}$$

then the sampling series exhibits Gibbs' phenomenon on the right-hand side of 0 for the function f (and similarly on the left-hand side). We shall simply say "Gibbs right" and "Gibbs left" for these two cases if they hold for any function with such a jump at 0. We shall see later that these are independent of the particular function.

There is a possible source of ambiguity in our series (S) at points of discontinuity. By changing the value of $f(0)$, we could change Gibbs right to Gibbs left and vice versa. This was avoided in [13] by assuming that $f(t) = f(t^+)$ for all $t\epsilon\mathbb{R}$. However, this assumption is unnecessarily restrictive and by eliminating it, we can sometimes avoid Gibbs left or right. We shall, however, always suppose that

$$f\left(0^-\right) \le f(0) \le f\left(0^+\right)$$

to avoid pathological behavior.

The sampling function $S(t)$ is related to the scaling function $\phi(t)$ by

$$\widehat{S}(w) = \frac{\widehat{\phi}(w)}{\sum_k \widehat{\phi}(w + 2\pi k)}, \quad w\epsilon\mathbb{R} \tag{2.3}$$

where $\widehat{f}$ denotes the Fourier transform of f. The denominator in (2.3) is assumed not to vanish. All such $S(t)$ have the properties [16]:

(i) $\int_{-\infty}^{\infty} S(t)dt = 1$
(ii) $\sum_{n=-\infty}^{\infty} S(t-n) = 1$
(iii) $\sum_k \widehat{S}(w + 2\pi k) = 1$
(iv) $\frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{S}(w)dw = 1$
(v) $S(t) = O(|t|^{-2})$

The last property is obtained from the fact that the second derivative of

$$\sum_k \widehat{\phi}(w + 2\pi k) = \sum_n \phi(n)e^{-iwn}$$

is in $L^2(0, 2\pi) \cap C$ and so is its reciprocal. Hence,

$$S(t) = \sum a_n \phi(t-n)$$

has coefficients such that $\{n^2 a_n\}\epsilon\ell^2$ [15].

2.1 The Shannon Case Revisited

The Shannon system, although it serves as a prototype, does not satisfy the hypotheses of the theorems about Gibbs' phenomenon in [13] and [14]. The formulae, however, are rather simple and may be used to show directly that Gibbs occurs for both sampling series and orthogonal series. In this particular case the sampling function $S(t) = (\sin\pi t)/\pi t = \phi(t)$, the orthonormal scaling function. However, the sampling approximation to a continuous function is not the same as the orthogonal projection since the coefficients need not be the same. Nonetheless, both cases can lead to Gibbs' phenomenon for functions with jump discontinuities at 0 and the overshoot calculated. Indeed in [14] it was shown that the overshoot is exactly the same as for Fourier series in the case of orthogonal approximations. We can also calculate it for sampling series.

Sampling Overshoot

We shall use the function h given by

$$h(t) = \begin{cases} sgn\, t - t & , 0 < |t| \le 1 \\ \alpha & , 0 = t \\ 0 & , 1 < |t| \end{cases}$$

to investigate Gibbs at $t = 0$. Its sampling approximation is given by

$$h_m^S(t) = \alpha S\left(2^m t\right) + \sum_{n=1}^{2^m-1} \left(1 - n2^{-m}\right)\left[S\left(2^m t - n\right) - S\left(2^m t + n\right)\right] . \tag{2.4}$$

If $h_m^S(t_m) \to \gamma^+ > h(0^+)$ (or $h_m^S(-t_m) \to \gamma^- < h(0^-)$) where $t_m \downarrow 0$, we have Gibbs right (or Gibbs left). By taking $t_m = 2^{-m-1}$, we find that

$$\begin{aligned} h_m^S\left(2^{-m-1}\right) &= \alpha S\left(\frac{1}{2}\right) + \sum_{n=1}^{2^m-1}\left(1 - n2^{-m}\right)\left[S\left(\frac{1}{2} - n\right) - S\left(\frac{1}{2} + n\right)\right] \\ &= \alpha\frac{2}{\pi} + \sum_{n=1}^{2^m-1}\left(1 - n2^{-m}\right)\frac{(-1)^n}{\pi}\left(\frac{1}{\frac{1}{2} - n} - \frac{1}{\frac{1}{2} + n}\right) \\ &= \frac{2\alpha}{\pi} + \frac{2}{\pi}\sum_{n=1}^{2^m-1}(-1)^n\left(\frac{1}{1 - 2n} - \frac{1}{1 + 2n}\right) + \frac{2}{2^m\pi}\sum_{n=1}^{2^m-1}\frac{(-1)^n n^2}{n^2 - \frac{1}{4}} \\ &= \frac{2\alpha}{\pi} + \frac{2}{\pi}\left[\frac{-1}{1-2} - \frac{(-1)^{2^m-1}}{1 + 2^{m+1} - 2}\right] + \frac{2}{2^m\pi}\left[-1 + \frac{1}{2}\sum_{n=1}^{2^m-1}\frac{1}{n^2 - \frac{1}{4}}\right] \\ &\to \frac{2}{\pi}(\alpha + 1) \quad \text{as} \quad m \to \infty . \end{aligned} \tag{2.5}$$

Similarly we have

$$h_m^S\left(-2^{-m-1}\right) \longrightarrow \frac{2}{\pi}(\alpha - 1) \text{ as } m \longrightarrow \infty .$$

Thus, we have Gibbs right whenever $\alpha > \frac{\pi}{2} - 1$ and in particular for h continuous on the right ($\alpha = 1$). In this case the overshoot is $\frac{4}{\pi} - 1$, which is greater than that for the orthogonal approximation.

