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Single Wavelets in n-Dimensions

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ABSTRACT. Under very minimal regularity assumptions, it can be shown that $2^n - 1$ functions are needed to generate an orthonormal wavelet basis for $L^2(\mathbb{R}^n)$. In a recent paper by Dai et al. it is shown, by abstract means, that there exist subsets K of $\mathbb{R}^n$ such that the single function ψ, defined by $\hat{\psi} = \chi_K$, is an orthonormal wavelet for $L^2(\mathbb{R}^n)$. Here we provide methods for constructing explicit examples of these sets. Moreover, we demonstrate that these wavelets do not behave like their one-dimensional counterparts.

1. Introduction

A function $\psi \in L^2(\mathbb{R})$ is said to be an *orthonormal wavelet* if the collection

$$\left\{\psi_{j,k}(x) = 2^{j/2}\psi\left(2^j x - k\right) : j, k \in \mathbb{Z}\right\} \tag{1.1}$$

is an orthonormal basis for $L^2(\mathbb{R})$. More generally, a family of functions $\psi^1, \dots, \psi^M$ in $L^2(\mathbb{R}^n)$ is called a *wavelet collection*, or a *wavelet family*, if

$$\left\{\psi^i_{j,k}(x) = 2^{nj/2}\psi^i\left(2^j x - k\right) : j \in \mathbb{Z}, k \in \mathbb{Z}^n, i = 1, \dots, M\right\} \tag{1.2}$$

is an orthonormal basis for $L^2(\mathbb{R}^n)$.

We say a wavelet ψ for $L^2(\mathbb{R})$ is a *minimally supported frequency* (MSF) wavelet if $\hat{\psi} = \chi_K$ for some set K in $\mathbb{R}$ (here $\hat{\psi}(\xi) = \int \psi(x)e^{-ix\xi}dx$ represents the Fourier transform of ψ). The simplest and most well-known example is the Shannon wavelet: $\hat{\psi} = \chi_K$, where $K = [-2\pi, -\pi] \cup [\pi, 2\pi]$. One of the reasons for studying MSF wavelets is their usefulness in providing counterexamples to conjectures about wavelets. In particular, the Journé wavelet $K = [-\frac{32}{7}\pi, -4\pi] \cup [-\pi, -\frac{4}{7}\pi] \cup [\frac{4}{7}\pi, \pi] \cup [4\pi, \frac{32}{7}\pi]$, was the first known non-MRA wavelet (MRAs are treated in the next section).

Auscher [1] has proven that every wavelet collection $\psi^1, \dots, \psi^M$ for $L^2(\mathbb{R}^n)$ whose members satisfy a weak smoothness and decay condition on the Fourier transform side must come from a multiresolution analysis (MRA). In particular, we will see that this implies that $M = q(2^n-1)$, $n \geq 1$,

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for some $q \in \mathbb{Z}^+$. This result casts some doubt on the existence of a single function $\psi \in L^2(\mathbb{R}^n)$, $n > 1$, such that $\psi_{j,k}$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$ is an orthonormal basis for $L^2(\mathbb{R}^n)$.

In [3], however, it is shown that there exists a set K in $\mathbb{R}^n$ such that $\hat{\psi} = \chi_K$ (χ_K denotes the characteristic function of K) defines an orthonormal wavelet for $L^2(\mathbb{R}^n)$. In fact, the authors prove the existence of such sets for wavelets with dilations more general than the usual dyadic ones. In light of Auscher's result, it is clear that it is the lack of smoothness of the Fourier transform of MSF wavelets that makes this possible.

In this article we provide explicit examples of such sets. In the one-dimensional case there are many examples of simple MSF wavelets (e.g., the Shannon wavelet). In several dimensions, however, the MSF wavelets all seem to be quite complicated. In particular, all of our examples exhibit fractal-like qualities. In addition to their complexity, these wavelets do not satisfy many properties that one-dimensional wavelets are known to satisfy.

The problem of constructing explicit examples of dyadic wavelet sets in higher dimensions was proposed to the two authors independently by Guido Weiss in the spring of 1996 after he became aware of the existence of such sets from the paper [3].

2. MRAs for $L^2(\mathbb{R}^n)$

A *multiresolution analysis* (MRA) for $L^2(\mathbb{R}^n)$, $n \geq 1$, is a sequence of closed subspaces V_j, $j \in \mathbb{Z}$, of $L^2(\mathbb{R}^n)$ satisfying

$$V_j \subset V_{j+1} \text{ for all } j \in \mathbb{Z}\,; \tag{2.1}$$

$$f \in V_j \text{ if and only if } f(2\cdot) \in V_{j+1} \text{ for all } j \in \mathbb{Z}\,; \tag{2.2}$$

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}\,; \tag{2.3}$$

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}^n)\,; \tag{2.4}$$

There exists a function $\phi \in V_0$ such that $\{\phi(\cdot - k) : k \in \mathbb{Z}^n\}$ is an orthonormal basis for V_0. (2.5)

The function ϕ is called the *scaling function* of the MRA. It is also possible to define an MRA with several scaling functions. This is done by assuming the existence of a finite family of functions $\phi_1, \dots, \phi_q \in V_0$ such that $\{\phi_i(\cdot - k) : k \in \mathbb{Z}^n, i = 1, \dots, q\}$ is an orthonormal basis for V_0.

Define W_j to be the orthogonal complement of the space V_j in V_{j+1}. We say that a wavelet collection $\psi^1, \dots, \psi^M$ is associated with an MRA if there exists an MRA such that $\{\psi^i(\cdot - k) : k \in \mathbb{Z}^n, i = 1, \dots, M\}$ is an orthonormal basis for W_0. For an explanation of how wavelets arise from an MRA, see [8] and [10] (see [2] for the multidimensional case). We now develop some properties of MRAs that we will need.

Since $\phi_{-1,0} \in V_{-1} \subset V_0$ it follows from (2.5) that there exists a $[0, 2\pi]^n$-periodic function m_0 such that

$$\hat{\phi}(2\xi) = m_0(\xi)\hat{\phi}(\xi)\,. \tag{2.6}$$

Similarly, $\psi^i \in W_0 \subset V_1$ so that $\psi^i_{-1,0} \in V_0$ and hence

$$\widehat{\psi^i}(2\xi) = m_i(\xi)\hat{\phi}(\xi) \tag{2.7}$$

for some $[0, 2\pi]^n$-periodic function m_i. The orthonormality of the $\mathbb{Z}^n$-shifts of ϕ implies that

$$\sum_{k\in\mathbb{Z}^n} \left|\hat{\phi}(\xi + 2\pi k)\right|^2 = 1 \tag{2.8}$$

