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Applications of Generalized Perron Trees to Maximal Functions and Density Bases

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ABSTRACT. In this article we give some new necessary conditions for subsets of the unit circle to give collections of rectangles (by means of orientations) which differentiate L^p-functions or give Hardy–Littlewood type maximal functions which are bounded on L^p, $p > 1$. This is done by proving that a well-known method, the construction of a Perron Tree, can be applied to a larger collection of subsets of the unit circle than was earlier known. As applications, we prove a partial converse of a well-known result of Nagel et al. [6] regarding boundedness of maximal functions with respect to rectangles of lacunary directions, and prove a result regarding the cardinality of subsets of arithmetic progressions in sets of the type described above.

1. Introduction

An important problem in harmonic analysis is the question of the differentiability of integrals in $\mathbb{R}^2$, or more generally $\mathbb{R}^n$. One formulation of this problem is the following: Consider a set $A \subset \mathbb{T}$, the unit circle in $\mathbb{R}^2$, and view A as a selection of directions. If A_x is the collection of all rectangles in $\mathbb{R}^2$ containing x and oriented in one of the directions in A, is it true that for "all" f

$$\lim_{\substack{diam R \to 0 \\ R \in A_x}} \frac{1}{|R|} \int_R f(y)\, dy = f(x) \text{ a.e. ?}$$

If this is true for some class of functions, we say that A differentiates that class of functions. Closely related to this is the problem of the L^p-boundedness of the corresponding Hardy–Littlewood maximal operator M_A, defined by

$$M_A(f) = \sup_{R \in A_x} \frac{1}{|R|} \int_R |f(y)|\, dy .$$

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If A differentiates characteristic functions, then A is called a density basis. When M_A is of strong type (p, p) for some $1 < p < \infty$, then A is said to be a Max(p) set (or that A has the Max(p) property). (The cases $p = 1$ or $p = \infty$ are not of much interest since M_A is never of strong type $(1, 1)$ [9, X.2.3] and obviously always of type ∞.)

Not all sets are Max(p) sets or density bases. Indeed, any set that is dense in a subset of $\mathbb{T}$ of positive Lebesgue measure is neither. (c.f. [3, p. 228] or Fefferman's proof that the characteristic function of the ball is not an L^p multiplier [2].) Also, in a recent article Katz showed that the classic Cantor set is not a Max(p) set for $p \leq 2$ [5]. In contrast, Nagel et al. [6], improving upon earlier results of Strömberg [11] and Cordoba and Fefferman [1], have shown that if A is a lacunary set of directions, then A has both properties. As a partial converse to this we use the Perron tree construction to show that if the set of directions $A = \{\theta_j\}$ is either a Max(p) set or a density basis, and satisfies some regularity conditions (for instance, if the sequence $\{\frac{1}{\theta_j} - \frac{1}{\theta_{j-1}}\}$ is increasing), then the set A must be lacunary. Our work improves upon a related result found independently by Stokolos [10].

However, it is not necessary for a set to be lacunary in order to be a Max(p) set as Sjögren and Sjölin [8] have shown. In the second part of the article we refine the Perron tree construction and use it to find necessary conditions for general sets of directions to have either of the two properties. As an application we find bounds for the cardinality of the intersection of a Max(p) set or density basis with an arithmetic progression.

2. Lacunary Sequences

Let $A = \{\theta_j\}_{j=1}^{2^k} \subset [0, \pi/2)$ be a decreasing sequence and consider the collection of triangles in Figure 1.

