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Sampling of Paley-Wiener Functions on Stratified Groups

Isaac Pesenson

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ABSTRACT. We consider a generalization of entire functions of spherical exponential type on stratified groups. An analog of the Paley–Wiener theorem is given. We also show that every spectral entire function on a stratified group is uniquely determined by its values on some discrete subgroups. The main result of the article is reconstruction formula of spectral entire functions from their values on discrete subgroups using Lagrangian splines.

1. Introduction and Statements of Main Results

The classical Shannon-Whittaker sampling theorem states that if $f \in L^2(R)$ and its Fourier transform $\hat{f}$ has support in $[-\omega, \omega]$, then f is completely determined by its values at points $n\Omega$, where $\Omega = \pi/\omega$ and in L^2-sense

$$f(t) = \sum f(n\Omega)\frac{\sin(\pi(t - n\Omega))}{\pi(t - n\Omega)} .$$

Functions $f \in L^2(R)$ with property $\text{supp}\hat{f} \subset [-\omega, \omega]$ form the Paley-Wiener class PW_ω. The Paley-Wiener theorem states that f is in PW_ω if and only if f is an entire function of exponential type ω.

Different kind of generalizations of the Shannon-Whittaker formula can be found in Benedetto's survey [1].

We introduce an appropriate generalization of entire functions of spherical exponential type which we call spectral entire functions of exponential type. Our goal is to show that the reconstruction of sampled spectral entire functions of exponential type is possible as long as the distance between points from a sampling sequence is small enough. The reconstruction formula involves the notion of a spline.

The consideration in the present article is subelliptic in the sense that the central role belongs to a certain subelliptic operator. The case of corresponding elliptic theory on manifolds was considered by the author in [11] and [12].

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The known proof of the Shannon-Whittaker Formula uses the fact that functions e^{int} form orthonormal basis in $L_2([-\pi, \pi])$. Our explanation of this phenomenon is different: an entire function of exponential type can be reconstructed from its values on certain discrete sets because it satisfies the Bernstein inequality. Our method applied to the classical case gives the result which is slightly weaker than the known Shannon-Whittaker Theorem in the sense that the distance between points from a sampling sequence should be small enough. But the method of reconstruction that is based on Approximation Theorem 3 seems to be new.

Recall that a nilpotent group Lie G is stratified if its Lie algebra is a direct sum of $V_1, V_2, ..., V_n$ where $[V_i, V_j] = V_{i+j}$, if $i+j \le n$ and $[V_i, V_j] = 0$, if $i+j > n$. As a manifold, such a group can be identified with Euclidean space and the invariant measure is the usual Lebesgue measure. The number $Q = \sum j(dim V_j)$ is called the homogeneous dimension of G.

Every stratified algebra Lie admits one parameter group of diffeomorphisms

$$\delta_t\,(v_1 + ... + v_n) = tv_1 + ... + t^n v_n,\; v_i \in V_i$$

which are called dilations. The homogeneous norm is introduced by the formula

$$\left|\sum_{i=1}^{n} v_i\right| = \left(\sum_{i=1}^{n} |v_i|^{(2n!/i)}\right)^{1/2r!},\; v_i \in V_i\,.$$

Let $X_1, ..., X_m$ be a basis for V_1. Then sub-Laplacian $D = -X_1^2 - ... - X_m^2$ is a second order self-adjoint and positive definite hypoelliptic operator which is homogeneous with respect to the above dilations.

Using sub-Laplacian D one can introduce the Sobolev scale of spaces with the norm $\|f\|_{S^\sigma(G)} = \|(I+D)^{\sigma/2}f\|$, $\sigma \ge 0$. As was shown by Folland [3] (see also [2, 4, 8, 9, 10]) this norm is equivalent to the norm $\|f\| + \|D^{\sigma/2}f\|$ and if $\sigma = r$ is an integer to the norm

$$\|f\| + \sum_{1\le i_1,...,i_r \le m} \left\|X_{i_1}...X_{i_r}f\right\|\,.$$

For negative σ spaces $S^\sigma(G)$ can be introduced using duality. The full scale $S^\sigma(G)$, $-\infty < \sigma < \infty$ serves the sub-Laplacian D in the same way as standard Sobolev spaces $H^\sigma(R^d)$, $-\infty < \sigma < \infty$ serve standard Laplacian Δ.

