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Unitary Mappings Between Multiresolution Analyses of $L^2(\mathbf{R})$ and a Parametrization of Low-Pass Filters

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ABSTRACT. This article provides classes of unitary operators of $L^2(\mathbf{R})$ contained in the commutant of the Shift operator, such that for any pair of multiresolution analyses of $L^2(\mathbf{R})$ there exists a unitary operator in one of these classes, which maps all the scaling functions of the first multiresolution analysis to scaling functions of the other. We use these unitary operators to provide an interesting class of scaling functions. We show that the Dai-Larson unitary parametrization of orthonormal wavelets is not suitable for the study of scaling functions. These operators give an interesting relation between low-pass filters corresponding to scaling functions, which is implemented by a special class of unitary operators acting on $L^2([-\pi, \pi))$, which we characterize. Using this characterization we recapture Daubechies' orthonormal wavelets bypassing the spectral factorization process.

1. Introduction and Preliminaries

One of the two issues of this article is to deal with the following problem. Given an arbitrary multiresolution analysis (abbreviated MRA) can we find unitary operators that map a scaling function of the given MRA to a scaling function of any other MRA? Once we can determine all such unitary operators we have all possible MRAs. The corresponding problem for orthonormal wavelets and their generalization complete wandering vectors was studied by Dai and Larson [4]. In our first main result, Theorem 1, we prove that given an arbitrary MRA there exists a class of unitary operators such that each member of this class maps a scaling function of the given MRA to a scaling function of some MRA and vice versa. Thus we obtain an operator-theoretic characterization of scaling functions. For an MRA $\mathcal{M} = \{V_j\}_j$ we adopt $V_{j-1} \subseteq V_j$. For the definitions and a very careful and comprehensive introduction to MRA theory the reader should refer to [11]

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Definition 1.

A square-integrable function ϕ defined on $\mathbf{R}$ is a scaling function for the MRA $\mathcal{M}$ if $\{\phi(.-n) : n \in \mathbf{Z}\}$ is an orthonormal basis of V_0.

We use our operator-theoretic characterization of scaling functions to characterize the class of Borel subsets K of $\mathbf{R}$ such that $\frac{1}{\sqrt{2\pi}}\chi_K$ is the Fourier transform of a scaling function. Due to this result it turns out that only a restrictive class of unitary operators mapping orthonormal wavelets to orthonormal wavelets map scaling functions to scaling functions (see discussion following Corollary 1, Corollary 2, and Example 1). Thus, although each scaling function is associated with an orthonormal wavelet, the classes of unitary operators provided by Dai and Larson are not the proper tool for the study of scaling functions. Corollary 1 gives the form of unitary operators mapping Shannon scaling function to every other scaling function. A careful look at this result reveals that determining these unitary operators is precisely the same with finding the scaling functions. This shows that our characterization cannot lead us to the construction of all scaling functions. Let $\{V_j\}_j$ be an arbitrary MRA and ϕ be a scaling function for this MRA. Since V_0 is contained in V_1 for a.e. γ in $[-\pi, \pi)$ we have

$$\sqrt{2}\hat{\phi}(2\gamma) = m(\gamma)\hat{\phi}(\gamma) \tag{1.1}$$

The 2π-periodic function m defined uniquely by the previous equation is the *low-pass filter* corresponding to the scaling function ϕ. Equation (1.1) is often called *two-scale relation.* MRA theory proves that for a low-pass filter m corresponding to a scaling function the next equality is true:

$$|m(\gamma)|^2 + |m(\gamma+\pi)|^2 = 2 \quad a.e. \tag{1.2}$$

We want to consider the low-pass filters corresponding to scaling functions as a subclass of a bigger class of such objects. Thus we introduce the next definition.

Definition 2.

A 2π-periodic Borel measurable function m satisfying (1.2) is called low-pass filter.

Every low-pass filter is not necessarily associated with a scaling function and when it is so we specify it in order to avoid confusion. The term low-pass filter was first adopted by engineers. We assign to the term low-pass filter a wider meaning than usually given to it in the engineering literature. The engineers usually consider continuous low-pass filters m such that $m(0) = \sqrt{2}$. We want to consider the most general class of them. This justifies our definition.