Similarly for the ψ^i. Define $\square = \mathbb{Z}^n \cap [0, 1]^n$. That is, $\square$ consists of the 2^n lattice points contained in the unit cube. Then

$$\begin{aligned} 1 &= \sum_{k\in\mathbb{Z}^n} \left|\hat{\phi}(2\xi + 2\pi k)\right|^2 \\ &= \sum_{k\in\mathbb{Z}^n} |m_0(\xi + \pi k)|^2 \left|\hat{\phi}(\xi + \pi k)\right|^2 \\ &= \sum_{l\in\square} \sum_{k\in\mathbb{Z}^n} |m_0(\xi + \pi l + 2\pi k)|^2 \left|\hat{\phi}(\xi + \pi l + 2\pi k)\right|^2 \\ &= \sum_{l\in\square} |m_0(\xi + \pi l)|^2 \sum_{k\in\mathbb{Z}^n} \left|\hat{\phi}(\xi + \pi l + 2\pi k)\right|^2 \\ &= \sum_{l\in\square} |m_0(\xi + \pi l)|^2 . \end{aligned}$$

Similarly for the ψ^i. Fix i, then the shifts of ψ^i are all orthogonal to ϕ so that

$$0 = \sum_{k\in\mathbb{Z}^n} \widehat{\psi^i}(\xi + 2\pi k)\overline{\hat{\phi}(\xi + 2\pi k)} .$$

Therefore, we obtain

$$\begin{aligned} 0 &= \sum_{k\in\mathbb{Z}^n} \widehat{\psi^i}(2\xi + 2\pi k)\overline{\hat{\phi}(2\xi + 2\pi k)} \\ &= \sum_{k\in\mathbb{Z}^n} m_i(\xi + \pi k)\,\overline{m_0(\xi + \pi k)} \left|\hat{\phi}(\xi + \pi k)\right|^2 \\ &= \sum_{l\in\square} \sum_{k\in\mathbb{Z}^n} m_i(\xi + l\pi + 2\pi k)\overline{m_0(\xi + l\pi + 2\pi k)} \left|\hat{\phi}(\xi + l\pi + 2\pi k)\right|^2 \\ &= \sum_{l\in\square} m_i(\xi + l\pi)\overline{m_0(\xi + l\pi)} . \end{aligned}$$

Thus, the vectors $\vec{m}_i(\xi) \equiv \{ m_i(\xi + l\pi) \}_{l\in\square}$ and $\vec{m}_0(\xi) \equiv \{ m_0(\xi + l\pi) \}_{l\in\square}$ are orthonormal in $l^2(\square)$. The same is true for the vectors $\vec{m}_{i_1}(\xi)$ and $\vec{m}_{i_2}(\xi)$ for $i_1 \neq i_2$. Thus, the matrix with rows equal to $\vec{m}_i(\xi)$, $i = 0, \dots, 2^n - 1$, is a square orthogonal matrix. This implies, in particular, that the first column vector has norm 1. That is

$$1 = \sum_{i=0}^{2^n-1} |m_i(\xi)|^2 .$$

Therefore, we have

$$\left|\hat{\phi}(\xi)\right|^2 = \left|\hat{\phi}(2\xi)\right|^2 + \sum_{i=1}^{2^n-1} \left|\widehat{\psi^i}(2\xi)\right|^2 .$$

The usual iteration argument (see [8]) then gives

$$\left|\hat{\phi}(\xi)\right|^2 = \sum_{i=1}^{2^n-1} \sum_{j=1}^{\infty} \left|\widehat{\psi^i}\left(2^j\xi\right)\right|^2 . \tag{2.9}$$

Therefore, using (2.8) we obtain

$$1 = \sum_{i=1}^{2^n-1} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \widehat{\psi^i} \left(2^j (\xi + 2k\pi) \right) \right|^2 .$$

Note that this equality was derived using the scaling function of the MRA although there is no longer an explicit reference to it. It was observed by Auscher [1], Wang [12] and Gripenberg [7] that this equality, in fact, characterizes those wavelet families that arise from an MRA.

For a given wavelet family $\psi^1, \dots, \psi^M$ we define the *dimension function*

$$D(\xi) \equiv \sum_{i=1}^{M} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \widehat{\psi^i} \left(2^j (\xi + 2k\pi) \right) \right|^2 . \tag{2.10}$$

The theorem of Auscher, Gripenberg, and Wang is then:

Theorem 1.

Let $\psi_1, \dots, \psi_M$ be a wavelet collection for $L^2(\mathbb{R}^n)$. Then this collection is associated with an MRA (with a single scaling function) iff $D(\xi) \equiv 1$ for a.e. ξ.

We now mention two interesting properties of the dimension function. First, it is shown in [1] that the function D is a.e. integer-valued [$D(\xi)$ represents the dimension of a certain finite-dimensional vector space]. Second, using Plancherel's theorem and only the fact that $\|\psi^i\| = 1$, we obtain

$$\begin{aligned}
\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} D(\xi) d\xi &= \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \sum_{i=1}^{M} \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^n} \left| \widehat{\psi^i} \left(2^j (\xi + 2k\pi) \right) \right|^2 d\xi \\
&= \frac{1}{(2\pi)^n} \sum_{i=1}^{M} \sum_{j=1}^{\infty} \int_{\mathbb{R}^n} \left| \widehat{\psi^i} \left(2^j \xi \right) \right|^2 d\xi \\
&= \sum_{i=1}^{M} \sum_{j=1}^{\infty} 2^{-nj} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left| \widehat{\psi^i}(\xi) \right|^2 d\xi \\
&= M \sum_{j=1}^{\infty} 2^{-nj} = \frac{M}{2^n - 1} .
\end{aligned}$$

This implies the following.

Corollary 1.

If $\psi^1, \dots, \psi^M$ is a wavelet collection associated with an MRA, with a single scaling function, then $M = 2^n - 1$.

If we assume that $|\widehat{\psi^i}|$ is continuous and $|\widehat{\psi^i}(\xi)| = O(|\xi|^{-\frac{n}{2}-\alpha})$ at ∞ for some $\alpha > 0$ and $i = 1, \dots, M$, then it can be shown that the dimension function must be continuous. Since it is a.e. integer-valued, it must be identically q for some $q \in \mathbb{Z}^+$. This implies that $M = q(2^n - 1)$. In summary:

Corollary 2.

Single function wavelets for $L^2(\mathbb{R}^n)$, $n > 1$, do not arise from an MRA and must have irregular Fourier transforms.

3. Wavelets in $L^2(\mathbb{R}^n)$

It has been known for several years that wavelets for $L^2(\mathbb{R})$ can be characterized by two simple equations involving their Fourier transforms. It has recently been shown [4, 11] that these equations generalize to higher dimensions.

Theorem 2.