This does not however imply that Gibbs' phenomenon fails to exist on the right for $\alpha \le \frac{\pi}{2} - 1$. In order to show that it does, we consider other sequences of the form $t_m = a2^{-m}$ for some $a > 0$. Then by calculations similar to (2.5) we find that

$$\begin{aligned} h_m^S\left(a2^{-m}\right) &\longrightarrow \alpha S(a) + \sum_{n=1}^{\infty}[S(a - n) - S(a + n)] \\ &= \alpha S(a) - S(a) + 1 - 2\sum_{n=1}^{\infty} S(a + n) \end{aligned} \tag{2.6}$$

Thus, we have Gibbs right if the last expression in (2.6) > 1, i.e., if

$$(\alpha - 1)S(a) > 2\sum_{n=1}^{\infty} S(a + n) .$$

For intervals in which $S(a)$ is positive we find that

$$\alpha > 1 + 2\sum_{n=1}^{\infty}\frac{S(a + n)}{S(a)} = \xi(a) \tag{2.7}$$

is sufficient for Gibbs right, with the opposite inequality giving it for negative $S(a)$. If $S(a) = 0$, then a is a positive integer, and (2.6) is equal to 1, so that Gibbs right does not occur. The right-hand side of (2.7) may be expressed for $S(a) \neq 0$ as

$$\begin{aligned}
\xi(a) &= 1+2\sum_{n=1}^{\infty}\frac{\sin\pi a(-1)^n}{\pi(a+n)}\cdot\frac{\pi a}{\sin\pi a} \\
&= 1+2a\sum_{n=1}^{\infty}\frac{(-1)^n}{a+n} = 1+2a\sum_{n=0}^{\infty}\frac{(-1)^n}{a+n}-\frac{2a}{a} \\
&= -1+2a\sum_{n=0}^{\infty}\frac{(-1)^n}{a+n} = -1+2a\beta(a) \qquad (2.8)
\end{aligned}$$

where $\beta(a)=\sum_{n=0}^{\infty}\frac{(-1)^n}{a+n}=\int_0^1\frac{t^{a-1}}{1+t}dt=\int_0^{\infty}\frac{e^{-aw}}{1+e^{-w}}dw$ for $a>0$. We use this to find (2.7)

$$\begin{aligned}
\xi(a) &= -1+2a\int_0^{\infty}\frac{e^{-aw}}{1+e^{-w}}dw \\
&= -1+2a\frac{e^{-aw}}{-a}\frac{1}{1+e^{-w}}\Big|_0^{\infty}-2a\int_0^{\infty}\frac{e^{-aw}}{-a}\frac{e^{-w}(-dw)}{\left(1+e^{-w}\right)^2} \\
&= -1+\frac{2a}{2a}+\frac{2a}{a}\int_0^{\infty}\frac{e^{-aw}dw}{\left(e^{-w/2}+e^{w/2}\right)^2} \\
&= 2\int_0^{\infty}\frac{e^{-aw}}{4\cosh^2 w/2}dw>0\,. \qquad (2.9)
\end{aligned}$$

From this expression it is also clear that $\xi(a)$ converges to zero monotonically as $a \longrightarrow \infty$. From (2.8) we see that $\xi(0)=1$ and $\xi(1)=2\log 2-1$.

Now $S(a)$ is positive when $a\epsilon(2n,2n+1), n=0,1,\ldots,$ and for each $\alpha>0$ we can find an a such that $S(a)>0$ and $\alpha>\xi(a)$. Similarly for each $\alpha<2\log 2-1$, we can find an a such that $S(a)<0$ and $\alpha<\xi(a)$. Hence, for all values of α, Gibbs right exists and by a symmetric argument so does Gibbs left.

We can use these results to obtain similar ones for other functions with a jump discontinuity at 0. Indeed let f be such a function such that $f\epsilon C^1[(-\infty,0)\cup(0,\infty)]$ and suppose both f and f' can be extended to $L^2(\mathbb{R})$ by assigning some value at zero. Then g given by

$$\begin{aligned}
g(t) &= f(t)-f\left(0^+\right)h(t)-th(t)\left[f'\left(0^+\right)+f\left(0^+\right)\right], \quad t>0 \\
g(t) &= f(t)+f\left(0^-\right)h(t)+th(t)\left[f'\left(0^-\right)-f\left(0^-\right)\right], \quad t<0 \\
g(0) &= 0\,,
\end{aligned}$$

is continuous on all of $\mathbb{R}$ and $g\epsilon L^2(\mathbb{R})$ while g' is continuous near zero and $g'\epsilon L^2(\mathbb{R})$. Thus, $g\epsilon H^1(\mathbb{R})$, the Sobolev space.

Lemma 1.

Let $g\epsilon H^1(\mathbb{R})$, then the Shannon sampling expansion of g,

$$g_m(t)=\sum_n g\left(n2^{-m}\right)S\left(2^m t-n\right), \quad m\epsilon\mathbb{Z},\ t\epsilon\mathbb{R}\,,$$

converges uniformly to $g(t)$ on $\mathbb{R}$ as $m \longrightarrow \infty$.

Proof. The error is given by

$$g_m(t)-g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\widehat{g}_m(w)-\widehat{g}(w)\right]e^{iwt}dw$$

$$= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\left[\widehat{g}_m(w) - \widehat{g}^*(w) + \widehat{g}^*(w) - g(w)\right]e^{iwt}dt$$
$$-\frac{1}{2\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\}\widehat{g}(w)e^{iwt}dw$$

where $\widehat{g}^*(w) = \sum_{k=-\infty}^{\infty}\widehat{g}(w+2^m 2\pi k)$ is the periodic extension of $\widehat{g}(w)$. Note that $\widehat{g}_m$ has support in $[-2^m\pi, 2^m\pi]$ and

$$\begin{aligned}\widehat{g}_m(w) &= \sum_{n=-\infty}^{\infty} g\left(2^{-m}n\right)e^{-iw2^{-m}n}\widehat{S}\left(w2^{-m}\right)2^{-m}\\ &= \widehat{g}^*(w)\widehat{S}\left(w2^{-m}\right) = \widehat{g}^*(w),\ |w| < 2^m\pi\ ,\end{aligned}$$

by the Poisson summation formula. Hence, we have

$$\begin{aligned}|g_m(t) - g(t)| &\le \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\left|\widehat{g}^*(w) - \widehat{g}(w)\right|dw + \frac{1}{2\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\}|\widehat{g}(w)|\,dw\\ &= \frac{1}{2\pi}\int_{-2^m\pi}^{2^m\pi}\left|\sum_{k\neq 0}\widehat{g}\left(w+2^m 2\pi k\right)\right|dw + \frac{1}{2\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\}|\widehat{g}(w)|\,dw\\ &= \frac{1}{\pi}\left\{\int_{-\infty}^{-2^m\pi} + \int_{2^m\pi}^{\infty}\right\}|\widehat{g}(w)|\,dw\ ,\end{aligned} \tag{2.10}$$

and since $\widehat{g}\epsilon L^1(\mathbb{R})$

$$\left(\int|\widehat{g}(w)|\,dw \le \left\{\int|\widehat{g}(w)|^2\left(w^2+1\right)dw\int\left(w^2+1\right)^{-1}dw\right\}^{\frac{1}{2}}\right)$$

the last expression in (2.10) $\to 0$ as $m \to \infty$. □

Corollary 1.