Theorem 2 shows that for any two low-pass filters associated with scaling functions there exists a unitary operator which belongs to a specific Von-Neumman algebra acting on $L^2([-\pi, \pi))$ mapping one low-pass filter to the other. We can use this theorem to find low-pass filters associated with MRAs. In fact Theorem 2 provides the a unitary pametrization of the complete class of low-pass filters. The significance of this theorem is that we show that each of the unitary operators characterized in Theorem 1, mapping a scaling function to any other scaling function produces a relation between the low-pass filters corresponding to these scaling functions.

We use low-pass filters to construct scaling functions. For a very large subclass of low-pass filters the infinite product (3.1) defines a square-integrable function which satisfies the two-scale relation (1.1). Such a function is not automatically a scaling function. There exist conditions on the low-pass filter guaranteeing that the infinite product (3.1) produces a scaling function but there are still enough open questions related to this issue. We will discuss all these in Section 3 of the present article. Nevertheless the infinite product (3.1) cannot give all scaling functions (see Example 2). We include a discussion on this example concluding with an interesting problem. Finally we give a generic example (Example 3) showing how we can use Theorem 2 to recapture low-pass filters that are trigonometric polynomials.

It can easily be verified that if ϕ is a scaling function and μ is a 2π-periodic unimodular function, then $\mu\hat{\phi}$ is the Fourier transform of another scaling function for the same MRA. For the rest of this article, all subspaces are closed. Let ψ be any square-integrable function defined on $\mathbf{R}$. We define $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$, for all integers j, k. If X is a subset of a Hilbert space, we define $[X]$ to be the linear span of X and by $[X]^-$ the closure of $[X]$. Two important unitary operators acting on a multiresolution analysis of $L^2(\mathbf{R})$ are the dilation and shift operators defined, by the equations $Df(t) = \sqrt{2}f(2t)$ and $Sf(t) = f(t-1)$, $f \in L^2(\mathbf{R})$, respectively.

2. Unitary Mappings Between MRAs

Another issue of this article is to provide certain classes of unitary operators which map the scaling function of one MRA to the scaling function of another MRA. Since the zero-indexed subspace of every MRA is the only subspace that has an orthonormal basis consisted of the integer translations of a single function [16], it is natural to search within a class of unitary operators that preserve this property. This leads us to the unitary operators in the commutant of the shift operator. Let $\mathcal{M} = \{V_j : j \in \mathbf{Z}\}$, $\mathcal{N} = \{\tilde{V}_j : j \in \mathbf{Z}\}$ be two MRAs with ϕ, χ their scaling functions and ψ, $\tilde{\psi}$ orthonormal wavelets, respectively. The operator U defined by

$$\begin{aligned} U\phi_{0,n} &= \chi_{0,n} n \in \mathbf{Z} \\ U\psi_{j,k} &= \tilde{\psi}_{j,k}\ j \geq 0\ ,\ k \in \mathbf{Z} \end{aligned}$$

is a unitary operator of $L^2(\mathbf{R})$.

Theorem 1.

Let U be a unitary operator defined as in the preceding paragraph. Then the following hold:

$$SU = US \tag{2.1}$$

$$UDg = DUg \quad \text{for every } g \text{ in } \left[\psi_{j,k} : j \geq 0,\ \ k \in \mathbf{Z}\right]^- \tag{2.2}$$

Conversely every unitary operator defined in $L^2(\mathbf{R})$ satisfying the preceding properties defines an MRA generated by $U\phi$ and $U\psi$ is a wavelet associated with this MRA.

Proof. We include only the proof of the converse implication because the other direction is obvious. Suppose that U is a unitary operator satisfying properties (2.1) and (2.2). Let $\mathcal{M} = \{V_j : j \in \mathbf{Z}\}$ be a given MRA generated by the scaling function ϕ and ψ be its orthonormal wavelet. Set $\tilde{V}_0 := U(V_0)$ and $\tilde{V}_j := D^j(\tilde{V}_0)$. We will prove that $\{\tilde{V}_j : j \in \mathbf{Z}\}$ is an MRA generated by $U\phi$ which will be denoted by $\mathcal{N}$.