The following is a brief description of our main results. First of all we introduce an abstract definition of entire functions of exponential type.

Let E be a Hilbert space with the norm $\|.\|$ and D a self-adjoint positive definite operator in E. According to the spectral theory [6] there exists a direct integral of Hilbert spaces $X = \int X(\lambda)dm(\lambda)$ and a unitary operator F from E onto X, which transforms domain of D^k onto $X^k = \{x \in X | \lambda^k x \in X\}$ with norm

$$\|x(\lambda)\|_{X^k} = \left(\int_0^\infty \lambda^{2k}\|x(\lambda)\|^2_{X(\lambda)}dm(\lambda)\right)^{1/2}$$

besides $F(D^k f) = \lambda^k(Ff)$, if f belongs to the domain of D^k. As is known, X is the set of all m-measurable functions $\lambda \to x(\lambda) \in X(\lambda)$, for which the norm

$$\|x\|_X = \left(\int_0^\infty \|x(\lambda)\|^2_{X(\lambda)}dm(\lambda)\right)^{1/2}$$

is finite.

We will say that a vector f from E is a spectral entire function of exponential type ω (ω-SE function) if its "Fourier transform" Ff has support in $[0, \omega]$. The $E_\omega(D)$ will denote the set of all ω -SE functions. The next theorem can be considered an abstract version of the Paley-Wiener Theorem.

Theorem 1.

The following conditions are equivalent:

a) a vector f belongs to $E_\omega(D)$;

b) a vector satisfies the Bernstein inequality

$$\|D^k f\| \leq \omega^k \|f\|$$

for every natural k;

c) for any $h \in E$ the complex valued function of one variable $t \in (-\infty, \infty)$

$$F(t) = \left\langle e^{itD} f, h \right\rangle = \int_M e^{itD} f \bar{h} d\mu$$

is an entire function of exponential type ω which is bounded on the real line, i.e., it has analytic extension to the complex plane C and there exists a constant $a = a(h)$ such that

$$|F(z)| \leq a e^{\omega |z|}, z \in C ;$$

d) the abstract function $e^{itD} f$ has a continuation to the plane as an entire function and there exists a constant b such that

$$\|e^{izD} f\| \leq b e^{\omega |z|}, z \in C .$$

In the case of R^1, this definition being applied to $\frac{d^2}{dt^2}$ gives standard band-limited functions. In the case of $R^n, n > 1$ and Laplace operator Δ it gives what is known as entire functions of spherical exponential type that belong to $L_2(R^n)$.

In the situation of a stratified group G we use sub-Laplacian D in the space $L_2(G)$. It is a self-adjoint positive definite operator. We apply the above construction to the operator D and it gives us the notion of a SE-function on the group G .

We will also assume that Lie algebra has rational structure constants. This assumption implies existence of a discrete co-compact subgroup Γ which is invariant under dilations. We will use notation $\Gamma_j = \delta_{2^j} \Gamma$. Let $B(x, r)$ be a ball in homogeneous metric $\rho(g, h) = |X - Y|, g = expX, h = expY$ with center $x \in G$ and radius r. Here exp is the exponential map from Lie algebra onto a corresponding group. Suppose that $\left\{B(x_\gamma, r)\right\}_{x_\gamma \in \Gamma}$ is a cover of G. It is clear that this cover has a finite multiplicity M in the sense that every ball from this family has non-empty intersections with no more than M other balls from the same family. Since metric $\rho(x, y)$ is homogeneous, the family of balls $\{B(x_\gamma, 2^j r)\}_{x_\gamma \in \Gamma_j}$ will also be a cover of G of the same multiplicity M.

Given a subgroup Γ_j and a sequence $\{s_\gamma\} \in l_2$ we will be interested to find a function $s_{k,j} \in S^{2k}(G), k > Q/4$ such that

a) $s_{k,j}(x_\gamma) = s_\gamma, x_\gamma \in \Gamma_j$;

b) function $s_{k,j}$ minimizes functional $u \to \|D^k u\|$.

The same problem for functional $u \to \|u\|_{S^{2k}(G)}, u \in S^{2k}(G), k > Q/4$ can be solved easily.