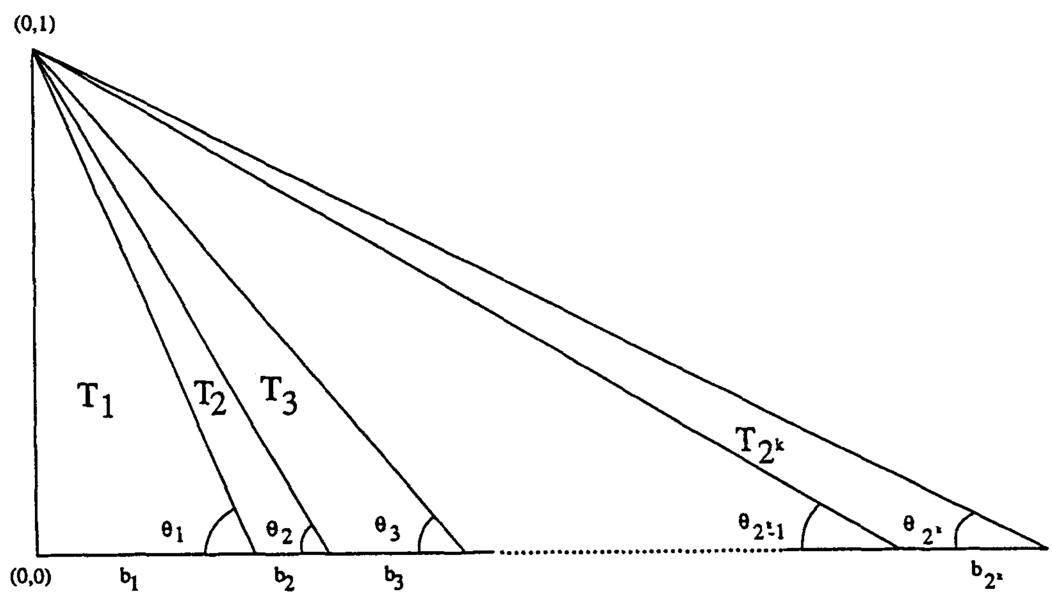


FIGURE 1. Set of triangles.

If from the set of triangles we can construct a Perron tree $E = \bigcup_{i=1}^{2^k} \tilde{T}_i$ such as Rademacher does for the special case $b_1 = b_2 = \ldots = b_{2^k} = 1$ (see [7], [9, X.1], or [3, 8.1]), with $\tilde{T}_i$ a translate of T_i and $m(\bigcup_{i=1}^{2^k} \tilde{T}_i) < \epsilon m(\bigcup_{i=1}^{2^k} T_i)$, then, as is reasoned on [3, p. 224–226], E has the property that

$$m\left\{x : M_A(\chi_E)(x) > \frac{1}{100}\right\} \geq \frac{m(E)}{\epsilon}. \tag{2.1}$$

If M_A is of strong type (p, p) for some $p < \infty$ (i.e., A is a Max(p) set), then (2.1) cannot hold for ϵ arbitrarily small (depending only on the operator norm of M_A). If A is a density basis, the Busemann–Feller criterion (see [3, 6.4.3]) again implies that (2.1) cannot hold for ϵ sufficiently small (depending on A). Thus, if such a construction exists for all $\epsilon > 0$, the set A cannot be Max(p) for any $p < \infty$ and cannot be a density basis.

We turn now to the construction of the (generalized) Perron tree (P.T.), taking into consideration the variation of b_j. We shall see (in this section and the next) that under certain restrictions on the variations the P.T. construction will still go through.

We begin with two adjacent triangles as shown in Figure 2.

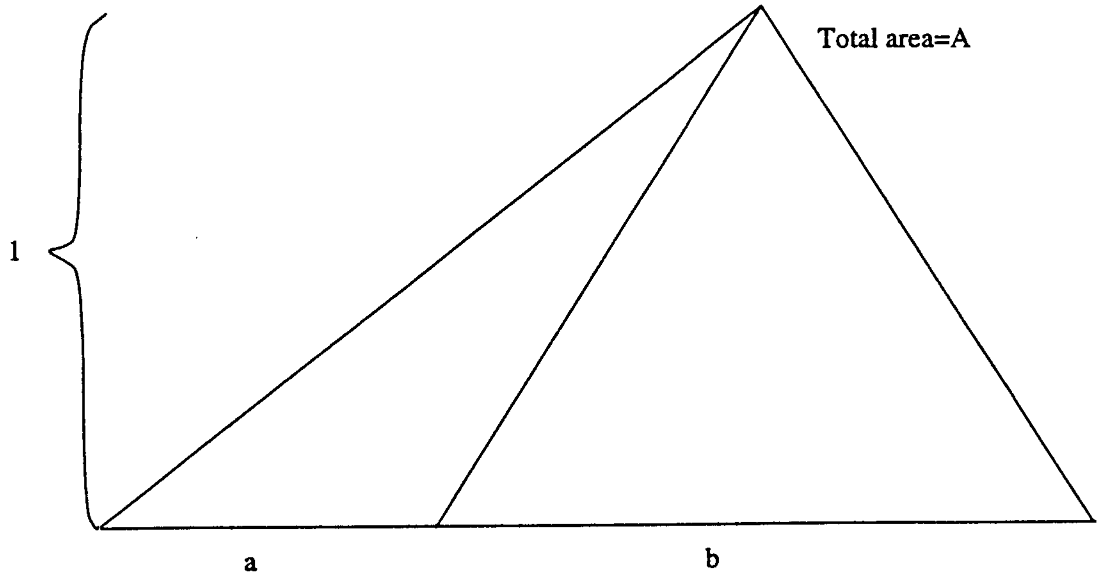


FIGURE 2. Two adjacent triangles.

This situation is transformed by the basic operations in the P.T. construction, described in [3, p. 202], to the situation in Figure 3.