A collection of functions $\psi^1, \ldots, \psi^M \in L^2(\mathbb{R}^n)$, with $\|\psi^i\| = 1$, is a wavelet collection if and only if

$$\sum_{i=1}^{M}\sum_{j\in\mathbb{Z}}\left|\widehat{\psi^i}\left(2^j\xi\right)\right|^2 = 1, \quad \text{for a.e. } \xi \in \mathbb{R}^n , \tag{3.1}$$

and

$$\sum_{i=1}^{M}\sum_{j=0}^{\infty}\widehat{\psi^i}\left(2^j\xi\right)\overline{\widehat{\psi^i}\left(2^j(\xi+2m\pi)\right)} = 0, \quad \text{for a.e. } \xi \in \mathbb{R}^n, \quad m \in \mathbb{Z}^n \setminus 2\mathbb{Z}^n . \tag{3.2}$$

We will also need the following equations: For a given function $\psi \in L^2(\mathbb{R}^n)$ the set $\{\psi_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$ is orthonormal in $L^2(\mathbb{R}^n)$ if and only if

$$\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}(\xi+2\pi k)\right|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n , \tag{3.3}$$

and

$$\sum_{k\in\mathbb{Z}^n}\hat{\psi}\left(2^j(\xi+2\pi k)\right)\overline{\hat{\psi}(\xi+2\pi k)} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n , \quad j \geq 1 . \tag{3.4}$$

We say that a set K in $\mathbb{R}^n$ is a *wavelet set* if the function ψ defined by $\hat{\psi} = \chi_K$ is a wavelet for $L^2(\mathbb{R}^n)$. Using the above equations, explicit conditions can be given for a set to be a wavelet set.

Let us define a *partition* of a measurable set $A \subset \mathbb{R}^n$ as a countable collection $\{A_r\}$ of measurable subsets of A, such that $A_r \cap A_s$ has measure zero if $r \neq s$ and such that $\cup_r A_r = A$ up to a set of measure zero.

Theorem 3.

A measurable set K in $\mathbb{R}^n$ is a wavelet set if and only if

(a) $\left\{K + 2\pi k : k \in \mathbb{Z}^n\right\}$ *is a partition of* $\mathbb{R}^n$ *and*

(b) $\left\{2^j K : j \in \mathbb{Z}\right\}$ *is a partition of* $\mathbb{R}^n$.

Proof. Let $\hat{\psi} = \chi_K$, then $\hat{\psi}$ only assumes the values 1 and 0. In this case we will see that ψ is a wavelet if and only if (3.1) and (3.3) are satisfied. For such a ψ, these equations are equivalent to the two partition equations.

If ψ is a wavelet, then (3.1) and (3.3) are clearly satisfied. Conversely, we will see that (3.1) and (3.3) imply (3.2). Suppose $2^j\xi \in \operatorname{supp}\hat{\psi}$. Then $|\hat{\psi}(2^j\xi)| = 1$. Then, by (3.3), it follows that

$$\left|\hat{\psi}\left(2^j\xi + 2\pi k\right)\right| = 0, \quad \text{for all } k \in \mathbb{Z}^n, k \neq 0 .$$

In particular, $\hat{\psi}(2^j(\xi + 2\pi m)) = 0$ for every $m \in \mathbb{Z}^n \setminus 2\mathbb{Z}^n$ whenever $j \geq 0$. Thus, (3.2) is satisfied and, hence, ψ is a wavelet. □

Remark 1.

Condition (a) in the above theorem implies that the Lebesgue measure of K, $|K|$, is $(2\pi)^n$.

□

The problem of constructing wavelets can now be treated from a purely geometric viewpoint. Note that the first partition equation suggests that K resembles a cube. The second partition equation suggests that K resembles an annulus. In terms of certain equivalence relations, this is true.

Definition 1. *([3])*

We say two sets A and B are δ-congruent if there are two partitions $\{A_l\}_l$ and $\{B_l\}_l$ of A and B, respectively, and a sequence $\{j_l\}$ in $\mathbb{Z}$ such that $A_l = 2^{j_l} B_l$ for all l.

Two sets A and B are τ-congruent if there are two partitions $\{A_l\}_l$ and $\{B_l\}_l$ of A and B, respectively, and a sequence $\{k_l\}$ in $\mathbb{Z}^n$ such that $A_l = B_l + 2\pi k_l$ for all l.

Let A be a neighborhood of the origin and define $I = 2A \setminus A$. It is not difficult to show that $\{2^j I\}_{j\in\mathbb{Z}}$ is a partition of $\mathbb{R}^n$.

Theorem 4.

A measurable set K in $\mathbb{R}^n$ is a wavelet set if and only if K is δ-congruent to I and K is τ-congruent to $T \equiv [-\pi, \pi]^n$.

Proof. Suppose there are partitions $\{K_l\}$ and $\{K^l\}$ of K and partitions $\{I_l\}$ and $\{T_l\}$ of I and T, respectively, such that $K_l = 2^{j_l} I_l$ and $K^l = T_l + 2\pi k_l$ for sequences $\{j_l\} \subset \mathbb{Z}$ and $\{k_l\} \subset \mathbb{Z}^n$. Then we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \chi_K\left(2^j x\right) &= \sum_{j\in\mathbb{Z}} \chi_{2^j K}(x) \\ &= \sum_l \sum_{j\in\mathbb{Z}} \chi_{2^{j+j_l} I_l}(x) \\ &= \sum_{j\in\mathbb{Z}} \chi_{2^j I}(x) \\ &= 1 \text{ (a.e.)}.\end{aligned}$$

Similarly, we have $\sum_{k\in\mathbb{Z}^n} \chi_K(x + 2\pi k) = 1$ (a.e.). Thus, by Theorem 3, K is a wavelet set.

Conversely, suppose K satisfies the two partition equations of Theorem 3. Define

$$\begin{aligned}K_l &= K \cap \left(2^{-j_l} I\right) \\ K^l &= K \cap (T - 2\pi k_l)\end{aligned}$$

where $\{j_l\}$ and $\{k_l\}$ are enumerations of $\mathbb{Z}$ and $\mathbb{Z}^n$, respectively. Clearly $\{K_l\}$ and $\{K^l\}$ are partitions of K. Since $\{2^{j_l} K\}$ is a partition of $\mathbb{R}^n$, we see that the collection of sets

$$\left(2^{j_l} K\right) \cap I = 2^{j_l} K_l$$

is a partition of I. Thus, K is δ-congruent to I. Similarly, $\{K^l + 2\pi k_l\}$ is a partition of T and, hence, K is τ-congruent to T. □

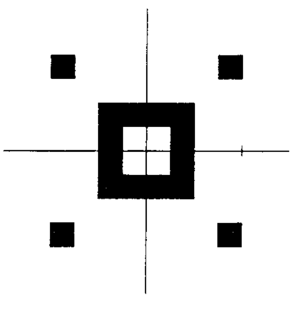
In their paper, Dai et al. prove that there exist sets K in $\mathbb{R}^n$ that are δ-congruent to $\{|x| < 2\} \setminus \{|x| < 1\}$ and are τ-congruent to $[-\pi, \pi]^n$. Moreover, they prove such a result for more general dilations and translations. That is, the definitions of δ-congruency and τ-congruency can be very general. However, their proof is abstract and does not provide an explicit construction. To construct explicit examples of these sets we will use two inductive methods based on Theorem 4.