First we have to prove the inclusion $\tilde{V}_0 \subseteq \tilde{V}_1$. By property (2.2) we have $UD(V_0^{\perp}) = DU(V_0^{\perp})$. Since U and D are unitary operators, they preserve orthogonal complements. Thus, we have

$$[UD\,(V_0)]^{\perp} = [DU\,(V_0)]^{\perp}$$

equivalently $U(V_1) = \tilde{V}_1$. But $V_1 = W_0 \oplus V_0$. Therefore,

$$\tilde{V}_1 = U\,(W_0) \oplus \tilde{V}_0 \tag{2.3}$$

This equation implies $\tilde{V}_0 \subseteq \tilde{V}_1$. By the definition of $\mathcal{N}$ and the latter inclusion we have that $\mathcal{N}$ is an increasing sequence of subspaces of $L^2(\mathbf{R})$. By property (2.1) $\{(U\phi)_{0,n} : n \in \mathbf{Z}\}$ is an orthonormal basis of $\tilde{V}_0$. Proposition 5.3.1 in [11] shows $\bigcap_{j\in\mathbf{Z}} \tilde{V}_j = \{0\}$. By induction, using properties (2.2) and (2.3) we show for $j \in \mathbf{Z}^+$ $\tilde{V}_{j+1} = U(W_j) \oplus \tilde{V}_j$, thus

$$\sum_{l=0}^{j-1} \oplus U\,(W_l)\ \oplus \tilde{V}_0\ = \tilde{V}_j$$

Therefore $U(V_j) = \tilde{V}_j$ for each $j > 0$, thus $\bigcup_{j\in\mathbf{Z}} \tilde{V}_j$ is dense in $L^2(\mathbf{R})$. □

Remark 1.

If P is the orthogonal projection onto V_0 then condition (2.2) of the previous theorem is equivalent to

$$(DU - UD)(I - P) = 0 \qquad \square$$

The merit of the previous theorem is that given an arbitrary MRA it provides an abstract characterization of *every* scaling function of *every* other MRA by determining the unitary operators in this class. The unitary operator U defined in the paragraph preceding Theorem 1 mapping ϕ to χ is defined modulo the wavelets ψ and $\tilde{\psi}$. Since an infinite number of orthonormal wavelets, are associated with an MRA we conclude that the correspondence between the set of all scaling functions and each of these classes of unitary operators of Theorem 1 is not one to one. It is worth noticing that given an orthonormal wavelet ψ, then the class of bounded operators A in $\{S\}'$ such that $(AD - DA)(I - P) = 0$ is a SOT-closed linear subspace of $\mathcal{B}(L^2(\mathbf{R}))$. This subspace is a left module over the abelian selfadjoint algebra $\{D, S\}'$.

Theorem 1 has also been obtained in a rather complicated form by Dai and Lu in [5, Corollary 3.6], independently, but was published first in [17]. It is easy to establish the equivalence between the two results.

Let $\mathcal{F}$ denote the Fourier Transform on $L^2(\mathbf{R})$. We define the Fourier transform on $L^1(\mathbf{R})$ by the formula:

$$\hat{f}(\gamma) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-it\gamma} dt \qquad f \in L^1(\mathbf{R})$$

Let M be defined on $L^2(\mathbf{R})$ by the equation $Mf(t) = e^{it} f(t)$ a.e. If S and D are the Shift and Dilation operators, respectively, it is easy to see $\mathcal{F}S\mathcal{F}^{-1} = M^*$ and $\mathcal{F}D\mathcal{F}^{-1} = D^*$. Throughout the article, given an arbitrary bounded operator A acting on $L^2(\mathbf{R})$ we use $\hat{A}$ to denote the operator $\mathcal{F}A\mathcal{F}^{-1}$.

Let $\mathcal{M}$ be the Shannon MRA generated by the Shannon scaling function $\phi := (2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{[-\pi,\pi)})$. The Shannon wavelet ψ which is associated with this MRA is given by $\hat{\psi} = (2\pi)^{-1/2} \chi_{[-2\pi,-\pi)\cup[\pi,2\pi)}$. It is easy to see $\mathcal{F}(V_0) = L^2([\pi, -\pi))$ and $\mathcal{F}([\psi_{k,n} : n \in \mathbf{Z}]^-) = L^2([-2^{k+1}\pi, -2^k\pi) \cup [2^k\pi, 2^{k+1}\pi))$, $k \geq 0$. Let $\mathcal{N}$ be an arbitrary MRA. Let $\tilde{\phi}$ and $\tilde{\psi}$ be its scaling function and orthonormal wavelet, respectively. Next we will find the form of the unitary operators U mapping ϕ to $\tilde{\phi}$ and ψ to $\tilde{\psi}$ satisfying conditions (2.1) and (2.2) of the previous theorem. Obviously

$$L^2(\mathbf{R}) = \sum_{k=0}^{\infty} \bigoplus L^2\left(\left[-2^{k+1}\pi, -2^k\pi\right) \cup \left[2^k\pi, 2^{k+1}\pi\right)\right) \bigoplus L^2([\pi, -\pi)) . \tag{2.4}$$