Simple geometry gives that triangles I and III are similar, as are triangles II and IV, and the triangles have areas:

$$\begin{aligned} |I| &= (1-\alpha)^2 \frac{b}{a+b} A, \quad |II| = (1-\alpha)^2 \frac{a}{a+b} A \\ |III| &= (1-\alpha)^2 \frac{a^2}{b^2} \frac{b}{a+b} A, \quad |IV| = (1-\alpha)^2 \frac{b^2}{a^2} \frac{a}{a+b} A. \end{aligned}$$

Thus, the excess triangles have area totalling

$$|I| + |II| + |III| + |IV| = (1-\alpha)^2 A \left(1 + \frac{a^2 - ab + b^2}{ab}\right) = (1-\alpha)^2 A \left(\frac{a}{b} + \frac{b}{a}\right).$$

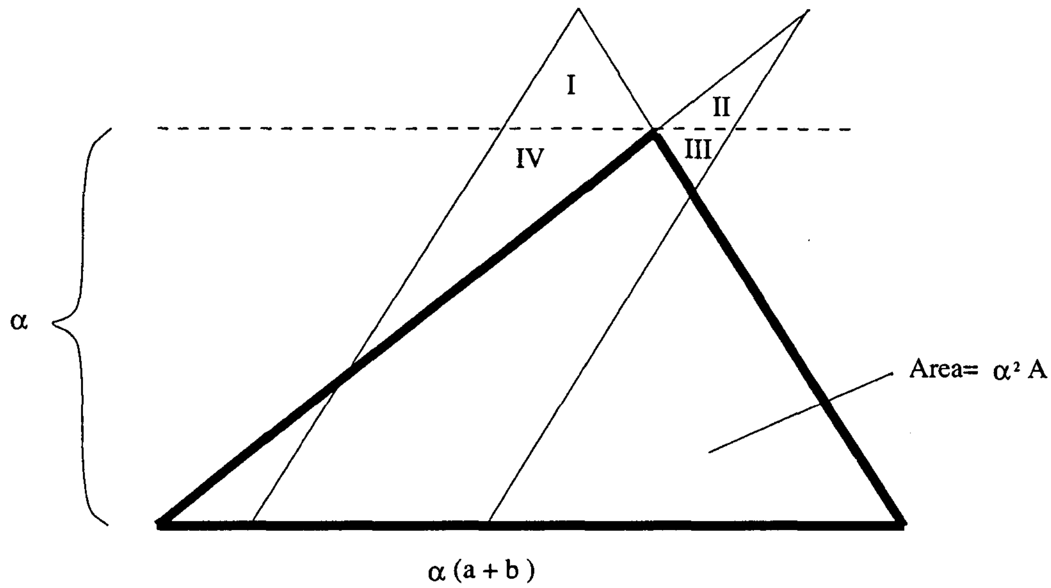


FIGURE 3. Situation obtained using the basic operations in the P.T. construction.

The factor $(1-\alpha)^2$ is the same as in the classical situation. The prize we pay for having bases with variable length is the factor $(\frac{a}{b}+\frac{b}{a})$ (which in the classical case always reduces to its minimum value 2).

Continuing these transformations as in the classical construction, we easily see that what is needed, in order to go through with the P.T. construction, is control of this factor.

It is clear that if the factor $(\frac{a}{b}+\frac{b}{a})$ is uniformly bounded in all the steps of the P.T. construction, with the bound, say c, independent of the number of steps, then after k steps we will have constructed $\tilde{T}_i$, $i=1,\ldots,2^k$ satisfying

$$m\left(\bigcup_{i=1}^{2^k}\tilde{T}_i\right) \le \left(\alpha^{2k}+c(1-\alpha)\right) m\left(\bigcup_{i=1}^{2^k} T_i\right).$$

By choosing α and k suitably we can arrange for

$$m\left(\bigcup_{i=1}^{2^k}\tilde{T}_i\right) \le \epsilon m\left(\bigcup_{i=1}^{2^k} T_i\right),$$

for any specified $\epsilon > 0$.

The case when the factor $(\frac{a}{b}+\frac{b}{a})$ is not uniformly bounded needs more care, and will be discussed in the next section. The bounded case suffices for a partial converse to [6] when the directions $\{\theta_j\}_1^\infty$ have certain regularity properties.

Recall that a decreasing sequence $\{\theta_j\}$ of positive numbers is called lacunary if $\inf \frac{\theta_j}{\theta_{j+1}} \ge q > 1$.

Theorem 1.

Let $\{\theta_j\}_1^\infty \subset [0,\frac{\pi}{2})$ be a decreasing sequence and suppose there is some $0 < c \le 1$ such that

$$\cot\theta_j - \cot\theta_{j-1} \ge c \max_{1\le i\le j}(\cot\theta_i - \cot\theta_{i-1}) \tag{2.2}$$

(where we set $\theta_0 = \frac{\pi}{2}$). If $\{\theta_j\}$ is a Max(p) set for some $p < \infty$ or a density basis, then $\{\theta_j\}$ is lacunary.