Let f be as above; then the Shannon expansion of f exhibits Gibbs' phenomenon on both the right and the left.

3. General Wavelet Sampling Series

In the last section we saw that for the Shannon system, the existence of Gibbs' phenomenon for a function with a jump discontinuity at zero holds whatever the value of the function at 0. In this section we attempt to get similar results for other wavelet sampling series. In [13], Gibbs' phenomenon for these sampling series was studied under the hypothesis that $f(0) = f(0^+)$. As was seen by the example in the last section, this is much too restrictive since Gibbs can occur for all choices of $f(0)$.

We shall require that the sampling function $S(t)$ satisfies the conditions (i) to (v) following (2.3). These conditions are implied by the conditions (2.1) on the scaling function $\phi(t)$, from which $S(t)$ may be constructed. Because of these conditions on $S(t)$, which are not satisfied by the Shannon function of Section 2.1, we are able to get local convergence results without the assumption of differentiality.

These results should be compared to those for orthogonal series. The conditions in Theorem 1 will be analogous to the conditions found in [9] for the orthogonal case, except that the integral in the latter case is replaced by a series in the former. In [14] it was shown that the integral condition for orthogonal wavelets is satisfied for all standard wavelets, in fact for all wavelets with continuous scaling functions that satisfy

$$\phi(x) = O\left(|x|^{-\beta}\right) \qquad |x| \to \infty$$

for some $\beta > 1$. We have been unable to obtain such a result for the interpolating series. If there is such a result it would have to be more restrictive because, as we shall see, the interpolating series for ${}_2\phi(t)$ does not exhibit Gibbs while its orthogonal series does.

Theorem 1.

Let ϕ be a scaling function satisfying (2.1) and let S be the associated sampling function given by (2.3); let $f \in L^2(\mathbb{R})$ be continuous except for a jump discontinuity at zero where $f(0^-) \le f(0) \le f(0^+)$. Then the sampling series (S) exhibits Gibbs' phenomenon on the right of 0 (respectively, left of 0) if and only if

$$\left[f(0) - f\left(0^+\right)\right] S(a) > \left[f\left(0^+\right) - f\left(0^-\right)\right] \sum_{n=1}^{\infty} S(a+n) \tag{3.1}$$

for some $a > 0$ (respectively,

$$\left[f\left(0^-\right) - f(0)\right] S(a) > \left[f\left(0^+\right) - f\left(0^-\right)\right] \sum_{n=1}^{\infty} S(a-n)$$

for some $a < 0$).

The proof involves two lemmas.

Lemma 2.

Let $f \in L^\infty(\mathbb{R})$ be locally BV and continuous on $(-a, a)$, $a > 0$; let f_m be the sum of the sampling series (S). Then for each $[-b, b] \subseteq (-a, a)$, $f_m \to f$ uniformly in $[-b, b]$.

Proof. The sampling approximation f_m is given by

$$\begin{aligned} f_m(x) &= \sum_n f\left(n2^{-m}\right) S\left(2^m x - n\right) \\ &= \int_{-\infty}^{\infty} \sum_n S\left(2^m x - n\right) \delta\left(t - n2^{-m}\right) f(t) dt \\ &= \int_{-\infty}^{\infty} r_m(x, t) f(t) dt \,. \end{aligned}$$

We find that the measure $r_m(x, t)$ satisfies

(i) $$\int_{-\infty}^{\infty} r_m(x, t) dt = \sum_n S\left(2^m x - n\right) = 1, \; x \in \mathbb{R}, \; m \in \mathbb{Z}$$

(ii) $$\int_{-\infty}^{\infty} |r_m(x, t)|\, dt \le \sum_n \left|S\left(2^m x - n\right)\right| \le C < \infty, \; x \in \mathbb{R}, \; m \in \mathbb{Z}$$

(iii) For each $\gamma > 0$, $\int_{|x-t| \ge \gamma} |r_m(x, t)|\, dt \longrightarrow 0$ as $m \longrightarrow$, uniformly for $x \in \mathbb{R}$

It is clear that (i) and (ii) hold since $S(x) = O(|x|^{-2})$. To obtain (iii), we observe that

$$\int_{|x-t| \ge \gamma} |r_m(x, t)|\, dt \;\le\; \sum_n \left|S\left(2^m x - n\right)\right| \int_{|x-t| \ge \gamma} \delta\left(t - n2^{-m}\right) dt$$

$$\begin{aligned} &= \sum_{|x-n2^{-m}|\geq\gamma} \left|S\left(2^m x - n\right)\right| = \sum_{|2^m x-n|\geq 2^n\gamma} \left|S\left(2^m x - n\right)\right| \\ &\leq \sum_{|2^m x-n|\geq\gamma 2^m} \frac{1}{|2^m x - n|^2 + 1} \\ &\leq \sum_n \left(\frac{1}{|2^m x - n|^2 + 1}\right)^{2/3} \left(\frac{1}{|\gamma 2^m|^2 + 1}\right)^{1/3} \\ &\leq C\left(2^{-m}\gamma^{-1}\right)^{2/3} \end{aligned}$$

These three properties are all that is needed to prove the convergence since

$$\begin{aligned} f_m(x) - f(x) &= \int_{-\infty}^{\infty} r_m(x,t)[f(t) - f(x)]dt \\ &= \int_{|x-t|<\gamma} + \int_{|x-t|\geq\gamma} = I_1 + I_2 \, . \end{aligned}$$

Since f is continuous on $(-a, a)$, it is uniformly continuous on closed subintervals. For $\gamma < a - b$, we have

$$|I_1| \leq \int_{|x-t|<\gamma} |r_m(x,t)|\,|f(x) - f(t)|dt$$

Now given $\epsilon > 0$, choose γ such that $|f(x) - f(t)| < \epsilon$ for $|x - t| < \gamma < a - b$ and $x\epsilon[-b, b]$. Then I_1 satisfies

$$|I_1| \leq \int_{-\infty}^{\infty} |r_m|\,(x,t)|dt\epsilon \leq C\epsilon$$

while I_2 satisfies, by (iii)

$$|I_2| \leq \int_{|x-t|\geq\gamma} |r_m(x,t)|\, 2 \, \| f \|_\infty \, dt \leq \epsilon \text{ for } m \geq m_0 \, .$$

Hence, by first choosing γ and then m_0 we see

$$|f_m(x) - f(x)| \leq C\epsilon + \epsilon$$

for $m \geq m_0$, and $x\epsilon[-b, b]$. □

We now can use a simpler standard function because of this lemma. We take h_α to be

$$h_\alpha(t) : \begin{cases} sgn\, t, & 0 < |t| \leq 1 \\ \alpha, & t = 0 \\ 0 & 1 \leq |t| \end{cases} \tag{3.2}$$

We use h_α to get rid of the jump discontinuity of f at 0.