Method 1. Start with the cube $K_1 \equiv [-\pi, \pi]^n$. To construct the set K_{m+1} from K_m we are only allowed to translate pieces of K_m by $2\pi k$ for some $k \in \mathbb{Z}^n$. In this way we are assured that every set K_m is τ-congruent to $K_1 = [-\pi, \pi]^n$. After each step we assess how much overlap there is between the various dyadic dilations of K_m. The set K_{m+1} is constructed so as to reduce the overlap occurring between these dilates. The goal of this process is to arrive (after countably many steps) at a set K that is also δ-congruent to $2A \setminus A$, where A is some neighborhood of the origin.

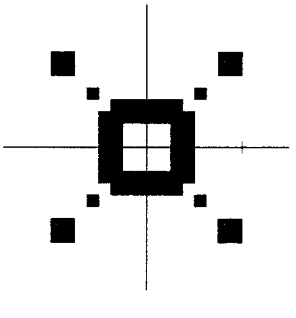
Method 2. Start with an annulus $K_1 \equiv 2A \setminus A$ where A is some neighborhood of the origin. To construct the set K_{m+1} from the set K_m, we are only allowed to dilate pieces of K_m by 2^j for some $j \in \mathbb{Z}$. Thus, we are assured that every set K_m is δ-congruent to $2A \setminus A$. After each step we assess how much overlap there is between the various $2\pi\mathbb{Z}^n$-translations of K_m. The set K_{m+1} is constructed so as to reduce the overlap occurring between these translates. The goal is to arrive (after countably many steps) at a set K that is also τ-congruent to $[-\pi, \pi]^n$.

It is interesting to note that these two methods can produce wavelet sets with different properties. In the next section, we will use the second method to construct wavelet sets that have the origin as a limit point, thus producing wavelets whose Fourier transforms are not continuous at the origin.

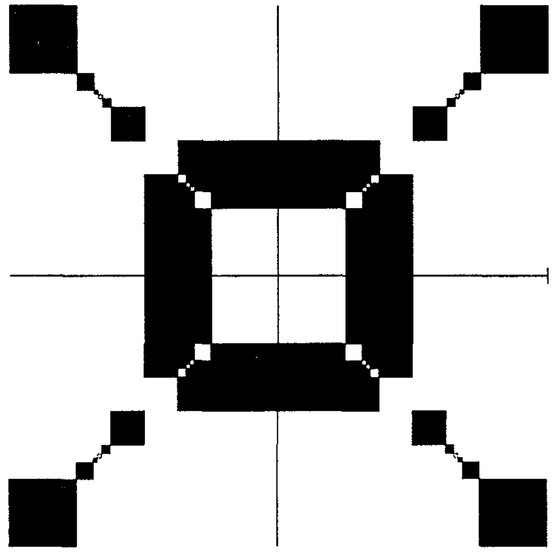
We now use Method 1 to construct a wavelet set in $\mathbb{R}^2$. This construction generalizes to higher dimensions (in $\mathbb{R}$, it produces the Shannon wavelet). Let $K_1 = [-\pi, \pi]^2$. To construct K_2 we move four subcubes of K_1. Translate the cube $[0, \frac{\pi}{2}]^2$ by the vector $(-2\pi, -2\pi) \in 2\pi\mathbb{Z}^2$. That is, a cube in the lower left corner of the first quadrant is moved into the third quadrant. Do the same for every other quadrant in a symmetrical way:



Note that $2^{-1}K_2$ overlaps K_2, for example, in the cube $[\frac{3}{4}\pi, \pi]^2$. Thus, to construct the set K_3 we should translate (recall that only $2\pi\mathbb{Z}^n$-translation is allowed) this cube to avoid overlap. Translate $[\frac{3}{4}\pi, \pi]^2$ by $(-2\pi, -2\pi)$. Do the same, in a symmetric manner, for the overlap in the other three quadrants:



Continue this process and let K be the resulting set:

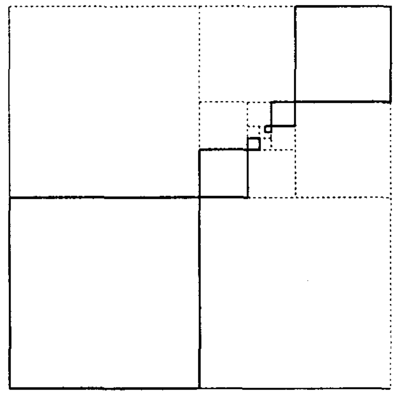


The process outlined above is not a rigorous proof that K is a wavelet set. We provide such a proof now.

We explicitly define the portion of K lying in the first quadrant. The rest of K is defined by symmetry. Define disjoint cubes Q_m and $R_m, m \geq 1$, as follows:

$$Q_m = \left[\frac{4-\left(\frac{1}{4}\right)^{m-2}}{6}\pi, \frac{4-\left(\frac{1}{4}\right)^{m-1}}{6}\pi\right]^2, \tag{3.5}$$

$$R_m = \left[\frac{2+\left(\frac{1}{4}\right)^{m}}{3}\pi, \frac{2+\left(\frac{1}{4}\right)^{m-1}}{3}\pi\right]^2. \tag{3.6}$$



Define

$$S = \left(\bigcup_{i=1}^{\infty} Q_i\right) \cup \left(\bigcup_{i=1}^{\infty} R_i\right), \tag{3.7}$$

$$A = [0,\pi]^2 \setminus S, \tag{3.8}$$

$$B = -S + (2\pi, 2\pi) \tag{3.9}$$

[here $(\cdot,\cdot)$ denotes an ordered pair point in the plane]. The portion of K that lies in the first quadrant is defined to be $A \cup B$. The rest of K is defined by symmetry. We now show that K satisfies the hypotheses of Theorem 4.

To show K is δ-congruent to $[-\pi,\pi]^2 \setminus [-\frac{\pi}{2},\frac{\pi}{2}]^2$ it suffices, by symmetry, to prove:

Claim 1.
$\left(2^{-1}B\right) \cup A = [0,\pi]^2 \setminus [0,\frac{\pi}{2}]^2$.

Proof. For this it suffices to show that

$$\frac{1}{2}(-Q_m + (2\pi,2\pi)) = R_m \quad, m \geq 1, \tag{3.10}$$

$$\frac{1}{2}(-R_m + (2\pi,2\pi)) = Q_{m+1} \quad, m \geq 1. \tag{3.11}$$

Indeed, if (3.10) and (3.11) hold, then

$$\begin{aligned}\left(2^{-1}B\right) \cup A &= 2^{-1}(-S + (2\pi,2\pi)) \cup A \\ &= (Q_2 \cup Q_3 \cup \ldots) \cup (R_1 \cup R_2 \cup \ldots) \cup A \\ &= [0,\pi]^2 \setminus Q_1 = [0,\pi]^2 \setminus \left[0,\frac{\pi}{2}\right]^2.\end{aligned}$$