Proof. This proof is an adaption to our situation of the proof of Theorem 3.5 in [4]. Suppose $\{\theta_j\}$ is a Max(p) set for some $p < \infty$ or a density basis. As discussed at the beginning of the section, there exists some $\epsilon > 0$ so that there is no P.T. construction using directions from $\{\theta_j\}$ satisfying (2.1).

Pick an $\alpha < 1$ satisfying $6(1-\alpha) < \frac{\epsilon}{2}$ and take k so that $\alpha^{2k} < \frac{\epsilon}{2}$. We will show that if $\{\theta_j\}$ is not lacunary, then there are directions $\{\theta_{i_1}, \dots, \theta_{i_{2k}}\}$ with all factors $(\frac{a}{b} + \frac{b}{a})$ at most 6. Our choice of α and k ensures that $\alpha^{2k} + 6(1-\alpha) < \epsilon$; hence, the corresponding P.T. construction will give a set E satisfying (2.1) which contradicts our assumption.

We begin by choosing M such that $[2^{M-1}c] \geq 2^{k+1}$. (Here $[x]$ denotes the integer part of x.)

First, we see that (2.2) implies

$$\cot\theta_j = \sum_1^j (\cot\theta_i - \cot\theta_{i-1}) \geq cj\cot\theta_1 ;$$

hence, $\theta_j \to 0$ as $j \to \infty$. Thus, we can choose J_0 so that for all $j \geq J_0$,

$$\frac{\theta_{j-1}}{\theta_j} \geq \left(1 - \frac{1}{2^M}\right)\frac{\cot\theta_j}{\cot\theta_{j-1}}$$

because $\theta\cot(\theta) = \theta\frac{\cos\theta}{\sin\theta} \to 1$ as $\theta \to 0$. We claim that $\{\theta_j\}$ is not only lacunary, but in fact

$$\frac{\theta_{j-1}}{\theta_j} \geq 1 + \frac{1}{2^M}, \quad \forall j \geq J_0 ,$$

for suppose not, say $\frac{\theta_{J-1}}{\theta_J} < 1 + \frac{1}{2^M}$ for some $J \geq J_0$. Then

$$\frac{\cot\theta_J - \cot\theta_{J-1}}{\cot\theta_{J-1}} < \left(\frac{1}{1-2^{-M}}\right)\left(1+2^{-M}\right) - 1 = \frac{2}{2^M - 1} .$$

Set $d = \cot\theta_J - \cot\theta_{J-1}$. Then

$$\cot\theta_J = d + \cot\theta_{J-1} > d2^{M-1} ,$$

so

$$\cot\theta_J - i\frac{d}{c} > 0 \text{ if } i \leq \left[2^{M-1}c\right] .$$

For each $i = 1, 2, \dots, [2^{M-1}c] \geq 2^{k+1}$, let K_i be chosen such that

$$\cot\theta_J - i\frac{d}{c} \in \left(\cot\theta_{K_i-1} \, , \, \cot\theta_{K_i}\right] .$$

Notice that the integers K_i are distinct since

$$\begin{aligned}\left(\cot\theta_J - i\frac{d}{c}\right) - \left(\cot\theta_J - (i+1)\frac{d}{c}\right) = \frac{d}{c} &= \frac{\cot\theta_J - \cot\theta_{J-1}}{c} \\ &\geq \cot\theta_j - \cot\theta_{j-1}\end{aligned}$$

for $j \leq J$.

Now construct the P.T. with directions $\{\theta_{K_{2i}}\}_{i=1}^{2^k}$ (see Figure 4). Notice that the base lengths of the corresponding triangles are between d/c and $3d/c$ since there are always exactly two terms from the arithmetic progression,

$$\left\{\cot\theta_J - i\frac{d}{c}, \; i = 1, \dots \left[2^{M-1}c\right]\right\}$$