Lemma 3.

Let g be given by

$$g(t) := \begin{cases} f(t) - f\left(0^+\right) h_\alpha(t), & t > 0 \\ 0, & t = 0 \\ f(t) + f\left(0^-\right) h_\alpha(t), & t < 0 \end{cases}$$

then $g_m(t) \to g(t)$ uniformly for $t\epsilon[-1/2, 1/2]$ as $m \to \infty$.

The proof of this lemma follows directly from Lemma 2 if we observe that $g(t)$ is continuous on $(-1, 1)$.

The value of $h_\alpha(t)$ at $t = 0$ did not enter into the definition of $g(t)$ in this lemma. However, since it will turn out to the important, we define $h_\alpha(0) = \alpha$ to be the proportional value

$$\alpha = \frac{f(0) - \frac{f(0^+)+f(0^-)}{2}}{\frac{f(0^+)-f(0^-)}{2}} \tag{3.3}$$

Proof of the theorem. Let t_m be a positive sequence such that $t_m \to 0$ as $m \to \infty$. Then, since $g_m(t_m) \to 0$, we need only consider $h_m(t_m)$ in studying Gibbs right. (Gibbs left is analogous.)

If Gibbs right exists at 0, then there is such a sequence $\{t_m\}$ such that $h_m(t_m) \to \gamma^+ > 1$, and hence

$$1 < h_m(t_m) = \left[\sum_{n=1}^{2^m} S\left(2^m t_m - n\right) + \alpha S\left(2^m t_m\right) - \sum_{n=1}^{2^m} S\left(2^m t_m + n\right)\right]$$

for $m \geq m_0$. We now take $a = 2^{m_0} t_{m_0}$ and obtain

$$1 < h_{m_0}\left(2^{-m_0} a\right) = \sum_{n=1}^{2^{m_0}} S(a-n) + \alpha S(a) - \sum_{n=1}^{2^{m_0}} S(a+n)\,.$$

Moreover, by taking m_0 even larger if necessary we can deduce that

$$1 < \sum_{n=1}^{\infty} S(a-n) + \alpha S(a) - \sum_{n=1}^{\infty} S(a+n)\,. \tag{3.4}$$

This condition is also sufficient for Gibbs right since the right-hand side is equal to $lim_{m\to\infty} h_m(a2^{-m})$. This inequality may be expressed by using the fact that $\sum_{n\epsilon\mathbb{Z}} S(a-n) = 1$, as

$$1 < 1 - S(a) - 2\sum_{n=1}^{\infty} S(a+n) + \alpha S(a)$$

or

$$(\alpha - 1)S(a) > 2\sum_{n=1}^{\infty} S(a+n)\,. \tag{3.5}$$

By replacing α in (3.5) by the expression in (3.3), we obtain the first conclusion (3.1). The second is obtained by using the corresponding inequality for Gibbs left,

$$(\alpha + 1)S(a) < -2\sum_{n=1}^{\infty} S(a-n)\,. \quad \square \tag{3.6}$$

Corollary 2.

Let S and f be as in the theorem and let $S(t) \geq 0$; then the sampling series (S) does not exhibit Gibbs' phenomenon whatever the choice of $f(0)$ (satisfying $f(0^-) \leq f(0) \leq f(0^+)$).

Proof. Since $\alpha < 1$, the left side of (3.5) would be negative and the right positive for $S(t) \geq 0$. Hence, the inequality cannot hold for any value of $a > 0$. Similarly (3.6) cannot hold for $\alpha > -1$. $\square$

Example 1. The piecewise linear spline with $S(t) = (1 - |t|)\chi_{[-1,1]}(t)$ satisfies the hypothesis of the Corollary and hence the sampling series does not exhibit Gibbs' phenomenon. This is in contrast to the mean square wavelet approximation which does exhibit Gibbs' phenomenon [14]. We shall see later that the same is true for the Daubechies wavelets with four taps [4].

Remark 1.

In the special case $\alpha = 1$, corresponding to f continuous on the right at 0, the condition for Gibbs right is $\sum_{n=1}^{\infty} S(a+n) < 0$. This can be expressed as

$$1 - \sum_{n=0}^{\infty} S(a-n) < 0$$

or

$$\sum_{n=0}^{\infty} S(a-n) > 1$$

which is the condition for Gibbs right in [13].

The condition (3.1) unfortunately is not easy to check. We next introduce a simpler sufficient condition for Gibbs right. It involves $\int_{-1}^{1} S(t)dt$, which $= 1$ for the linear spline case which has no Gibbs, but > 1 for the Shannon case which does. □

Lemma 4.

Let $S(t)$ be an even sampling function such that $S(t) > 0$ for $|t| < 1$ and

$$\int_{-1}^{1} S(t)dt = \gamma > 1 \, ;$$

let f and S satisfy the conditions of Theorem 1. Then there is a $\delta > 0$, such that if

$$f\left(0^+\right) - \delta < f(0) \leq f\left(0^+\right) \, ,$$

the sampling series exhibits Gibbs right at 0.

Proof. We use the well-known fact that

$$\sum_{n=-\infty}^{\infty} S(t-n) = \int_{-\infty}^{\infty} S(t)dt = 1 \, ,$$

and let $S_{\pm}$ denote the continuous functions

$$S_{\pm}(t) = \sum_{n=1}^{\infty} S(t \pm n) \, .$$

Then we have

$$\int_0^1 S_+(t)dt = \int_0^1 \sum_{n=1}^{\infty} S(t+n)dt = \int_1^{\infty} S(t)dt$$

and

$$\int_{-1}^{0} S_-(t)dt = \int_{-\infty}^{-1} S(t)dt \, .$$

Hence, by the symmetry of $S(t)$ we find that

$$1 = \int_{-\infty}^{\infty} S(t)dt = \int_{-1}^{1} S(t)dt + 2\int_0^1 S_+(t)dt$$

and

$$\frac{1-\gamma}{2} = \int_0^1 S_+(t)dt \, .$$

By the mean value theorem there is an $a \epsilon (0, 1)$ such that

$$\frac{1-\gamma}{2} = S_+(a) = \sum_{n=1}^{\infty} S(a+n) .$$

The expression (3.5) then becomes

$$(\alpha - 1)S(a) > 1 - \gamma$$

or, since $S(a) > 0$,

$$\alpha > 1 - \frac{\gamma - 1}{S(a)} . \tag{3.7}$$

This gives Gibbs right for $1 - \frac{\gamma-1}{S(a)} < \alpha \leq 1$ for the standard function $h(t)$ which has $h(0) = \alpha$. The proof in the theorem gives us the result for other functions. □

Corollary 3.