Pick a ball $B(0, r)$ of very small radius r and then by translations construct a family of pair ways disjoint balls $B(x_\gamma, r), x_\gamma \in \Gamma_j$. In the ball $B(0, r)$ we consider any function $\varphi_0 \in C_0^\infty(B(0, r))$ such that $\varphi_0(0) = 1$. Using translations we construct similar functions φ_γ in balls $B(x_\gamma, r)$. Because of invariance all these functions have the same Sobolev norm

$$\left\|\varphi_\gamma\right\|_{S^k(G)} = \left\|\varphi_\gamma\right\| + \sum_{1 \leq i_1 \leq \ldots \leq i_k \leq m} \left\|X_{i_1} X_{i_2} \ldots X_{i_k} \varphi_\gamma\right\| .$$

It is clear that for any sequence $\{s_\gamma\} \in l_2$ the formula

$$f = \sum s_\gamma \varphi_\gamma$$

defines a function from $S^k(G)$. Let Pf denote the orthogonal projection of this function f (in the Hilbert space $S^{2k}(G)$ with natural inner product) on the subspace $U^{2k}(\Gamma_j) = \{f \in S^{2k}(G) | f(x_\gamma) = 0\}$ with $S^{2k}(G)$-norm. Then the function $g = f - Pf$ will be a unique solution of the above minimization problem for the functional $u \to \|u\|_{S^{2k}(G)}, k > Q/4$.

The problem with functional $u \to \|D^k u\|$ is that it is not a norm. But fortunately we are able to show that for all natural $k > Q/4$ and all integer j the norm

$$\left\|D^k f\right\| + \left(\sum_{x_\gamma \in \Gamma_j} |f(x_\gamma)|^2\right)^{1/2}$$

is equivalent to the norm $\|f\|_{S^{2k}(G)}$. So, the above procedure can still be applied to the Hilbert space $S^{2k}(G)$ with the inner product

$$< f, g >= \sum_{x_\gamma \in \Gamma_j} f\left(x_\gamma\right) g\left(x_\gamma\right) + \left\langle D^{k/2} f, D^{k/2} g\right\rangle$$

and it clearly proves existence and uniqueness of the solution of our minimization problem for the functional $u \to \|D^k u\|, k > Q/4$.

The proofs of all main results in the present article are based on the following inequalities.

Theorem 2.

There exist a $j_0 \in Z$ and a constant $C_0 \geq 0$ such that for $j \leq j_0$ and every $f \in S^{2k}(G), k = 2^l Q, l = 1, 2, \dots,$ the following inequality takes place:

$$\|f\| \leq 2^l C_0 \left(\sum_{x_\gamma \in \Gamma_j} \left|f\left(x_\gamma\right)\right|^2\right)^{1/2} + \left(C_0 2^{j/2Q}\right)^k \left\|D^k f\right\| .$$

In particular for $f \in U^k(\Gamma_j)$

$$\|f\| \leq \left(C_0 2^{j/2Q}\right)^k \left\|D^k f\right\| .$$

For the given $f \in S^{2k}(G), k > Q$, the $s_{k,j}(f) \in S^{2k}(G)$ will be the function that minimizes $u \to \|D^k u\|$ and takes the same values on Γ_j, i.e., $s_{k,j}(f)|_{\Gamma_j} = f_{\Gamma_j}$. Since D is invariant with respect to translations it is clear that $s_{k,j}(f) = \sum_{x_\gamma \in \Gamma_j} f(x_\gamma) L_{k,j}(x x_\gamma^{-1})$ where $L_{k,j}(x) \in S^{2k}(G)$ is the function that minimizes the same functional and $L_{k,j}(0) = 1$, and is zero at all other points of Γ_j. In the classical case, such functions are called Lagrangian splines.

We prove the following approximation theorem.

Theorem 3.

There exists $c_0 > 0$ such that for $j \leq j_0$ the following estimate takes place

$$\left\|f - s_{k,j}(f)\right\| \leq \left(c_0 2^{j/2Q}\right)^k \left\|D^k f\right\|, f \in S^{2k}(G), k = 2^l Q, l = 1, 2, \dots .$$

Proof. If $f \in S^{2k}(G), k = 2^l Q$, then $f - s_{k,j}(f) \in U^{2k}(\Gamma_j)$ and according to Theorem 2 we have

$$\left\|f - s_{k,j}(f)\right\| \leq \left(C_0 2^{j/2Q}\right)^k \left\|D^k \left(f - s_{k,j}(f)\right)\right\| .$$

Using minimization property of $s_{k,j}(f)$ we obtain

$$\left\|f - s_{k,j}(f)\right\| \leq \left(c_0 2^{j/2Q}\right)^k \left\|D^k f\right\|, k = 2^l Q ,$$

where $c_0 = 2C_0$ and the constant C_0 is from Theorem 2. □

Using Theorem 3 and the Bernstein inequality from Theorem 1 we immediately come to the following uniqueness and reconstruction theorem.