If m belongs to $L^2([\pi, -\pi))$, then extend m 2π-periodically over the whole real line. It is easy to verify $\hat{U}m = \sqrt{2\pi} m \hat{\tilde{\phi}}$. For m in $L^2([-2^{k+1}\pi, -2^k\pi) \cup [2^k\pi, 2^{k+1}\pi))$ we have $\hat{U}m = \sqrt{2\pi} m \hat{\tilde{\psi}}(2^{-k}.)$, with m $2^k 2\pi$-periodically extended over the whole real line. If g is an arbitrary function in $L^2(\mathbf{R})$ then it can be decomposed with respect to the latter direct sum in $g = h + \sum_{k=0}^{\infty} g_k$, where h is in $L^2([-\pi, \pi))$ and each g_k in $L^2([-2^{k+1}\pi, -2^k\pi) \cup [2^k\pi, 2^{k+1}\pi))$. Then we have

$$\hat{U}g = \sqrt{2\pi}\left[h\hat{\tilde{\phi}} + \sum_{k=0}^{\infty} g_k \hat{\tilde{\psi}}\left(2^{-k}.\right)\right] \tag{2.5}$$

It is easy to establish the converse implication. Thus, we proved the following.

Corollary 1.

Let $\mathcal{N}$ be an arbitrary MRA. Let $\tilde{\phi}$ and $\tilde{\psi}$ be its scaling function and orthonormal wavelet, respectively. Then the unitary operator U given by (2.5) mapping ϕ to $\tilde{\phi}$ and ψ to $\tilde{\psi}$ satisfies *conditions (2.1) and (2.2) of the previous theorem. Conversely, let U be a unitary operator satisfying conditions (2.1) and (2.2). If we set $\hat{U}\chi_{[-\pi,\pi)} := \sqrt{2\pi}\hat{\tilde{\phi}}$ and $\hat{U}\chi_{[-2\pi,-\pi)\cup[\pi,2\pi)} := \sqrt{2\pi}\hat{\tilde{\psi}}$, then $\tilde{\phi}$ is a scaling function and $\tilde{\psi}$ is a wavelet associated with the MRA $\tilde{\phi}$ generates.*

Dai and Larson [4] characterize the set of all unitary operators U such that given any orthonormal wavelet ψ, $U\psi$ is again an orthonormal wavelet. For a given orthonormal wavelet ψ the *local commutant* $\mathcal{C}_\psi(D,S)$ is the set of all bounded operators A satisfying $AD^nS^m\psi = D^nS^mA\psi$ for all integers m and n. They prove that for every orthonormal wavelet η there exists a unique unitary operator U such that $U\psi = \eta$ and $U \in \mathcal{C}_\psi(D,S)$. Also $\mathcal{C}_\psi(D,S) \subseteq \{D\}'$. Thus a unitary operator U satisfying conditions (2.1) and (2.2) U belongs to the local commutant $\mathcal{C}_\psi(D,S)$ if and only if $U \in \{D,S\}'$.

The following proposition which is an application of Theorem 1 characterizes the Borel subsets K of $\mathbf{R}$ such that $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_K)$ is a scaling function for an MRA. Following [7] we call a scaling function ϕ MSF (Minimally Supported in the Frequency) if $|\hat{\phi}| = (2\pi)^{-1/2}\chi_K$, where K is a Borel subset of $\mathbf{R}$. We use λ to denote the Lebesgue measure on $\mathbf{R}$. Therefore the following proposition essentially characterizes all MSF-scaling functions (see Remark 2). An *MSF or s-elementary* wavelet is wavelet ψ such that $|\hat{\psi}| = (2\pi)^{-1/2}\chi_L$, where L is a Borel subset of $\mathbf{R}$. The set L is called *wavelet set*. The term s-elementary wavelet was introduced by Dai and Larson in [4].