Let $S(t)$ and $f(t)$ be as in the lemma, and let $f(t)$ be continuous on the right (respectively left) at 0. Then the sampling series exhibits Gibbs right (respectively left) at 0.

The result for Gibbs left follows from the symmetry.

Remark 2.

In many examples of wavelet systems, $S(t)$ is a convex function on [-1,1]. Since $S(0) = 1$, then $\int_{-1}^{1} S(t)dt > 1$ and the hypothesis holds. □

Example 2. The Meyer wavelets have a scaling function $\phi(t)$ whose Fourier transform $\widehat{\phi}(w)$ has support on $[-\pi - \epsilon, \pi + \epsilon]$ for some $0 < \epsilon \leq \frac{\pi}{3}$ and $\widehat{\phi}(w) = 1$ for $w \epsilon [-\pi + \epsilon, \pi - \epsilon]$. The same conditions hold for $S(t)$ since $\widehat{S}(w) = \widehat{\phi}(w)/\widehat{\phi}^*(w)$. Thus, it is possible to show that $\widehat{S}$ must be of the form

$$\widehat{S}(w) = \int_{w-\pi}^{w+\pi} h \tag{3.8}$$

where h is some function ≥ 0 with support on $[-\epsilon, \epsilon]$ such that $\int h = 1$. We suppose that h and hence $\widehat{S}$ is symmetric.

We may find $S(t)$ by using the inverse Fourier transform which gives us

$$\begin{aligned} S(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \widehat{S}(w) e^{iwt} dw \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{w-\pi}^{w+\pi} h \right) e^{iwt} dw \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (h(w-\pi) - h(w+\pi)) \frac{e^{iwt}}{it} dw \\ &= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(w)e^{i(w+\pi)t} - h(w)e^{i(w-\pi)t}}{2it} dw \\ &= \int_{-\infty}^{\infty} h(w) e^{iwt} \frac{sin \pi t}{\pi t} dw \\ &= \frac{sin \pi t}{\pi t} \int_{-\epsilon}^{\epsilon} h(w) e^{iwt} dw \\ &= \frac{sin \pi t}{\pi t} \left(1 + \int_{-\epsilon}^{\epsilon} h(w) \left(e^{iwt} - 1 \right) dw \right) \end{aligned} \tag{3.9}$$

We already know that $\int_{-1}^{1} \frac{\sin \pi t}{\pi t} dt > 1$; in fact, this is exactly the overshoot for Fourier series $\simeq 1.18$. Hence, if we can show the last integral in (3.9) to be sufficiently small in magnitude, we will have shown that Gibbs' phenomenon exists.

Let σ^2 denote the second moment of $h(w)$,

$$\sigma^2 = \int_{-\epsilon}^{\epsilon} w^2 h(w) dw .$$

Then we have

$$\begin{aligned} \int_{-1}^{1} S(t)dt &= 2\int_0^1 \frac{\sin \pi t}{\pi t} 2 \int_0^{\epsilon} h(w) \cos wt dw dt \\ &\geq 4\int_0^1 \frac{\sin \pi t}{\pi t} \int_0^{\epsilon} h(w)\left(1 - \frac{w^2 t^2}{2}\right) dw dt \end{aligned}$$

since $\cos wt \geq 1 - \frac{w^2 t^2}{2}$ for $|wt| \leq \pi/3$. Furthermore, the second integral satisfies

$$4\int_0^1 \int_0^{\epsilon} h(w) \frac{w^2}{2} t^2 \frac{\sin \pi t}{\pi t} dw dt = \sigma^2 \int_0^1 \frac{t}{\pi} \sin \pi t dt = \frac{\sigma^2}{\pi^2} .$$

We can find a bound on σ^2 since

$$\sigma^2 = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} w^2 h(w) dw \leq \left(\frac{\pi}{3}\right)^2 \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} h(w) dw$$

and hence

$$\int_{-1}^{1} S(t)dt \geq 1.179 - \frac{\pi^2}{\pi^2 9} > 1 ,$$

i.e., Gibbs holds for all symmetric Meyer wavelets for functions continuous on the right or the left at 0.

Example 3. The Daubechies wavelets with support on $[0, 3]$ are defined by the solution to the dilation equations

$$\phi(t) = \sqrt{2} \sum_{k=0}^{3} c_k \phi(2t - k) \tag{3.10}$$

where

$$\begin{aligned} c_0 &= \nu(\nu - 1)/D \\ c_1 &= (1 - \nu)/D \\ c_2 &= (\nu + 1)/D \\ c_3 &= \nu(\nu + 1)/D \end{aligned}$$

and

$$D = \sqrt{2}\left(\nu^2 + 1\right), \quad \nu \in \mathbb{R} .$$

The standard case that has a vanishing first wavelet moment corresponds to $\nu = -\frac{1}{\sqrt{3}}$. The sampling function for $\nu < 0$ is [15, p. 139]

$$S(t) = \frac{2\nu}{\nu - 1} \sum_{n=0}^{\infty} \left(\frac{1+\nu}{1-\nu}\right)^n \phi(t - n + 1) . \tag{3.11}$$

Since the $S(t)$ is not symmetric, we cannot use Lemma 4 but must use (3.5) directly. In order to do so, we must evaluate

$$\sum_{n=1}^{\infty} S(a+n)\,.$$

We try $a = \frac{1}{2}$. Then

$$\begin{aligned} S\left(\frac{1}{2}-n\right) &= \frac{2\nu}{\nu-1}\sum_{k=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^{k}\phi\left(\frac{1}{2}-n-k+1\right) \\ &= \frac{2\nu}{\nu-1}\sum_{j=n}^{1}\left(\frac{1+\nu}{1-\nu}\right)^{j-n}\phi\left(\frac{3}{2}-j\right) \end{aligned} \tag{3.12}$$