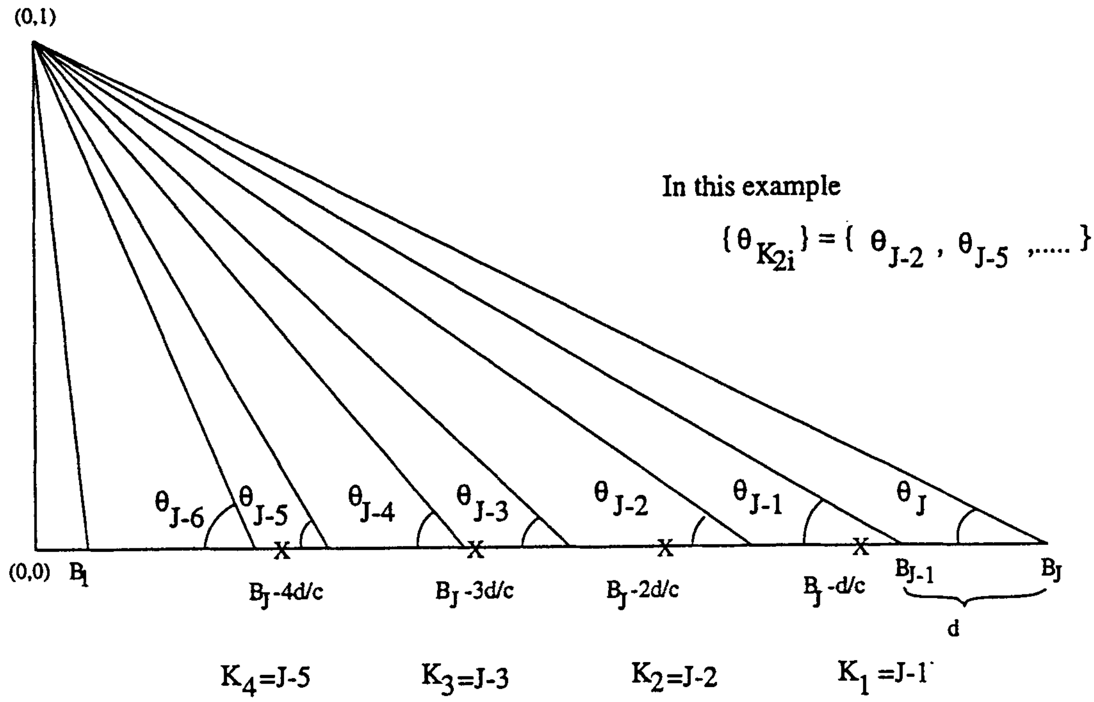


FIGURE 4. P.T. with $\{\theta_{K_{2i}}\}_{i=1}^{2^k}$.

on the base of each of the corresponding triangles. Consequently, each factor $(\frac{a}{b} + \frac{b}{a})$ is at most 6. Since there are 2^k triangles so constructed, we have obtained the desired contradiction.

Thus,

$$\frac{\theta_{j-1}}{\theta_j} \geq \left(1 + \frac{1}{2^M}\right), \ \forall j \geq J_0$$

and this certainly suffices to prove that $\{\theta_j\}$ is lacunary. □

Remark 1.

Theorem 3.5 of [4] could similarly be generalized to the following. □

Proposition 1.

Let $E = \{n_j\}_{j=0}^{\infty} \subset \mathbb{N}$ be an increasing sequence and suppose there is some $c > 0$ so that $n_{j+1} - n_j \geq c \max_{i \leq j} (n_{i+1} - n_i)$ for all j. If the square function

$$S_E(f)(x) \equiv \left(\sum_j \left| \sum_{n \in [n_j, n_{j+1})} \hat{f}(n) e^{inx} \right|^2\right)^{\frac{1}{2}}$$

is of strong type (p, p) for some $p < 2$, then E is a lacunary set.

We next state some corollaries of Theorem 1. In order to prove the first corollary we need an elementary lemma.

Lemma 1.

For $\theta \neq \beta \in [0, \frac{\pi}{2})$

$$\frac{2}{\pi} \leq \frac{\cot(\theta) - \cot(\beta)}{\frac{1}{\theta} - \frac{1}{\beta}} \leq \left(\frac{\pi}{2}\right)^2 .$$

Proof. It is routine to see that

$$\frac{\cot(\theta)-\cot(\beta)}{\frac{1}{\theta}-\frac{1}{\beta}}=\frac{\sin(\beta-\theta)}{\beta-\theta}\left(\frac{\theta}{\sin\theta}\right)\left(\frac{\beta}{\sin\beta}\right).$$

Now observe that $\beta-\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$ and

$$\frac{2}{\pi}\le\frac{\sin x}{x}\le 1 \text{ if } |x|\le\frac{\pi}{2}. \qquad \square$$

Remark 2.

In fact, it is clear from the first equality in the proof that given any $\epsilon>0$, there exists θ_0 so that if $-\theta_0<\theta\neq\beta<\theta_0$, then

$$1-\epsilon<\frac{\cot(\theta)-\cot(\beta)}{\frac{1}{\theta}-\frac{1}{\beta}}<1+\epsilon. \qquad \square$$

Corollary 1.

Suppose $\{\theta_j\}\subset[0,\frac{\pi}{2})$ is a decreasing sequence and that there exists $0<c\le 1$ such that

$$\frac{1}{\theta_j}-\frac{1}{\theta_{j-1}}\ge c\max_{1\le i\le j}\left(\frac{1}{\theta_i}-\frac{1}{\theta_{i-1}}\right) \tag{2.3}$$

(setting $\theta_0=\frac{\pi}{2}$). If $\{\theta_j\}$ is a Max(p) set for some $p<\infty$ or a density basis, then $\{\theta_j\}$ is lacunary.