We will prove (3.10):

$$\begin{aligned}\frac{1}{2}(-Q_m + (2\pi,2\pi)) &= \frac{1}{2}\left(\left[-\left(\frac{4-\left(\frac{1}{4}\right)^{m-1}}{6}\right)\pi, -\left(\frac{4-\left(\frac{1}{4}\right)^{m-2}}{6}\right)\pi\right]^2 + (2\pi,2\pi)\right) \\ &= \frac{1}{2}\left[\left(2-\frac{4-\left(\frac{1}{4}\right)^{m-1}}{6}\right)\pi, \left(2-\frac{4-\left(\frac{1}{4}\right)^{m-2}}{6}\right)\pi\right]^2 \\ &= \frac{1}{2}\left[\frac{8+\left(\frac{1}{4}\right)^{m-1}}{6}\pi, \frac{8+\left(\frac{1}{4}\right)^{m-2}}{6}\pi\right]^2 \\ &= \left[\frac{8+\left(\frac{1}{4}\right)^{m-1}}{12}\pi, \frac{8+\left(\frac{1}{4}\right)^{m-2}}{12}\pi\right]^2 \\ &= \left[\frac{2+\left(\frac{1}{4}\right)^{m}}{3}\pi, \frac{2+\left(\frac{1}{4}\right)^{m-1}}{3}\pi\right]^2 \\ &= R_m.\end{aligned}$$

The proof of (3.11) is similar. □

To show K is τ-congruent to $[-\pi, \pi]^2$ we will show that

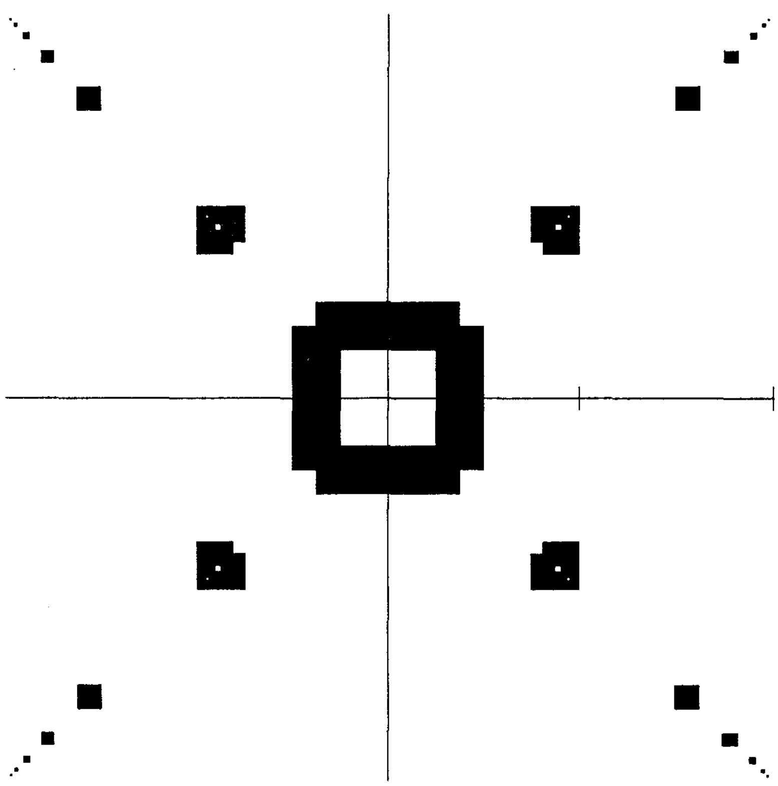
$$(-B + (2\pi, 2\pi)) \cup A = [0, \pi]^2 . \tag{3.12}$$

(note that $-B$ is a portion of K lying in the third quadrant). This equality follows trivially from the definition of B. Indeed,

$$\begin{aligned} (-B + (2\pi, 2\pi)) \cup A &= (S - (2\pi, 2\pi) + (2\pi, 2\pi)) \cup A \\ &= S \cup A = [0, \pi]^2 . \end{aligned}$$

This completes the proof that K is a wavelet set.

In one dimension the preceding construction yields the Shannon wavelet ($K = [-2\pi, -\pi] \cup [\pi, 2\pi]$). It can be shown, using results from [9] (see Theorem 7) that the only one-dimensional wavelet whose Fourier transform is supported in $[-2\pi, 2\pi]$ is the Shannon wavelet (up to modulation). Therefore, to construct more general wavelets (ones that are not extensions of the Shannon wavelet), we cannot have $K \subset [-2\pi, 2\pi]^n$. The following wavelet set is constructed in a similar way to our first example. Indeed, K_1 and K_2 are the same but we then start to translate outside of the cube $[-2\pi, 2\pi]^n$. The limiting set K is contained in $[-4\pi, 4\pi]^n$:



This construction also works in one dimension. It gives an example of a new band-limited MSF one-dimensional wavelet. This new wavelet can be visualized by projecting onto the x-axis the intersection of the above wavelet with the diagonal $\Delta = \{ (x, y) : x = y \}$.

4. Hole in the Middle

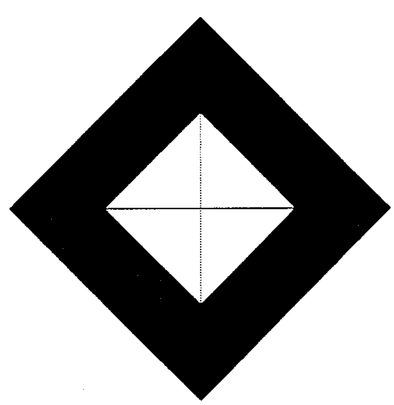
Suppose, for the moment, that ψ is a wavelet whose Fourier transform is continuous at the origin. Then the finiteness of the sum

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}\left(2^j\xi\right)\right|^2 = 1$$

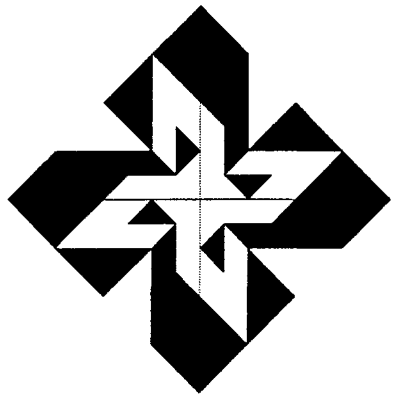
implies that $\hat{\psi}(0) = 0$. Sometimes even more can be said. For example, suppose ψ is a wavelet for $L^2(\mathbb{R})$ such that $\hat{\psi}$ is supported in $[-\frac{8\pi}{3}, \frac{8\pi}{3}]$ (no regularity assumptions). Then it is shown in [9] that $\hat{\psi}$ must vanish in the neighborhood $[-\frac{2\pi}{3}, \frac{2\pi}{3}]$ of the origin (there is a hole in the middle of the support of $\hat{\psi}$). More generally, if $\hat{\psi}$ is supported in $[-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$, for $0 < \alpha \leq \pi$, then $\hat{\psi}$ must vanish on $[-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha]$ (see Theorem 7). One may conjecture that any band-limited (i.e., compactly supported Fourier transform) wavelet must vanish in a neighborhood of the origin. This is not the case. Indeed, Garrigós [6] has constructed a wavelet set that is supported in the interval $[-\pi, 4\pi]$ that has 0 as a limit point. Thus, in one dimension, suitable restrictions on the size of the support of $\hat{\psi}$ imply that $\hat{\psi}$ must vanish in a neighborhood of the origin.