We may evaluate $\phi(\frac{3}{2}-j)$ again by using (3.10). It gives us

$$\begin{aligned} \phi\left(\frac{1}{2}\right) &= \sqrt{2}\sum c_k\phi(1-k) = \sqrt{2}c_0\phi(1) \\ &= \frac{\sqrt{2}\nu(\nu-1)}{\sqrt{2}\left(\nu^2+1\right)}\frac{(\nu-1)}{2\nu}\frac{(\nu-1)^2}{2\left(\nu^2+1\right)} \\ \phi\left(\frac{3}{2}\right) &= \sqrt{2}\sum c_k\phi(3-k) = \sqrt{2}\left(c_1\phi(2)+c_2\phi(1)\right) \\ &= \left(\frac{1-\nu}{\nu^2+1}\right)\left(\frac{\nu+1}{2\nu}\right)+\left(\frac{1+\nu}{\nu^2+1}\right)\left(\frac{\nu-1}{2\nu}\right)=0 \\ \phi\left(\frac{5}{2}\right) &= \sqrt{2}\sum c_k\phi(5-k) = \sqrt{2}c_3\phi(2) \\ &= \frac{\nu(\nu+1)}{\left(\nu^2+1\right)}\frac{(\nu+1)}{2\nu} = \frac{(\nu+1)^2}{2\left(\nu^2+1\right)} \end{aligned}$$

where $\phi(1)$ and $\phi(2)$ are also found from (3.10) and the relation $\phi(1)+\phi(2)=1$. Hence, we find by (3.12) that

$$\begin{aligned} S\left(\frac{1}{2}\right) &= \frac{2\nu}{\nu-1}\left\{1\phi\left(\frac{3}{2}\right)+\left(\frac{1+\nu}{1-\nu}\right)\phi\left(\frac{1}{2}\right)\right\} \\ &= \frac{2\nu}{\nu-1}\left\{\left(\frac{1+\nu}{1-\nu}\right)\frac{(\nu-1)^2}{2\left(\nu^2+1\right)}\right\} = -\frac{\nu(\nu+1)}{\nu^2+1}\,. \end{aligned}$$

We also have

$$\begin{aligned} \sum_{n=1}^{\infty} S\left(\frac{1}{2}+n\right) &= \sum_{n=1}^{\infty}\frac{2\nu}{\nu-1}\sum_{k=0}^{n+1}\left(\frac{1+\nu}{1-\nu}\right)^{k}\phi\left(\frac{3}{2}+n-k\right) \\ &= \frac{2\nu}{\nu-1}\sum_{n=1}^{\infty}\left\{\left(\frac{1+\nu}{1-\nu}\right)^{n+1}\phi\left(\frac{1}{2}\right)+\left(\frac{1+\nu}{1-\nu}\right)^{n-1}\phi\left(\frac{5}{2}\right)\right\} \\ &= \frac{2\nu}{\nu-1}\sum_{n=1}^{\infty}\left\{\left(\frac{1+\nu}{1-\nu}\right)^{n+1}\frac{(\nu-1)^2}{2\left(\nu^2+1\right)}+\left(\frac{1+\nu}{1-\nu}\right)^{n-1}\frac{(\nu+1)^2}{2\left(\nu^2+1\right)}\right\} \\ &= \frac{\nu}{(\nu-1)\left(\nu^2+1\right)}\sum_{n=1}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^{n-1}2(1+\nu)^2 \end{aligned}$$

$$= \frac{(1+\nu)^2}{1+\nu^2}$$

For $\nu < -1$, $S(\frac{1}{2}) < 0$, and hence (3.5) becomes $\alpha < -\frac{1}{\nu}$. We always have Gibbs right in this case. The case $\nu \geq -1$ is inconclusive.

Example 4. The cubic spline $\theta_3(t)$ has support on [0, 4]. Its sampling series is given by [15, p. 139]

$$\begin{aligned} S_3(t) &= \sqrt{3}\left\{\sum_{n=0}^{\infty}\left(\sqrt{3}-2\right)^{n+1}\theta_3(t-n+1)\right. \\ &\quad \left. + \sum_{n=1}^{\infty}\left(\sqrt{3}-2\right)^{n-1}\theta_3(t+n+1)\right\} \end{aligned}$$

Thus, we need to evaluate the integrals

$$\int_0^1 \theta_3,\ \int_0^2 \theta_3,\ \int_1^3 \theta_3,\ \int_2^4 \theta_3,\ \int_3^4 \theta_3$$

which because of the symmetry of θ_3 are easily found. Since $\theta_3(t) = \theta_3(4-t)$ and $\theta_3(t) = t^3/6$ for $0 \leq t \leq 1$, we find

$$\begin{aligned} \int_0^1 \theta_3 &= \frac{1}{24} = \int_3^4 \theta_3\,, \\ \int_0^2 \theta_3 &= \int_2^4 \theta_3 = \frac{1}{2}\,, \\ \int_1^3 \theta_3 &= 1 - \frac{1}{12} = \frac{11}{12}\,. \end{aligned}$$

Then we find

$$\begin{aligned} \int_{-1}^{1} S_3(t)dt &= \sqrt{3}\left\{\sum_{n=0}^{\infty}\left(\sqrt{3}-2\right)^{n+1}\int_n^{2-n}\theta_3(t)dt\right. \\ &\quad \left. + \sum_{n=1}^{\infty}\left(\sqrt{3}-2\right)^{n-1}\int_n^{2+n}\theta_3(t)dt\right\} \\ &= \sqrt{3}\left\{\left(\sqrt{3}-2\right)\int_0^2\theta_3 + \left(\sqrt{3}-2\right)^2\int_0^1\theta_3\right. \\ &\quad \left. + 1\int_1^3\theta_3 + \left(\sqrt{3}-2\right)\int_2^4\theta_3 + \left(\sqrt{3}-2\right)^2\int_3^4\theta_3\right\} \\ &= \sqrt{3}\left\{\left(\sqrt{3}-2\right)\frac{1}{2} + \left(\sqrt{3}-2\right)^2\frac{1}{24} + \frac{11}{12} + \left(\sqrt{3}-2\right)\frac{1}{2} + \left(\sqrt{3}-2\right)^2\frac{1}{24}\right\} \\ &= \sqrt{3}\left\{\left(\sqrt{3}-2\right) + \frac{\left(\sqrt{3}-2\right)^2}{12} + \frac{11}{12}\right\} \\ &= \sqrt{3}\{-.2679 + .0059 + .9166\} = 1.1339\,. \end{aligned}$$

Hence, $S_3(t)$ exhibits Gibbs' phenomenon at 0.