In [7] Fang and Wang characterize the MSF-wavelets associated with MRAs. Their proof is rather elaborate. The following proposition was motivated by their result.

Let F_1 and F_2 be Borel subsets of $\mathbf{R}$. We call F_1 2π *-translation congruent* to F_2 if and only if for almost every $x \in F_1$ there exists a unique integer $k(x)$ such that $(x - 2k(x)\pi)$ is contained in F_2 and the mapping $x \longrightarrow (x - 2k(x)\pi)$ is a bijection between F_1 and F_2 (modulo null sets). Obviously 2π-translation congruence is an equivalence relation.

Proposition 1.

Let K be a Borel subset of $\mathbf{R}$. Let $\phi := (2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_K)$. Then ϕ is a scaling function of an MRA if and only if the following conditions hold:

A) $\{K + 2r\pi : r \in \mathbf{Z}\}$ *is an a.e. partition of* $\mathbf{R}$.

B) $K \subseteq 2K$ *(modulo null sets). For* $L = 2K\backslash K$ *the set* $\{2^jL : j \geq 0\} \cup \{K\}$ *is a partition of* $\mathbf{R}$ *and* $\{L + 2r\pi : r \in \mathbf{Z}\}$ *is an a.e. partition of* $\mathbf{R}$.

Also the low-pass filter m corresponding to ϕ is the 2π-periodic extension of $\sqrt{2}\chi_{K/2}$ to the whole real line.

Proof. Let $\hat{\phi}$ be a scaling function such that $\hat{\phi} = (2\pi)^{-1/2}\chi_K$. We will prove that conditions (A) and (B) are true. The orthonormality of the set $\{S^n\phi : n \in \mathbf{Z}\}$ gives $\sum_r |\hat{\phi}(\gamma + 2r\pi)|^2 = 1/2\pi$ a.e. which implies $\sum_r \chi_K(\gamma + 2r\pi) = 1$ a.e. This proves (A).

Next the two-scale relation (1.1) gives

$$\chi_{K/2}(\gamma) = \frac{1}{\sqrt{2}}m(\gamma)\chi_K(\gamma) \quad a.e. \tag{2.6}$$

From this equation we obtain that modulo null sets $K/2$ is contained in K.

By partitioning $\mathbf{R}$ into $\{[-\pi,\pi) + 2r\pi : r \in \mathbf{Z}\}$ and $\{K + 2r\pi : r \in \mathbf{Z}\}$ we can define (modulo null sets) the measurable bijection $s : K \longrightarrow [-\pi,\pi)$ such that for almost every γ in K $s(\gamma) := \gamma + 2r(\gamma)\pi$, where $r(\gamma)$ is the unique integer such that $\gamma + 2r(\gamma)\pi$ belongs to $[-\pi,\pi)$. Since $(K/2) \subseteq K$ we can define $E := s(K/2)$.

Let m be the 2π-periodic extension of $\sqrt{2}\chi_E$ to the whole real line. We will verify that (2.6), which is (1.1) for this case, is true for a.e. γ in $\mathbf{R}$. Indeed, if γ does not belong to K then as we have already proved it does not belong to $K/2$, thus both sides of (2.6) are equal to zero. If γ belongs to K then $\gamma \in K/2$ if and only if $s(\gamma) \in E$, since s is a bijection between K and $[-\pi,\pi)$. But

$s(\gamma) - \gamma$ is an integral multiple of 2π. Since m is 2π-periodic we have

$$m(\gamma) = m(s(\gamma)) = \sqrt{2}\chi_E(s(\gamma))$$

thus (2.6) is true in this case as well. Therefore, (1.1) is satisfied for this particular 2π-periodic function m and the scaling function ϕ. By the uniqueness property the low-pass filters have to satisfy (1.1) we get that m is the low-pass filter corresponding to the scaling function ϕ.

Equation (1.2) gives $(E+\pi)\mathrm{mod}(2\pi) = [-\pi, \pi)\backslash E$, while we have $s(K\backslash(K/2)) = [-\pi, \pi)\backslash E$. Thus we obtain that $s(K\backslash(K/2))$ is 2π-congruent to $E + \pi$.