Theorem 4.

For the same constant $c_0 > 0$ as above

a) every ω-SE function $f \in E_\omega(D), \omega > 0$ is uniquely determined by its values on any set $\Gamma_j = \delta_{2^j}\Gamma$ as long as $j < -2Q\log_2(c_0\omega)$;

b) for every such Γ_j the sequence of splines $s_{k,j}(f)(x) = \sum_{x_\gamma\in\Gamma_j} f(x_\gamma)L_{k,j}(xx_\gamma^{-1}), k = 2^l Q, l = 1, 2, \ldots,$ converges to $f \in E_\omega(D)$ in $L^2(G)$-norm.

The rest of this article is devoted to the proof of Theorems 1 and 2.

2. Proof of Theorem 1

The goal of this section is to prove Theorem 1. The following lemma is evident.

Lemma 1.

a) The set $\bigcup_{\omega>0} E_\omega(D)$ is dense in E;

b) the $E_\omega(D)$ is a linear closed subspace in E.

We now prove that conditions a) and b) from Theorem 1 are equivalent.

Let f belong to the space $E_\omega(D)$ and $Ff = x \in X$. Then

$$\left(\int_0^\infty \lambda^{2k}\|x(\lambda)\|^2_{X(\lambda)}dm(\lambda)\right)^{1/2} = \left(\int_0^\omega \lambda^{2k}\|x(\lambda)\|^2_{X(\lambda)}dm(\lambda)\right)^{1/2} \le \omega^k\|x\|_X, k \in N\,,$$

which gives Bernstein inequality for f.

Conversely, if f satisfies Bernstein inequality, then $x = Ff$ satisfies $\|x\|_{X^k} \le \omega^k\|x\|_X$. Suppose that there exists a set $\sigma \subset [0, \infty] \setminus [0, \omega]$ whose m-measure is not zero and $x|_\sigma \neq 0$. We can assume that $\sigma \subset [\omega + \epsilon, \infty)$ for some $\epsilon > 0$. Then for any $k \in N$ we have

$$\int_\sigma \|x(\lambda)\|^2_{X(\lambda)}dm(\lambda) \le \int_{\omega+\epsilon}^\infty \lambda^{-2k}\left\|\lambda^k x(\lambda)\right\|^2_{X(\lambda)} d\mu \le \|x\|^2_X\,(\omega/\omega+\epsilon)^{2k}$$

which shows that or $x(\lambda)$ is zero on σ or σ has measure zero.

The implications $b) \to d) \to c)$ in Theorem 1 are evident. Therefore, it is enough to show the implication $c) \to b)$ which is a consequence of the following lemma.

Lemma 2.

Let D be a self-adjoint operator in a Hilbert space E. If for some $f \in E$ there exists an $\omega > 0$ such that the quantity

$$\sup_{k\in N}\left\|D^k f\right\|\omega^{-k} = R(f)$$

is finite, then $R(f) \le \|f\|$.

Proof. By the assumption $\|D^r f\| \le R(f)\omega^r, r \in N$. Now for any complex number z we have

$$\left\|e^{izD}g\right\| = \left\|\sum_0^\infty\left(i^r z^r D^r g\right)/r!\right\| \le R(f)\sum_0^\infty |z|^r\omega^r/r! = R(f)e^{|z|\omega}\,.$$

It implies that for any $h \in E$ the scalar function $(e^{izD}f, h)$ is an entire function of exponential type ω which is bounded on the real axis R^1 by the constant $\|h\|\|f\|$. An application of the Bernstein inequality gives

$$\left\| \left(e^{itD}D^k f, h\right) \right\|_{C(R^1)} = \left\| \left(\frac{d}{dt}\right)^k \left(e^{itD}f, h\right) \right\|_{C(R^1)} \le \omega^k \|h\|\|f\| .$$