4. How to Avoid Gibbs' Phenomenon

The sampling function $S(t)$ which exactly recovers $f \in V_0$ from its sampling expansion

$$f(t) = \sum_n f(n)S(t-n), \qquad f \in V_0$$

is unique for a given multiresolution analysis $\{V_m\}$. If, however, we are interested in finding a sampling series

$$f_0(t) = \sum_n f(n)u(t-n)$$

which associates with each $f \in L^2(\mathbb{R}) \cap C$ an element $f_0 \in V_0$, then we have more latitude. We still need to check that the dilations

$$f_m(t) = \sum f\left(2^{-m}n\right) u\left(2^m t - n\right) \tag{4.1}$$

converge to $f(t)$ as $m \to \infty$. If we can find a $u \in V_0$ such that

$$\begin{array}{lll} (i) & u(t) \geq 0, & t \in R, \\ (ii) & \sum_n u(t-n) = 1, & t \in \mathbb{R} \\ (iii) & u(t) = 0\left(|t|^{-1-\alpha}\right) \text{ as } t \to \infty,\ \alpha > 0\,, & \end{array} \tag{4.2}$$

then we have the desired result. Similar results appear in many other settings. For one that is close to ours see [2].

Theorem 2.

Let $u(t) \in V_0$ satisfy (4.2) and let f be a piecewise continuous bounded function in $L^2(\mathbb{R})$. Then f_m given by (4.1) satisfies

$$f_m(t) \to f(t) \quad as \quad m \to \infty$$

at each point of continuity of f and does not exhibit Gibbs' phenomenon.

Proof. Let t be a point of continuity; then we have

$$\begin{aligned} & |f_m(t) - f(t)| \\ = & \left| \sum_n f\left(2^{-m}n\right) u\left(2^m t - n\right) - f(t) \sum_n u\left(2^m t - n\right) \right| \\ \leq & \sum_{|t-2^{-m}n| \leq \delta} \left| f\left(2^{-m}n\right) - f(t) \right| u\left(2^m t - n\right) \\ & + \sum_{|t-2^{-m}n| \geq \delta} \left| f\left(2^{-m}n\right) - f(t) \right| u\left(2^m t - n\right) \\ \leq & \ \epsilon \sum_{|t-2^{-m}n| < \delta} u\left(2^m t - n\right) \\ & + 2\|f\|_\infty \sum_{|2^m t - n| \geq 2^m \delta} u\left(2^m t - n\right) \\ \leq & \ \epsilon + 2\|f\|_\infty \sum_{|2^m t - n| \geq 2^m \delta} c \left| \frac{1}{2^m t - n} \right|^{1+\alpha} \\ = & \ \epsilon + 0(1) \quad \text{as } m \to \infty \end{aligned} \tag{4.3}$$

where $\epsilon > 0$ is arbitrary and δ is such that $|f(t) - f(s)| < \epsilon$ whenever $|t - s| < \delta$. Thus, $f_m(t) \to f(t)$ as $m \to \infty$. To show that Gibbs' phenomenon does not hold, it suffices to show that

$$\sum_{n=0}^{\infty} u(t-n) \leq 1, \qquad t > 0$$

and

$$\sum_{n=1}^{\infty} u(t+n) \geq 0 \qquad t < 0\,.$$

But both of these inequalities follow from the fact that $\sum_{n=-\infty}^{\infty} u(t-n) = 1$ and $u(t) \geq 0$. □

Now all we need to do is find such a function for each type of wavelet subspace.

Example 5. For the Meyer wavelets of Example 1 as in [14], we may take

$$u(x) = \frac{1}{4}\phi^2\left(\frac{x}{4}\right)$$

Then $u \in V_0$ (since $\widehat{u}(w)$ has support $\left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$).

Furthermore $\widehat{u}(w)$ satisfies $\widehat{u}(0) = 1$ while $\widehat{u}(2\pi k) = 0,\ k \neq 0$. Thus, the periodic function is given by

$$\sum_{k=-\infty}^{\infty} u(x-k) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$$

its Fourier series. But the coefficients are

$$\begin{aligned} a_n &= \int_0^1 \sum_k u(x-k) e^{-i2\pi i n x} dx \\ &= \int_{-\infty}^{\infty} u(x) e^{-2\pi i n x} dx \\ &= \widehat{u}(2\pi n) = \delta_{0n} \end{aligned}$$

and hence $\sum_k u(x-k) = 1$.

Example 6. For the wavelets based on splines, the defining function $\theta_n(t)$ is nonnegative. Since its Fourier transform is simply

$$\widehat{\theta}_n(w) = \left(\frac{1 - e^{-iw}}{iw}\right)^{n+1},$$

it follows that $\widehat{\theta}_n(2\pi k) = \delta_{0k}$. Hence, by the same argument as in Example 5, this function satisfies

$$\sum_n \theta_n(t-k) = 1\,.$$

Thus, we can avoid Gibbs' by using the original B-splines.

Example 7. The Daubechies wavelets were left in the air in the last section for values of the parameter $\nu \geq -1$. We were unable to show that Gibbs' exists. There is a good reason for this; we have the following.

Lemma 5.

Let $S(t)$ be the sampling function given by (3.11) for $-1 < \nu < 0$; then $S(t) \geq 0$ (and $\sum S(t-n) = 1$).

Proof. The proof involves finding the dilation equation for $S(t)$ from that of $\phi(t)$ (3.10). Since

$$\begin{aligned} S(t) &= \frac{2\nu}{\nu-1}\sum_{n=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^n \phi(t-n+1) \\ &= \frac{\sqrt{2}\,2\nu}{\nu-1}\sum_{n=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^n \sum_{k=0}^{3} c_k\phi(2t-2n+2-k) \end{aligned} \tag{4.4}$$

we need only convert the right-hand side of (4.4) to an expression involving S. But this is easy since

$$\phi(t) = \frac{\nu-1}{2\nu}S(t-1) + \frac{\nu+1}{2\nu}S(t-2)$$

for $\nu < 0$. Thus, we have

$$\begin{aligned} S(t) &= \frac{\sqrt{2}\,2\nu}{\nu-1}\sum_{n=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^n \sum_{k}\left(c_{k+1}\frac{\nu-1}{2\nu} + c_k\frac{\nu+1}{2\nu}\right)S(2t-k-2n) \\ &= \frac{\sqrt{2}\,2\nu}{\nu-1}\sum_{n=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^n \sum_{k} d_k S(2t-2n-k) \\ &= \frac{\sqrt{2}\,2\nu}{\nu-1}\sum_{j=-1}^{\infty}\left(\sum_{n=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^n d_{j-2n}\right)S(2t-j)\,. \end{aligned} \tag{4.5}$$