Proof. By Lemma 1

$$\frac{2}{\pi}\left(\frac{1}{\theta_j}-\frac{1}{\theta_{j-1}}\right)\le\cot\theta_j-\cot\theta_{j-1}\le\left(\frac{\pi}{2}\right)^2\left(\frac{1}{\theta_j}-\frac{1}{\theta_{j-1}}\right).$$

Thus,

$$\cot\theta_j-\cot\theta_{j-1}\ge\frac{c}{4}\max_{1\le i\le j}(\cot\theta_i-\cot\theta_{i-1})$$

By Theorem 1 $\{\theta_j\}_1^\infty$ is lacunary. $\square$

For the next result, just take $c=1$.

Corollary 2.

Suppose $\{\theta_j\}$ is a decreasing sequence, and

$$\left\{\frac{1}{\theta_j}-\frac{1}{\theta_{j-1}}\right\}$$

is increasing. Then $\{\theta_j\}$ is a Max(p) set for some $p<\infty$ or a density basis iff $\{\theta_j\}$ is lacunary.

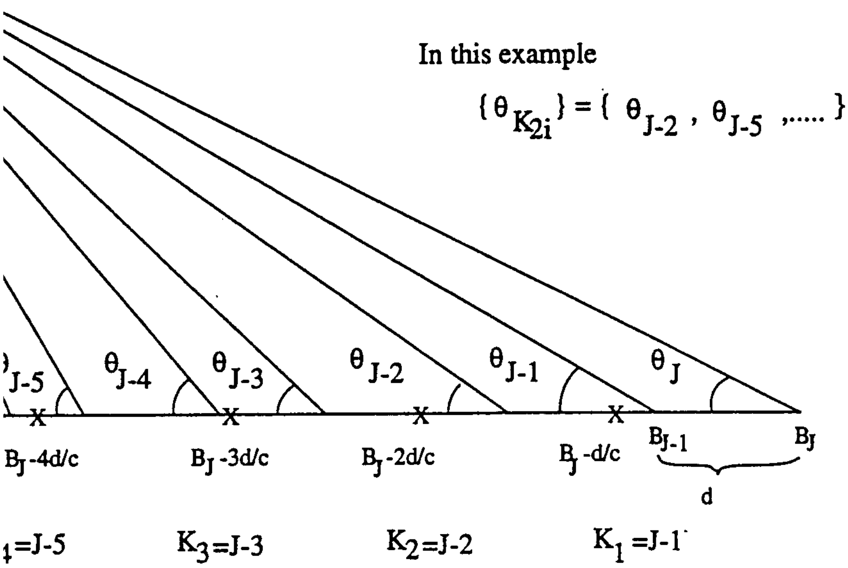
Corollary 3.

Suppose $\{\theta_j\}$ is a decreasing sequence with limit θ_0. If

$$\left\{\frac{1}{\theta_j-\theta_0}-\frac{1}{\theta_{j-1}-\theta_0}\right\}$$

is increasing and $\{\theta_j\}$ is a Max(p) set for some $p<\infty$, then $\{\theta_j-\theta_0\}$ is lacunary.

Proof. We need only observe that A is a Max(p) set iff $A-\theta_0$ is a Max(p) set, and apply the previous result. $\square$



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$_0 \subset \mathbb{N}$ *be an increasing sequence and suppose there is some* $c>0$ *so that* $(n_{i+1}-n_i)$ *for all* j. *If the square function*

$$S_E(f)(x) \equiv \left(\sum_j \left|\sum_{n\in[n_j,n_{j+1})} \hat{f}(n)e^{inx}\right|^2\right)^{\frac{1}{2}}$$

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tains its minimum in the

A consequence of the proposition is that if $H_k = o(k)$, then the P.T. construction can be used to show that certain sets of directions are not Max(p) sets. In terms of the directions $\{\theta_j\}_{j=1}^{\infty}$ we can obtain the following theorem:

Theorem 2.

Let $\{\theta_j\}_{j=1}^{\infty} \subset [0, \frac{\pi}{2})$ be a decreasing sequence. If (taking $\theta_o = \frac{\pi}{2}$)

$$\max_{\substack{n \geq 2^l \\ n+2^l \leq 2^k}} \left(\frac{\frac{1}{\theta_{n+2^l}} - \frac{1}{\theta_n}}{\frac{1}{\theta_n} - \frac{1}{\theta_{n-2^l}}} \right)^{\pm 1} = o(k)$$

then $\{\theta_j\}$ is neither a Max(p) set for any $p < \infty$ nor a density basis.

Proof. Our previous results show that we just need to check that this hypothesis gives $H_k = o(k)$.

First observe that at step l in the P.T. construction, $\left(\frac{b}{a}\right)^{\pm 1}$ is of the form

$$\left(\frac{\sum_{i=n+1}^{n+2^l} b_i}{\sum_{i=n-2^l+1}^{n} b_i} \right)^{\pm 1} ,$$

for appropriate choices of $n \geq 2^l$ and $n + 2^l \leq 2^k$.