We wish to show that this sort of result cannot be generalized to higher dimensions. This is done by constructing a wavelet set with essentially the smallest possible support such that 0 is a limit point of this set. We first make the observation that any wavelet set must accumulate near points in the lattice $2\pi\mathbb{Z}^n$ other than 0. It is clear that a wavelet set cannot be a neighborhood of the origin. This follows from the second partition equation in Theorem 3. However, the first equation in this theorem implies that a wavelet set is a neighborhood of the origin modulo $2\pi\mathbb{Z}^n$. Thus, the set must accumulate near non-zero lattice points. Consider the four closest lattice points to the origin in the plane: $(0, 2\pi)$, $(2\pi, 0)$, $(0, -2\pi)$, $(-2\pi, 0)$. It is not hard to see, using Theorem 3, that any wavelet set contained in the convex hull of these four points must accumulate near each of these lattice points. We now construct such a set which has 0 as a limit point. The construction follows Method 2.

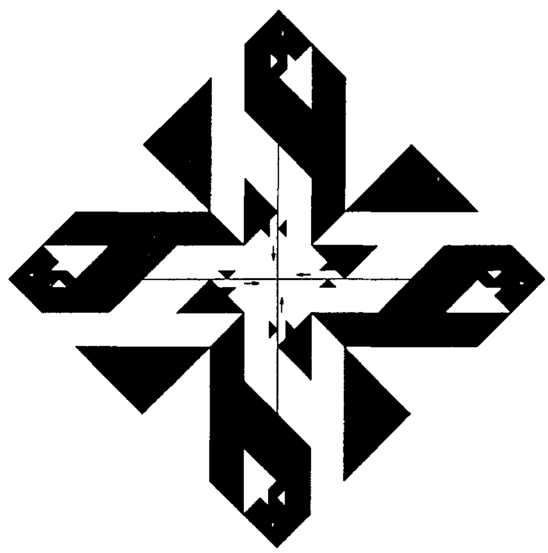
Let A be the convex hull of the points $(0, \pi)$, $(\pi, 0)$, $(0, -\pi)$, $(-\pi, 0)$ and set $K_1 = 2A \setminus A$:



Construct K_2 by dilating portions of this annulus by 2^{-1}:



To construct K_{m+1} from K_m, dilate the portions of K_m that overlap under the action of $2\pi\mathbb{Z}^n$ by smaller and smaller powers of two. This forces the origin to be a limit point of the resulting set:



We now state some specific results about the behavior of the Fourier transform of single function wavelets.

Theorem 5.

Let γ be a wavelet for $L^2(\mathbb{R}^n)$ (not necessarily an MSF wavelet). Then the intersection of $\operatorname{supp}\hat{\gamma}$ *and the cube $[-M, M]^n$ has strictly positive Lebesgue measure for* $M > \frac{\pi}{\sqrt[n]{2^n-1}}$.

Remark 2.

If γ is an MSF wavelet, there is an easy way to see this. Suppose the support of $\hat{\gamma}$ (the set K) does not intersect the cube $[-M, M]^n$. We know that the set K is δ-congruent to the annulus $[-2M, 2M]^n \setminus [-M, M]^n$. Since $K \cap [-M, M]^n$ has measure zero, it is an easy exercise to see that the measure of K must be larger than the measure of this annulus. However, the measure of K is always $(2\pi)^n$, thus we obtain the inequality $(2\pi)^n \geq (4M)^n - (2M)^n$. Solving the above inequality for M gives the desired result. □

Remark 3.

When we constructed our first wavelet set we commented that the construction works in all dimensions. The n-dimensional version of this set does not intersect the cube $[\frac{-\pi}{2}, \frac{\pi}{2}]^n$. *This is optimal for any construction that holds in all dimensions. Indeed, the limit as n approaches infinity of the expression* $\frac{\pi}{\sqrt[n]{2^n-1}}$ *is* $\frac{\pi}{2}$. □

Although a wavelet set K may have 0 as a limit point, we can say something about the concentration of K near the origin.

Theorem 6.

Let $K \subset \mathbb{R}^n$ *be such that* $\hat{\gamma} = \chi_K$ *is the Fourier transform of a single function wavelet. Then, for every* $k \geq 0$,

$$\left|K \cap 2^{-k}[-\pi, \pi]^n\right| \leq \frac{(2^n - 1)\pi^n}{2^{nk}}$$

so that $\hat{\gamma}$ *vanishes on a set of measure at least* $\frac{\pi^n}{2^{nk}}$ *contained in* $2^{-k}[-\pi, \pi]^n$.

If $K \subset [-2\pi, 2\pi]^n$, *then, for every* $k \geq 0$,

$$\left|K \cap 2^{-k}[-\pi, \pi]^n\right| \leq (2\pi)^n \frac{2^n - 2}{2^{n(k+1)} - 1} .$$

This implies, in particular, that $\hat{\gamma}$ *vanishes on a set of measure at least* $\frac{(2\pi)^n}{2^n-1}$ *contained in* $[-\pi, \pi]^n$.

Remark 4.

Suppose $K \subset [-2\pi, 2\pi]^2 \subset \mathbb{R}^2$, *then the above theorem states, in particular, that*

$$\left|K \cap [-\pi, \pi]^2\right| \leq \frac{2}{3}(2\pi)^2 .$$

That is, K does not cover more than $\frac{2}{3}$ *of the cube* $[-\pi, \pi]^2$. *This estimate is optimal since the wavelet set constructed in the previous section covers precisely* $\frac{2}{3}$ *of this cube.*

Another way to see this is as follows: If $K \subset [-2\pi, 2\pi]^2$, *then the set* $2^{-1}K \cup 2^{-2}K \cup \cdots$ *is disjoint from K and is contained in the cube* $[-\pi, \pi]^2$. *The measure of this set is*

$$|2^{-1}K| + |2^{-2}K| + \cdots = (2\pi)^2\left(\frac{1}{4} + \frac{1}{16} + \cdots\right) = \frac{(2\pi)^2}{3} . \qquad \square$$

Let $Q_1, \ldots, Q_{2^n}$ denote the 2^n quadrants in $\mathbb{R}^n$ and let $I_M = [-2M, 2M]^n \setminus [-M, M]^n$. Define functions $\psi^1, \ldots, \psi^{2^n}$ such that $\widehat{\psi_i} = \chi_{Q_i \cap I_M}$.

Lemma 1.