The last one gives for $t = 0$

$$\left|\left(D^k f, h\right)\right| \le \omega^k \|h\|\|f\| .$$

Choosing h such that $\|h\| = 1$ and $(D^k f, h) = \|D^k f\|$ we obtain the inequality $\|D^k f\| \le \omega^k \|f\|, k \in N$, which gives

$$R(f) = \sup_{k\in N} \left(\omega^{-k} \left\|D^k f\right\|\right) \le \|f\| .$$

Lemma 2 is proved. □

3. Proof of Theorem 2

Lemma 3.
If A is a self-adjoint operator in a Hilbert space and for some element f

$$\|f\| \le b + a\|Af\|, a > 0 ,$$

then for all $m = 2^l, l = 0, 1, 2, \ldots$

$$\|f\| \le mb + 8^{m-1}a^m \left\|A^m f\right\|$$

as long as f belongs to the domain of A^m.

Proof. For any self-adjoint operator B in a Hilbert space we have

$$\|f\| \le \|(I + \varepsilon i B)f\|$$

and the same for the operator $(I - \varepsilon i B)$. It gives

$$\varepsilon\|Bf\| \le \|(I - \varepsilon i B)f\| + \|f\| \le \left\|\left(I + \varepsilon^2 B^2\right) f\right\| + \|f\| \le \varepsilon^2 \left\|B^2 f\right\| + 2\|f\| .$$

So, for any f from the domain of B^2 we have the inequality

$$\|Bf\| \le \varepsilon \left\|B^2 f\right\| + 2/\varepsilon\|f\|, \varepsilon > 0 .$$

Our lemma is true for $m = 1$. If it is true for m, then applying the last inequality for $B = A^m$ we obtain

$$\|f\| \le mb + 8^{m-1}a^m \left(\varepsilon \left\|A^{2m} f\right\| + 2/\varepsilon\|f\|\right) .$$

Setting $\varepsilon = 8^{m-1}(a)^m 2^2$, we obtain

$$\|f\| \le 2mb + 8^{2m-1}(a)^{2m} \left\|A^{2m} f\right\| .$$

Lemma 3 is proved. □

We consider Sobolev spaces $S^{\sigma}(G)$ with the norm $\|f\|_{S^{\sigma}(G)} = \|f\| + \|D^{\sigma/2}f\|, \sigma > 0$ and for any open Ω in G we define the space $S^{\sigma}(\Omega)$ as the collection of all restrictions $g_{\Omega} = g|_{\Omega}, g \in S^{\sigma}(G)$ with the norm $\|g_{\Omega}\|_{S^{\sigma}(\Omega)} = \inf \|g\|_{S^{\sigma}(G)}$ where g runs over the set of all functions from $S^{\sigma}(G)$ whose restriction to Ω gives g_{Ω}. Let $B(\lambda, M) = \{B(x_\gamma, \lambda)\}$ be a cover of G of finite multiplicity M. We introduce a map

$$\begin{aligned} T_{B(\lambda,M)} &: S^{\sigma}(G) \to l_2\left(S^{\sigma}\left(B_\gamma\right)\right), \sigma \geq 0, \\ T_{B(\lambda,M)}(g) &= \left\{g_\gamma\right\}, g_\gamma = g|_{B(x_\gamma,\lambda)} \end{aligned}$$

where the Hilbert space on the right is defined as the set of all sequences $\{g_\gamma\}, g_\gamma \in S^{\sigma}(B(x_\gamma, \lambda))$ for which $(\sum_\gamma \|g_\gamma\|^2_{S^{\sigma}(B(x_\gamma,\lambda))})^{1/2} < \infty$.

Lemma 4.