The wavelet ψ associated to the scaling function ϕ is given by

$$\hat{\psi}(\gamma) = e^{i\gamma/2}\frac{1}{\sqrt{2\pi}}\overline{m(\gamma/2+\pi)}\hat{\chi}_K(\gamma/2)$$

By the preceding arguments if γ belongs to $2K$ then we get $m(\gamma/2 + \pi) = \sqrt{2}$ if $\gamma \in 2K\backslash K$, otherwise $m(\gamma/2+\pi) = 0$. Thus $|\hat{\psi}| = (2\pi)^{-1/2}\chi_{2K\backslash K}$. Once again orthonormality of the family $\{S^n\psi : n \in \mathbf{Z}\}$ gives $\{L + 2r\pi : r \in \mathbf{Z}\}$ is an a.e. partition of $\mathbf{R}$. Since $\{\phi_{0,n} : n \in \mathbf{Z}\} \cup \{\psi_{j,n} : j \geq 0,\ n \in \mathbf{Z}\}$ is an orthonormal basis for $L^2(\mathbf{R})$, we obtain that $\{2^jL : j \geq 0\} \cup \{K\}$ is an a.e. partition of $\mathbf{R}$.

Let us prove the converse implication. For g in $L^2(\mathbf{R})$ we have

$$g = h + \sum_{k=0}^{\infty} g_k$$

where h is the restriction of g on $[-\pi, \pi)$ and each g_k is the restriction of g on $[-2^{k+1}\pi, -2^k\pi) \cup [2^k\pi, 2^{k+1}\pi)$. Now define the operator U acting on $L^2(\mathbf{R})$ by the following equation

$$\hat{U}g = h\chi_K + \sum_{k=0}^{\infty} g_k\chi_{2^kL}$$

where g_k and h are now considered to be $2^k2\pi$ and 2π-periodically extended to the whole real line respectively. Since $\{L + 2r\pi : r \in \mathbf{Z}\}$ is an a.e. partition of $\mathbf{R}$ it is not hard to check that L is 2π-translation congruent to $[-2\pi, -\pi) \cup [\pi, 2\pi)$. We already have established such a congruence between K and $[-\pi, \pi)$. Combining these facts with the hypothesis that $\{2^jL : j \geq 0\} \cup \{K\}$ is an a.e. partition of $\mathbf{R}$ as well we get that $\hat{U}$ is a well-defined unitary operator. Obviously $\hat{U}$ commutes with M. It remains to prove $\hat{U}D^*g = D^*\hat{U}g$ for every g in $L^2((-\infty, -\pi) \cup [\pi, \infty))$. By the decomposition $g = \sum_{k=0}^{\infty} g_k$ where each g_k is in $L^2([-2^{k+1}\pi, -2^k\pi) \cup [2^k\pi, 2^{k+1}\pi))$, we have $D^*g = \sum_{k=0}^{\infty} D^*g_k$; thus, by the definition of $\hat{U}$ we obtain

$$D^*\hat{U}g = \sum_{k=0}^{\infty} (D^*g_k)\, \chi_{2^{k+1}L} = \hat{U}D^*g$$

This completes the proof. □

Remark 2.

The first paragraph of the previous proof gives that if ϕ is a scaling function such that $|\hat{\phi}| = (2\pi)^{-1/2}\chi_K$ for some Borel set K, then $\{K + 2r\pi : r \in \mathbf{Z}\}$ is an a.e. partition of $\mathbf{R}$. Thus K is 2π-translation congruent to $[-\pi, \pi)$. This proves that $\mathcal{F}^{-1}(|\hat{\phi}|)$ is a scaling function for the same MRA. Indeed if we consider the unimodular function μ defined on K by the equation $\hat{\phi} = \mu|\hat{\phi}|$ (μ is the phase of ϕ), then we can extend μ 2π-periodically over $\mathbf{R}$, since K is 2π-translation congruent to $[-\pi, \pi)$. Thus $\mathcal{F}^{-1}(|\hat{\phi}|)$ is a scaling function for the MRA ϕ generates. This argument shows that actually the previous proposition characterizes all MSF scaling functions.

A little after the acceptance of the present paper jointly with Sikic and Weiss we proved that $\{L + 2r\pi : r \in \mathbf{Z}\}$ is an a.e. partition of $\mathbf{R}$ in the previous proposition is condition redundant. It can be derived from condition (A) and from $K \subseteq 2K$. The proof of this result is non-trivial and will appear in a future paper.