If $0 < M \leq \pi$, *then the collection* $\psi^1, \ldots, \psi^{2^n}$ *generates a tight frame for* $L^2(\mathbb{R}^n)$. *This means that*

$$\|f\|^2 = \sum_{r=1}^{2^n} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} \left|\langle f, \psi^r_{j,k}\rangle\right|^2 \tag{4.1}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. It is shown in [4] that the two equations in Theorem 2 (without the assumption that $\|\psi^r\| = 1$) characterize tight frames of this form. Equation (3.1) is easily verified for this system. Equation (3.2) follows from the fact that the diameter of the sets $Q_i \cap I_M$ measured along the coordinate axes (i.e., in the norm $\|x\| = \max_{1\leq i\leq n}|x_i|$) is smaller than 2π. Hence, all of the summands in (3.2) vanish a.e. □

Note that

$$\|\psi^r\|^2 = \frac{1}{(2\pi)^n}\|\widehat{\psi^r}\|^2 = \frac{1}{(2\pi)^n}|Q_r \cap I_M| = \frac{M^n(2^n-1)}{(2\pi)^n}.$$

Let γ be a single wavelet for $L^2(\mathbb{R}^n)$. Every ψ^r can be expressed in terms of the system $\{\gamma_{j,k}\}$. Thus, we have that

$$\begin{aligned}
\frac{2^n M^n(2^n-1)}{(2\pi)^n} &= \sum_{r=1}^{2^n}\|\psi^r\|^2 = \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}}|\langle\psi^r,\gamma_{j,k}\rangle|^2 \\
&= \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}}2^{nj}\left|\int_{\mathbb{R}^n}\overline{\psi^r(x)}\gamma\left(2^jx-k\right)dx\right|^2 \\
&= \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}}2^{-nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\psi^r}(\xi)\hat{\gamma}\left(2^{-j}\xi\right)e^{-i2^{-j}k\xi}d\xi\right|^2
\end{aligned}$$

Changing variables in the integral and separating the sums for $j > 0$ and $j \le 0$, we have

$$\begin{aligned}
\frac{2^n M^n(2^n-1)}{(2\pi)^n} &= \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}}2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\psi^r}\left(2^j\xi\right)\hat{\gamma}(\xi)e^{-ik\xi}d\xi\right|^2 \\
&= \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j>0}2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\psi^r}\left(2^j\xi\right)\hat{\gamma}(\xi)e^{-ik\xi}d\xi\right|^2 \\
&\quad+\sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\le 0}2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\psi^r}\left(2^j\xi\right)\hat{\gamma}(\xi)e^{-ik\xi}d\xi\right|^2 \\
&\equiv S_1 + S_2 .
\end{aligned}$$

We first evaluate S_1. It is clear that $\cup_{r=1}^{2^n}\operatorname{supp}\widehat{\psi^r} = I_M$, and, if $j > 0$ and $M \le \pi$, that $\cup_{r=1}^{2^n}\operatorname{supp}\widehat{\psi^r}(2^j\cdot) = 2^{-j}I_M \subset [-\pi,\pi]^n$. Thus, we have that

$$\begin{aligned}
S_1 &= \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j>0}2^{nj}\left|\frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n}\widehat{\psi^r}\left(2^j\xi\right)\hat{\gamma}(\xi)e^{-ik\xi}d\xi\right|^2 \\
&= \sum_{r=1}^{2^n}\sum_{j>0}\frac{2^{nj}}{(2\pi)^n}\int_{[-\pi,\pi]^n}\left|\hat{\gamma}(\xi)\widehat{\psi^r}\left(2^j\xi\right)\right|^2 d\xi \\
&= \sum_{j>0}\frac{2^{nj}}{(2\pi)^n}\int_{2^{-j}I_M}|\hat{\gamma}(\xi)|^2 d\xi .
\end{aligned}$$

To estimate S_2 we use the fact that, for every $j \le 0$,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^n}2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{\gamma}(\xi)\widehat{\psi^r}\left(2^j\xi\right)e^{-ik\xi}d\xi\right|^2 \\
&\le \sum_{k\in\mathbb{Z}^n}2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{\gamma}(\xi)\widehat{\psi^r}\left(2^j\xi\right)e^{-i2^jk\xi}d\xi\right|^2 ,
\end{aligned}$$

since the second sum contains more terms than the first one.

Denote by ζ_M the function such that

$$\widehat{\zeta_M} = \left(1 - \chi_{[-M,M]^n}\right)\hat{\gamma}\ .$$

Then, using the fact that the ψ^r form a tight frame such that $\operatorname{supp}\widehat{\psi^r} \subset I_M$, we obtain

$$\begin{aligned} S_2 &\le \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\le 0} 2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\hat{\gamma}(\xi)\widehat{\psi^r}\left(2^j\xi\right)e^{-i2^jk\xi}d\xi\right|^2 \\ &= \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\sum_{j\in\mathbb{Z}} 2^{nj}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\zeta_M}(\xi)\widehat{\psi^r}\left(2^j\xi\right)e^{-i2^jk\xi}d\xi\right|^2 \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left|\widehat{\zeta_M}(\xi)\right|^2 d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\setminus[-M,M]^n}\left|\hat{\gamma}(\xi)\right|^2 d\xi\ . \end{aligned}$$

Furthermore, if $\operatorname{supp}\hat{\gamma} \subset [-2M,2M]^n$, then $\operatorname{supp}\widehat{\zeta_M} \subset [-2M,2M]^n \setminus [-M,M]^n$ and the above inequality becomes an equality, since in this case

$$S_2 = \sum_{r=1}^{2^n}\sum_{k\in\mathbb{Z}^n}\left|\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\zeta_M}(\xi)\widehat{\psi^r}(\xi)e^{-ik\xi}d\xi\right|^2 = (2\pi)^n\left\|\widehat{\zeta_M}\right\|^2\ .$$

Hence, we get the following result:

Proposition 1.

Suppose that γ is a single wavelet for $L^2(\mathbb{R}^n)$ and let $I_M = [-2M,2M]^n \setminus [-M,M]^n$, with $0 < M \le \pi$. Then

$$\sum_{j>0} 2^{nj}\int_{2^{-j}I_M}\left|\hat{\gamma}(\xi)\right|^2 d\xi \le (2M)^n\left(2^n-1\right) \le \sum_{j>0} 2^{nj}\int_{2^{-j}I_M}\left|\hat{\gamma}(\xi)\right|^2 d\xi + \int_{\mathbb{R}^n\setminus[-M,M]^n}\left|\hat{\gamma}(\xi)\right|^2 d\xi\ .$$

If $\operatorname{supp}\hat{\gamma} \subset [-2M,2M]^n$, *then*

$$(2M)^n\left(2^n-1\right) - (2\pi)^n = \sum_{j>0}\left(2^{nj}-1\right)\int_{2^{-j}I_M}\left|\hat{\gamma}(\xi)\right|^2 d\xi\ .$$

Proof. We have shown that $\frac{2^n M^n (2^n-1)}{(2\pi)^n} = (S_1 + S_2)$ and that

$$S_2 \le \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n\setminus[-M,M]^n}\left|\hat{\gamma}(\xi)\right|^2 d\xi$$

with equality if $\operatorname{supp}\hat{\gamma} \subset [-2M,2M]^n$. Thus the proposition holds, since

$$\int_{\mathbb{R}^n\setminus[-M,M]^n}\left|\hat{\gamma}(\xi)\right|^2 d\xi = (2\pi)^n - \sum_{j>0}\int_{2^{-j}I_M}\left|\hat{\gamma}(\xi)\right|^2 d\xi\ . \quad \square$$

We can now prove Theorem 5. Suppose γ is a wavelet for $L^2(\mathbb{R}^n)$ such that $\operatorname{supp}\hat{\gamma} \subset \mathbb{R}^n \setminus [-M,M]^n$. Then, by Proposition 1,

$$(2M)^n\left(2^n-1\right) \le \int_{\mathbb{R}^n\setminus[-M,M]^n}\left|\hat{\gamma}(\xi)\right|^2 d\xi = (2\pi)^n\ .$$

This implies that $M \le \frac{\pi}{\sqrt[n]{2^n-1}}$.