For any natural M and any $\sigma \geq 0$ there exists a $C = C(M, \sigma)$ such that for every cover $B(\lambda, M), \lambda > 0$,

$$\left\|T_{B(\lambda,M)}\right\| \leq C(M, \sigma) \max\left(1, \lambda^{-\sigma}\right).$$

Proof. Let $\theta \in C_0^{\infty}(R), \theta(t) = 1, |t| \leq 1, supp\, \theta \subset [-2, 2]$. We define $\theta_{\lambda,\gamma}(x) = \theta(\rho(0, \delta_{\lambda^{-1}}(xx_\gamma^{-1})), x \in G, \lambda > 0$. Then $\theta_\lambda \in C_0^{\infty}(G), \theta_\lambda(x) = 1, x \in B(x_\gamma, \lambda), supp\, \theta_\lambda \subset B(x_\gamma, 2\lambda)$. It is clear that $|X_{i_1} \ldots X_{i_j}\theta_\lambda(x)| \leq C(j, \theta)\lambda^{-j}$. Therefore, if $f \in S^k(G), k \geq 0$ is an integer, then

$$\begin{aligned} \left\|f|_{B(x_\gamma,\lambda)}\right\|^2_{S^k(B(x_\gamma,\lambda))} &\leq \|f\theta_\lambda\|^2_{S^k(G)} \\ &\leq \sum_{|j|\leq k} \int_{B(x_\gamma,2\lambda)} \left|X_{i_1}...X_{i_j}\left(f\theta_\lambda\right)(x)\right|^2 d\mu \\ &\leq C(k, \theta) \max\left(1, \lambda^{-2k}\right) \sum_{|j|\leq k} \int_{B(x_\gamma,2\lambda)} \left|X_{i_1}...X_{i_j}f(x)\right|^2 d\mu \end{aligned}$$

and then

$$\begin{aligned} \sum_\gamma \left\|f|_{B(x_\gamma,\lambda)}\right\|^2_{S^k(B(x_\gamma,\lambda))} &\leq C(k, \theta) \max\left(1, \lambda^{-2k}\right) \sum_\gamma \sum_{|j|\leq k} \int_{B(x_\gamma,2\lambda)} \left|X_{i_1}...X_{i_j}f(x)\right|^2 d\mu \\ &\leq C(k, M, \theta) \max\left(1, \lambda^{-2k}\right) \|f\|^2_{S^k(G)}. \end{aligned}$$

Thus, for natural $s = k$ the lemma is proved. A general case can be obtained by interpolation since for the complex interpolation functor $[.,.]_\theta$

$$\left[l_2\left(L_2\left(B_\gamma\right)\right), l_2\left(S^{\sigma}\left(B_\gamma\right)\right)\right]_\theta = l_2\left(S^{\theta\sigma}\left(B_\gamma\right)\right), 0 < \theta < 1. \qquad \square$$

We now prove Theorem 2.

Proof of Theorem 2. Let $\{B(x_\gamma, 1)\}_{x_\gamma\in\Gamma}$ be a cover of G of the finite multiplicity M. The cover $\{B(x_\gamma, 2^j)\}_{x_\gamma\in\Gamma_j}, \Gamma_j = \delta_{2^j}\Gamma$ also has the same multiplicity M. Let $\psi_\gamma, supp\psi_\gamma \subset B_\gamma$ be a corresponding partition of unity.

For a function f from $S^{\sigma}(S), \sigma > Q/2$ we consider decomposition

$$f(x) = \sum_\gamma f(x)\psi_\gamma(x) = \sum_\gamma f\left(x_\gamma\right)\psi_\gamma(x) + \sum_\gamma \left(f(x) - f\left(x_\gamma\right)\right)\psi_\gamma(x)$$

and then

$$\|f\|^2 \le C\left\{\sum_{\gamma}\left|f\left(x_\gamma\right)\right|^2 + \sum_{\gamma}\int_{B(x_\gamma,\lambda)}\left|f(x) - f\left(x_\gamma\right)\right|^2 d\mu\right\},$$

where C depends only on multiplicity M.

Since every vector field on the group G is a linear combination over C^∞ of the fields $[X_{i_1}, \ldots [X_{i_{k-1}}, X_{i_k}]\ldots]$, $1 \le k \le n$, $1 \le i_j \le m$, the Newton-Leibnitz formula gives

$$\left|f(x) - f\left(x_\gamma\right)\right|^2 \le$$
$$C4^j \sum_{k=1}^{n} \sum_{1\le i_1, i_2, \ldots i_k \le m}\left(\sup_{y\in B(x_\gamma, 2^j)}\left|X_{i_1}X_{i_2}\ldots X_{i_k}f(y)\right|\right)^2 .$$