Now, $b_i = \cot \theta_i - \cot \theta_{i-1}$ (recall Figure 1); hence,

$$\left(\frac{b}{a}\right)^{\pm 1} = \left(\frac{\cot \theta_{n+2^l} - \cot \theta_n}{\cot \theta_n - \cot \theta_{n-2^l}} \right)^{\pm 1} .$$

Since this ratio is comparable to

$$\left(\frac{\frac{1}{\theta_{n+2^l}} - \frac{1}{\theta_n}}{\frac{1}{\theta_n} - \frac{1}{\theta_{n-2^l}}} \right)^{\pm 1}$$

(See Lemma 1), H_k is indeed $o(k)$. □

One can use the "bounded factor" P.T. construction (or Corollary 1) to prove that a Max(p) set (or a density basis) does not contain arithmetic progressions of arbitrary length. This fact can be improved by applying the ideas of our previous theorem.

Theorem 3.

Suppose $A \subset [0, \frac{\pi}{2})$ is a Max(p) set for some $p < \infty$ or a density basis, and suppose f is a function satisfying $f(x) = o(\log x)$. There is an integer M_0, depending on A and f, such that A contains at most $2^M - f(2^M)$ terms from any arithmetic progression of length 2^M, $M \geq M_0$.

Proof. Choose $\epsilon > 0$ so that inequality (2.1) does not hold for M_A (for any set E).

Since $f(2^M)/M \to 0$ we can assume without loss of generality that $f(2^M) \leq 2^{M-2}$ for all M. Apply Lemma 2 to choose M_0 so that if $k \geq M_0$, and $1 \leq x_k \leq 12(1 + f(2^{k+2}))$ then

$$\left(\frac{x_k}{2k}\right)^{\frac{2k}{2k-1}} + \left(1 - \left(\frac{x_k}{2k}\right)^{\frac{1}{2k-1}}\right) x_k < \epsilon \tag{3.1}$$

Let $M \geq M_0 + 2$ and assume

$$\left\{\theta_j \,:\, j = 1, \ldots, 2^{M-1} - f\left(2^M\right)\right\} \subset A$$

is a decreasing sequence which is contained in the arithmetic progression of length 2^{M-1},

$$\left\{2^{M-1} d, \ldots, \left(2^M - 1\right) d\right\} .$$

We need to consider

$$\max_{\substack{n\geq 2^l\\ n+2^l\leq 2^k}}\left(\frac{\frac{1}{\theta_{n+2^l}}-\frac{1}{\theta_n}}{\frac{1}{\theta_n}-\frac{1}{\theta_{n-2^l}}}\right)^{\pm 1}=\max_{n,l}\left(\left(\frac{\theta_n-\theta_{n+2^l}}{\theta_{n-2^l}-\theta_n}\right)\frac{\theta_{n-2^l}}{\theta_{n+2^l}}\right)^{\pm 1}$$

Since the angles $\{\theta_j\}$ are chosen from an arithmetic progression of step size d, and at most $f(2^M)$ terms from this progression are omitted,

$$\theta_n-\theta_{n+2^l}\leq\left(2^l+f\left(2^M\right)\right)d\,,$$

and $\theta_{n-2^l}-\theta_n\geq 2^l d$.

Moreover, as $2^{M-1}d\leq\theta_j\leq(2^M-1)d$,

$$\frac{\theta_{n-2^l}}{\theta_{n+2^l}}\leq\frac{\left(2^M-1\right)d}{2^{M-1}d}\leq 2\,.$$

Hence,

$$\left(\frac{\theta_n-\theta_{n+2^l}}{\theta_{n-2^l}-\theta_n}\right)\left(\frac{\theta_{n-2^l}}{\theta_{n+2^l}}\right)\leq 2\left(1+\frac{f\left(2^M\right)}{2^l}\right)\leq 2\left(1+f\left(2^M\right)\right)\,.$$

As $A\subset[0,\frac{\pi}{2})$ we know (by Lemma 1) that

$$\frac{\cot\theta_{n+2^l}-\cot\theta_n}{\cot\theta_n-\cot\theta_{n-2^l}}\leq 4\left(\frac{\frac{1}{\theta_{n+2^l}}-\frac{1}{\theta_n}}{\frac{1}{\theta_n}-\frac{1}{\theta_{n-2^l}}}\right)\leq 8\left(1+f\left(2^M\right)\right)\,.$$

A similar argument gives the better bound $4\left(1+f(2^M)\right)$ for the reciprocal; hence, all the factors $\left(\frac{a}{b}+\frac{b}{a}\right)$ are at most $12\left(1+f(2^M)\right)$.