Using Proposition 1 we can now prove Theorem 6. Suppose $\hat{\gamma} = \chi_K$ is a wavelet for some $K \subset \mathbb{R}^n$. Let $I = I_\pi$. Then, Proposition 1 takes the form

$$\sum_{j>0} 2^{nj} \left|\left(2^{-j} I\right) \cap K\right| \le (2\pi)^n \left(2^n - 1\right) \le \sum_{j>0} 2^{nj} \left|\left(2^{-j} I\right) \cap K\right| + \left|K \setminus [-\pi, \pi]^n\right|$$

and, when $K \subset [-2\pi, 2\pi]^n$,

$$(2\pi)^n \left(2^n - 2\right) = \sum_{j>0} \left(2^{nj} - 1\right) \left|\left(2^{-j} I\right) \cap K\right| .$$

This implies that, for every $k \ge 0$,

$$2^{nk} \left|K \cap 2^{-k}[-\pi, \pi]^n\right| \le \sum_{j \ge k} 2^{nj} \left|K \cap 2^{-j} I\right| \le (2\pi)^n \left(2^n - 1\right) .$$

The second part of Theorem 6 is proved similarly.

5. Final Remarks

In [5] there is a construction of an unbounded wavelet set in one dimension. Like most constructions of wavelet sets in one dimension, the method in [5] does not readily generalize to higher dimensions. However, by using Method 1 one can easily construct unbounded wavelet sets in higher dimensions (and new ones for $L^2(\mathbb{R})$).

In [9] the authors obtain the following "classification" theorem for one-dimensional wavelets:

Theorem 7.

Suppose $\psi \in L^2(\mathbb{R})$ and $b = |\hat{\psi}|$ has support contained in

$$S_\alpha = \left[-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha\right], \quad 0 < \alpha \le \pi .$$

Then ψ is an orthonormal wavelet if and only if

1. $b^2(\xi) + b^2(\frac{1}{2}\xi) = 1$ *for a.e.* $\xi \in [-4\pi - \frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$;
2. $b(\xi) = 1$ *for a.e.* $\xi \in [2\pi - \frac{2}{3}\alpha, 4\pi - \frac{8}{3}\alpha]$;
3. $b^2(\xi) + b^2(\xi + 2\pi) = 1$ *for a.e.* $\xi \in [-\frac{4}{3}\alpha, -\frac{2}{3}\alpha]$;
4. $b(\xi) = b(\frac{1}{2}\xi + 2\pi)$ *for a.e.* $\xi \in [-\frac{8}{3}\alpha, -\frac{4}{3}\alpha]$;
5. $\hat{\psi}(\xi) = e^{ip(\xi)} b(\xi)$, *with* $p(\xi)$ *satisfying* $p(\xi) + p(2(\xi - 2\pi)) - p(2\xi) - p(\xi - 2\pi) = (2n(\xi) + 1)\pi$ *for a.e.* $\xi \in D_\alpha \bigcap (\operatorname{Supp} b) \bigcap \left(\frac{1}{2} \operatorname{Supp} b\right)$, *where* $D_\alpha = [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$ *and* $n(\xi)$ *is an integer-valued measurable function.*
6. $b(\xi) = 0$ *for a.e.* $\xi \in [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha] = H_\alpha$.

In particular, $|\hat{\psi}|$ is completely determined by its (arbitrary) values on $D_\alpha = [2\pi - \frac{4}{3}\alpha, 2\pi - \frac{2}{3}\alpha]$. The key to the proof of this theorem is showing that $\hat{\psi} = 0$ on $H_\alpha = [-\frac{2}{3}\alpha, 2\pi - \frac{4}{3}\alpha]$ when $\hat{\psi}$ is supported in $S_\alpha = [-\frac{8}{3}\alpha, 4\pi - \frac{4}{3}\alpha]$. In this case, the equations in Theorem 2 have a particular simple form. The results in the previous section show that no such simple "classification" theorem can hold for higher dimensional wavelets.

All of the wavelet sets constructed in this article are non-trivial in the sense that they cannot be written as a finite collection of polygons. We conjecture that this is always the case. That is, any dyadic wavelet in higher dimensions cannot be written as a finite collection of polygons. In [3] there are examples of dilation matrices A for which there exist wavelet sets consisting of two convex polygons (here the definition of wavelet is changed to have dilations by A^j instead of the usual dyadic dilations 2^j). Thus, an interesting problem would be to characterize those dilation matrices for which there exist "simple" wavelet sets.

References

[1] Auscher, P. (1995). Solution of two problems on wavelets, *J. Geom. Anal.*, 5(2), 181–236.

[2] David, G. (1991). Wavelets and singular integrals on curves and surfaces, *Lecture Notes in Mathematics 1465*, Springer-Verlag.

[3] Dai, X., Larson, D., and Speegle, D. *Wavelet sets in* $\mathbb{R}^n$, preprint.

[4] Frazier, M., Garrigós, G., Wang, K., and Weiss, G. *A characterization of functions that generate wavelet and related expansions*, preprint.

[5] Fang, X. and Wang, X. (1996). Construction of minimally supported frequency wavelets, *J. Fourier Anal. Appl.*, 2(4), 315–327.

[6] Garrigós, G. (1996). Personal communication.

[7] Gripenberg, G. (1995). A necessary and sufficient condition for the existence of a father wavelet, *Studia Math.*, **114**, 207–226.

[8] Hernández, E. and Weiss, G. (1996). *A First Course on Wavelets*, CRC Press, Boca Raton, FL.

[9] Hernández, E., Wang, X., and Weiss, G. (1996). Smoothing minimally supported frequency wavelets. Part I, *J. Fourier Anal. Appl.*, 2(4), 329–340.

[10] Meyer, Y. (1990). *Ondelettes et Opérateurs*, Hermann, Paris.

[11] Ron, A. and Shen, Z. *Affine systems in* $L^2(\mathbb{R}^d)$*: the analysis of the analysis operator*, preprint.

[12] Wang, X. *The study of wavelets from the properties of their Fourier transforms*, Ph.D. thesis, Washington University, St. Louis, MO.

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