' Applying anisotropic version of the Sobolev inequality [3] we obtain

$$\left|f(x) - f\left(x_\gamma\right)\right|^2 \le$$
$$C4^j \sum_{k=1}^{n} \sum_{1\le i_1, i_2, \ldots i_k \le m}\left(\sup_{y\in B(x_\gamma, 2^j)}\left|X_{i_1}X_{i_2}\ldots X_{i_k}f(y)\right|\right)^2 \le$$
$$C4^j \sum_{k=1}^{n} \sum_{1\le i_1, i_2, \ldots i_k \le m}\left\|X_{i_1}X_{i_2}\ldots X_{i_k}f\right\|^2_{S^{Q/2+\varepsilon}(B(x_\gamma,2^j))},$$

where $x \in B(x_\gamma, 2^j)$, $\varepsilon > 0$, $C = C(X_i, \ldots, X_m; \varepsilon)$.

An application of Lemma 4 gives

$$\sum_{\gamma}\int_{B(x_\gamma,2^j)}\left|f(x) - f\left(x_\gamma\right)\right|^2 d\mu \le$$
$$C\left(2^j\right)^{Q+2}\sum_{k=1}^{n}\sum_{1\le i_1, i_2, \ldots i_k\le m}\sum_{\gamma}\left\|X_{i_1}\ldots X_{i_k}f\right\|^2_{S^{Q/2+\varepsilon}(B(x_\gamma,2^j))} \le$$
$$C\left(2^j\right)^{2-2\varepsilon}\sum_{k=1}^{n}\sum_{1\le i_1,\ldots,i_k\le m}\left\|X_{i_1}\ldots X_{i_k}f\right\|^2_{S^{Q/2+\varepsilon}(G)} \le$$
$$C\left(2^j\right)^{2-2\varepsilon}\sum_{k=1}^{n}\sum_{1\le i_1,\ldots,i_k\le m}\left\|X_{i_1}\ldots X_{i_k}f\right\|^2_{S^{\sigma}(G)},$$

where $\sigma \ge Q/2 + \varepsilon$, C depends only on $X_i, \ldots, X_m$ on σ and on multiplicity M. Since

$$\left\|X_{i_1}X_{i_2}\ldots X_{i_k}f\right\| \le C\left\{\|f\| + \left\|D^{k/2+\sigma/2}f\right\|\right\},$$

we have for particular choice of $\varepsilon = 1/2$, $\sigma = 2Q - n$, $d \ge 2$

$$\|f\| \le C\left\{\left(\sum_{\gamma}\left|f\left(x_\gamma\right)\right|^2\right)^{1/2} + 2^{j/2}\left\|D^Q f\right\| + 2^{j/2}\|f\|\right\},$$

where C depends only on $X_1, \ldots X_m$ and multiplicity M. Thus, if j is smaller than some $j_0 = j_0(X_1, \ldots, X_m; M)$ it gives

$$\|f\| \le C\left\{\left(\sum_{\gamma}\left|f\left(x_\gamma\right)\right|^2\right)^{1/2} + C2^{j/2}\left\|D^Q f\right\|\right\}, C = C\left(X_1, \ldots, X_m; M\right).$$

Using Lemma 3 for $A = D^Q, b = \left(\sum_\gamma |f(x_\gamma)|^2\right)^{1/2}$ we obtain

$$\|f\| \le C2^l \left(\sum_\gamma |f(x_\gamma)|^2\right)^{1/2} + \left(C_0 2^{j/2}\right)^{2^l} \left\|D^{2^l Q} f\right\| ,$$

where $l = 0, 1, 2, \ldots, C_0 = 8C$. After all, for $f \in U^{2^l Q}(\Gamma_j), j < j_0$

$$\|f\| \le \left(C_0 2^{j/2}\right)^{2^l} \left\|D^{2^l Q} f\right\|, l = 0, 1, 2, \ldots .$$

Theorem 2 is proved. □

Lemma 5.