For $k=M-2$, construct the Perron tree from $\{\theta_j\}_{j=1}^{2^k}$ (we can do this since $2^k\leq 2^{M-1}-f(2^M)$) with proportionality parameter $\alpha=\frac{12(1+f(2^{k+2}))}{2k}$. If, as usual, H_k is the maximum value of the factors $\left(\frac{a}{b}+\frac{b}{a}\right)$ at all steps $1,2,\ldots,k$, then certainly $1\leq H_k\leq 12(1+f(2^{k+2}))$. Since $k\geq M_0$, (3.1) implies that (2.1) holds when E is this Perron tree, and this contradicts the choice of ϵ. This contradiction means we cannot have

$$\left\{\theta_j\,:\,j=1,\ldots,2^{M-1}-f\left(2^M\right)\right\}\subset A\bigcap\left\{2^{M-1}d,\ldots,\left(2^M-1\right)d\right\}\,.$$

To complete the proof, assume A contains the arithmetic progression $\{a+d,\ldots,a+2^Md\}$ for some $d>0$. As $A\subset[0,\frac{\pi}{2})$, $a\geq -d$, and thus, $\{d,2d,\ldots,(2^M-1)d\}\subset(A-a)\bigcap[0,\frac{\pi}{2})$. But $A-a$ is a Max(p) set (respectively, density basis) with the same choice of ϵ failing (2.1) as does for A. It follows that

$$\left|(A-a)\bigcap\left\{2^{M-1}d,\ldots,\left(2^M-1\right)d\right\}\right|\leq 2^{M-1}-f\left(2^M\right)$$

and thus A contains at most $2^M-f(2^M)$ terms from any arithmetic progression of length 2^M, $M\geq M_0+2$. □

Remark 5.

If, instead, we assumed the directions were chosen from a set of angles whose cotangents (i.e., the base lengths of the corresponding triangles in the P.T. construction) were in an arithmetic progression, then a similar result is true. Also, the angles do not need to be chosen from a strictly *arithmetic progression. By similar reasoning, one can more generally show that if A is a Max(p) set or a density basis, then*

$$\left|\left\{k = 1, 2, \ldots, 2^M \ : \ A \bigcap [a + kd, a + (k+1)d) \neq \emptyset\right\}\right| \leq 2^M - f\left(2^M\right)$$

for M sufficiently large. Another cardinality result for sets of angles of mutual distances at least d can be found in [8, 2.7]. □

In [3, Th. 8.6.1] the standard P.T. construction is used to show that $\{\frac{1}{n}\}_{n=1}^{\infty}$ is not a density basis (or a Max(p) set, as can be seen from the proof). This too can be improved.

Proposition 3.

Suppose $A \subset [0, \frac{\pi}{2})$ is a Max(p) set for some $p < \infty$ or a density basis, and $f(x) = o(\log x)$. For every $\alpha > 0$ there is an $N_0 = N_0(A, f)$ so that for all $N \geq N_0$

$$\left|A \bigcap \left\{\frac{1}{n^\alpha}\right\}_{n=1}^{2^N}\right| \leq 2^N - f\left(2^N\right) .$$

Proof. The details are similar to the previous theorem and are omitted. □

Remark 6.

The alert reader will have noticed that in Proposition 2 it is actually enough to have

$$\liminf \frac{H_k}{k} = 0 \tag{3.2}$$

in order to be able to show that for every $\epsilon > 0$ one can choose α, k so that $\alpha^{2k} + (1-\alpha)H_k < \epsilon$. Furthermore, this is all that is needed to make the P.T. construction work in, for example, Theorem 2. Our final result provides a partial converse to this. □

Proposition 4.

If the exact area of the set obtained from the Perron tree construction is

$$A\left(\alpha^{2j} + (1-\alpha)\tilde{H}_j\right) ,$$

where j is the number of iterations, A is the area of the initial triangle, and α is our choice of parameter in the construction, and if there is some $c > 0$ such that $\tilde{H}_j \geq cj$ for all j, then $\alpha^{2j} + (1-\alpha)\tilde{H}_j$ stays bounded away from zero for all choices of α, and the Perron tree construction fails.

Proof. If $\tilde{H}_j \geq cj$, then

$$\alpha^{2j} + (1-\alpha)\tilde{H}_j \geq \alpha^{2j} + (1-\alpha)cj .$$

Let

$$G_j(\alpha) = \alpha^{2j} + (1-\alpha)cj .$$

Then $G_j'(\alpha) = 2j\alpha^{2j-1} - cj$, so

$$G_j'(\alpha) = 0 \text{ if and only if } \alpha = \alpha_0 = \left(\frac{c}{2}\right)^{\frac{1}{2j-1}} .$$

The second derivate is positive, so α_0 is a (unique) minimum with value

$$G_j(\alpha_0) = \left(\frac{c}{2}\right)^{\frac{2j}{2j-1}} + \left(1 - \left(\frac{c}{2}\right)^{\frac{1}{2j-1}}\right) j ,$$

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