We first need to calculate the d_k of which there are only five non-zero values. They are from (3.10), since $d_k = c_{k+1}\frac{\nu-1}{2\nu} + c_k\frac{\nu+1}{2\nu}$, $D = \sqrt{2}(\nu^2+1)$

$$\begin{aligned} d_{-1} &= (\nu-1)^2/2D, \\ d_0 &= \frac{(\nu-1)}{2\nu}\frac{\left(\nu^2+1\right)}{D}, \\ d_1 &= 0, \\ d_2 &= (\nu+1)\left(\nu^2+1\right)/2\nu D, \\ d_3 &= (\nu+1)^2/2D\,. \end{aligned} \tag{4.6}$$

Thus, the scaling coefficients γ_j of $S(2t-j)$ in (4.5) are given by

$$\gamma_j = \frac{\sqrt{2}\,2\nu}{\nu-1}\sum_{n=0}^{\infty}\left(\frac{1+\nu}{1-\nu}\right)^n d_{j-2n}, \qquad j=-1,0,1,\cdots \tag{4.7}$$

For $j=-1$, we have

$$\gamma_{-1} = \frac{\nu(\nu-1)}{\left(\nu^2+1\right)}$$

while for the others we have

$$\begin{aligned} \gamma_0 &= 1 \\ \gamma_1 &= -\frac{\nu(1+\nu)}{\nu^2+1} \\ \gamma_2 &= 0 \\ \gamma_3 &= \frac{2\nu}{\nu-1}\frac{(\nu+1)^2}{\left(\nu^2+1\right)}, \end{aligned}$$

and in general for $j = 2p, \quad p \geq 2$,

$$\gamma_{2p} = \frac{\sqrt{2}\,2\nu}{\nu - 1}\left[\left(\frac{1+\nu}{1-\nu}\right)^p d_0 + \left(\frac{1+\nu}{1-\nu}\right)^{p-1} d_2\right] = \left(\frac{1+\nu}{1-\nu}\right)^{p-1} \gamma_2 = 0$$

and for $j = 2p + 1, p \geq 1$,

$$\begin{aligned} \gamma_{2p+1} &= \frac{\sqrt{2}\,2\nu}{\nu - 1}\left[\left(\frac{1+\nu}{1-\nu}\right)^{p+1} d_{-1} + \left(\frac{1+\nu}{1-\nu}\right)^p d_1 + \left(\frac{1+\nu}{1-\nu}\right)^{p-1} d_3\right] \\ &= \left(\frac{1+\nu}{1-\nu}\right)^{p-1} \gamma_3 = \left(\frac{1+\nu}{1-\nu}\right)^p \frac{(-2\nu)(\nu+1)}{(\nu^2+1)} . \end{aligned}$$

We now substitute these values in (4.5) to obtain (recall $D = \sqrt{2}(\nu^2 + 1)$)

$$\begin{aligned} S(t) &= \frac{\nu(\nu-1)}{2(\nu^2+1)} S(2t+1) + S(2t) \\ &+ \frac{-\nu(1+\nu)}{(1+\nu^2)} S(2t-1) \\ &+ \sum_{p=1}^{\infty} \frac{(-2\nu)(\nu+1)}{(\nu^2+1)} \left(\frac{1+\nu}{1-\nu}\right)^p S(2t-2p-1) . \end{aligned} \tag{4.8}$$

Since $-1 < \nu < 0$, all of the coefficients are positive. This enables us to deduce that $S(t) \geq 0$ for all dyadic rationals (since we know $S(k) = \delta_{0k}$), and hence by continuity for all real t. □

Corollary 4.

The sampling expansion does not exhibit Gibbs phenomenon for the Daubechies wavelet with scaling function ${}_2\phi(t)$.

This scaling function corresponds to $\nu = -\frac{1}{\sqrt{3}}$ [4, p. 235], which hence has a non-negative $S(t)$.

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References

[1] Battle, G.A. (1987). A block spin construction of ondelettes, Part I: Lemarié functions, *Comm. Math. Phys.*, **110**, 601–615.

[2] Butzer, P., Splettstősser, W., and Stens, R.L. (1988). The sampling theorem and linear prediction in signal analysis, *Jber. Dt. Math. Verein*, **90**, 1–70.

[3] Carslaw, H.S. (1925). A historical note on the Gibbs' phenomenon in Fourier's series and integrals, *Bull. Amer. Math. Soc.*, **31**, 420–424.

[4] Daubechies, I. (1992). *Ten Lectures on Wavelets*, CBMS-NSF Series in Appl. Math., SIAM Publ.

[5] Dym, H. and McKean, H.P. (1972). *Fourier Series and Integrals*, Academic Press, New York.

[6] Gibbs, J.W. (1989). Letter to the editor, *Nature (London)*, **59**, 606.

[7] Gomes, S. and Cortina, E. (1995). Some results on the convergence of sampling series based on convolution integrals, *SIAM J. Math. Anal.*, **26**, 1386–1402.

[8] Helmberg, G. (1994). The Gibbs phenomenon for Fourier interpolation, *J. Approx. Th.*, **78**, 41–63.
[9] Kelly, S. (1996). Gibbs phenomenon for wavelets, *Appl. Comp. Harm. Anal.*, **3**, 72–81.
[10] Lemarié, P.G. (1988). Ondelettes *à* localisation exponentielles, *J. Math. Pure et Appl.*, **67**, 227–236.
[11] Meyer, Y. (1990). *Ondelettes,* Hermann, Paris.
[12] Richards, F.B. (1991). A Gibbs phenomenon for spline functions, *J. Approx. Theory,* **66**, 334–351.
[13] Shim, H.-T. (1994). The Gibbs' phenomenon for wavelets, Ph.D. Thesis, University of Wisconsin–Milwaukee.
[14] Shim, H.-T. and Volkmer, H. (1996). On Gibbs' phenomenon for wavelet expansions, *J. Approx. Th.*, **84**, 74–95.
[15] Walter, G.G. (1994). *Wavelets and Other Orthogonal Wavelets with Applications,* CRC Press, Boca Raton, FL.
[16] Walter, G.G. and Zayed, A.I. (1996). *Multiresolution Analysis with Sampling Subspaces,* preprint.

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