For any natural $k > Q/4$ and any $\Gamma_j = \delta_{2^j}\Gamma, j \in Z$, the norm $\|f\|_{S^{2k}(G)}$ is equivalent to the norm

$$\left\|D^k f\right\| + \left(\sum_{x_\gamma \in \Gamma} |f(x_\gamma)|^2\right)^{1/2} .$$

Proof. The proof of Theorem 2 shows that for every natural $k > Q/4$ there exists a $j(k)$ such that for every $j \le j(k)$ there is a $C = C(k, j)$ for which

$$\|f\| \le C \left\{ \left\|D^k f\right\| + \left(\sum_{x_\gamma \in \Gamma_j} |f(x_\gamma)|^2\right)^{1/2} \right\} .$$

Now using homogeneity arguments one can easily show that for every natural $k > Q/4$ and every integer j there exists a $C = C(k, j)$ for which the above inequality takes place. In order to prove inverse inequality we consider $C_0^\infty(G)$ functions ϕ_γ with disjoint supports such that $\phi_\gamma(x_\gamma) = 1$. Using Sobolev embedding theorem we obtain for $k > Q/4$

$$\left(\sum_\gamma |f(x_\gamma)|^2\right)^{1/2} \le C_k \left(\sum_\gamma \|f\phi_\gamma\|^2_{S^{2k}(G)}\right)^{1/2} \le C_k \|f\|_{S^{2k}(G)}, k > Q/4 .$$

The proof of Lemma 5 is finished. □

In the introduction we explained how one can use these results to construct splines $s_{k,j}(f)$ and to prove uniqueness theorem and reconstruction formula.

4. Some Properties of the Functions $s_{k,j}$

We will show now that functions $s_{k,j}$ have the following remarkable property (see [5, 7]).

$$D^{2k} s_{k,j} = \sum_{x_\gamma \in \Gamma_j} \alpha_\gamma \delta(x_\gamma) ,$$

where $\delta(x)$ is the Dirac measure and $\{\alpha_\gamma\} \in l_2$.

Indeed, suppose that $s_{k,j} \in S^{2k}(G)$ is a solution to the minimization problem and $h \in U^{2k}(\Gamma_j)$. Then

$$\left\|D^k \left(s_{k,j} + \lambda h\right)\right\|^2 = \left\|D^k s_{k,j}\right\|_2^2 + 2Re\lambda \int_G D^k s_{k,j} D^k h d\mu + |\lambda|^2 \left\|D^k h\right\|_2^2 .$$

The function $s_{k,j}$ can be a minimizer only if for any $h \in U^{2k}(\Gamma_j)$

$$\int_M D^k s_{k,j} D^k h d\mu = 0 .$$

So, the function $\Phi = D^k s_{k,j} \in L_2(G)$ is orthogonal to $D^k U^{2k}(\Gamma_j)$. Let φ_γ be the same set of functions as above and $h \in C_0^\infty(G)$. Then the function $h - \sum h(x_\gamma)\varphi_\gamma$ belongs to the $U^{2k}(\Gamma_j) \cap C_0^\infty(G)$. Thus,

$$0 = \int_G \Phi \overline{D^k \left(h - \sum h_\gamma \varphi_\gamma\right)} d\mu = \int_G \Phi \overline{D^k h} d\mu - \sum \overline{h\left(x_\gamma\right)} \int_G \Phi \overline{D^k \varphi_\gamma} d\mu .$$

In other words

$$D^k \Phi = \sum_{x_\gamma \in \Gamma_j} \alpha_\gamma \delta\left(x_\gamma\right) ,$$

or

$$D^{2k} s_{k,j} = \sum_{x_\gamma \in \Gamma_j} \alpha_\gamma \delta\left(x_\gamma\right) ,$$

where $\delta(x)$ is the Dirac measure.

Moreover, for any integer $r > 0$

$$\sum_{\gamma=1}^{r} |\alpha_\gamma|^2 = \left\langle \sum_1^\infty \alpha_\gamma \delta\left(x_\gamma\right), \sum_1^r \alpha_\gamma \phi_\gamma \right\rangle \le C \left\| \sum_1^\infty \alpha_\gamma \delta\left(x_\gamma\right) \right\|_{S^{-2k}(G)} \left(\sum_1^r |\alpha_\gamma|^2 \right)^{1/2} ,$$

where C is independent on r. It shows that the sequence $\{\alpha_\gamma\}$ belongs to l_2.

It also can be shown that on the space of splines the norms $L_2(G)$, $S^{2k}(G)$ and $(\sum |f(x_\gamma)|^2)^{1/2}$ are equivalent.

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Department of Mathematics 038-16, Temple University, Philadelphia, PA 19122
e-mail: pesenson@math.temple.edu