A characterization of low-pass filters associated with MSF wavelets (thus with MSF scaling functions) in terms of the corresponding wavelet sets is also given in [10]. □

It is easy to obtain the Fang and Wang characterization of the MSF-wavelets associated with MRAs from the previous result.

Next assume that K_1, K_2 are two Borel sets such that $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{K_1})$ and $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{K_2})$ are scaling functions. Set $L_i = 2K_i \backslash K_i$, $i = 1, 2$. It is easy to obtain a simple form for the unitary operator satisfying properties (2.1) and (2.2) of Theorem 1 mapping $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{K_1})$ to $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{K_2})$ and wavelet $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{L_1})$ to wavelet $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{L_2})$. By the previous proposition $\{2^j L_1 : j \geq 0\} \cup \{K_1\}$ is a partition of $\mathbf{R}$. Define σ first by setting σ/K_1 for the mapping implementing the 2π-congruence between K_1 and K_2. Similarly set σ/L_1 for the mapping implementing the 2π-congruence between L_1 and L_2. Next set for every $t \in 2^j L_1$ $\sigma(t) = 2^j\sigma(2^{-j}t)$ $(j \geq 0)$. Using (2.5) it is not hard to verify $\hat{U}f = f \circ \sigma^{-1}$, $f \in L^2(\mathbf{R})$.

A similar formula holds for every unitary operator V mapping an MSF-wavelet to an MSF-wavelet (see, [4, Ch. 5]). The class of these Borel subsets of $\mathbf{R}$ is characterized in [4]. Using different techniques, Fang and Wang characterize the same class in [7], but the Dai-Larson approach is more comprehensible.

The problem of the connectivity of orthonormal wavelets in $L^2(\mathbf{R})$ and other function spaces was introduced by Weiss in [10]. Later but independently Dai and Larson asked the same question with respect to the topology of $L^2(\mathbf{R})$ [4]. However, the first result is due to Speegle. In [19], Speegle proves that the class of MSF-wavelets is a path-connected subset of the class of all wavelets with respect to the $L^2(\mathbf{R})$-norm. One may ask if this is also true for the class of all MSF scaling functions. Ever since there has been substantial progress in the study of this problem. After the acceptance of the present paper enough results and preprints came to our attention. It is beyond our purpose to discuss all of them even in brief. However, we consider useful to include that a research consortium (the WUTAM consortium) was formed involving mainly people from Washington University in St. Louis and Texas A&M University for the study of this and other relevant problems. This consortium will soon release a paper discussing all these issues in detail.

In [17] the author asks whether unitaries in the local-commutant $C_\psi(D, S)$ map scaling functions to scaling functions. The next proposition and the example following address this issue.

Corollary 2.

Suppose that $\psi = (2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{L_1})$ and $\tilde{\psi} = (2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{L_2})$ are wavelets associated with the MRAs $\mathcal{M}$ and $\mathcal{N}$, respectively. Let V be the unique unitary operator in $C_\psi(D, S)$ such that $V\psi = \tilde{\psi}$. Then V maps every scaling function of $\mathcal{M}$ to a scaling function of $\mathcal{N}$.

Proof. Set $K_i = \cup_{j=1}^{\infty} 2^{-j}L_i, i = 1, 2$. Using Proposition 1 one can see that $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_{K_i})$, $i = 1, 2$ are scaling functions for $\mathcal{M}$ and $\mathcal{N}$, respectively. Following [4] we have $\hat{V}f = f \circ \sigma^{-1}$, where σ/L_1 implements the 2π-congruence mapping L_1 onto L_2 and for every $j \in \mathbf{Z}$ and $\gamma \in 2^j L_1$ we set $\sigma(\gamma) := 2^j\sigma(2^{-j}\gamma)$. It is easy to see $\hat{V}\chi_{K_1} = \chi_{K_2}$. Let μ be an arbitrary 2π-periodic unimodular function. Then $(\mu \circ \sigma^{-1})\chi_{K_2}$ is again unimodular supported on K_2 which is 2π-translation congruent to $[-\pi, \pi)$, therefore we can consider this function extended 2π-periodically over $\mathbf{R}$. Thus, $(2\pi)^{-1/2}(\mu \circ \sigma^{-1})\chi_{K_2}$ is the Fourier transform of a scaling function for $\mathcal{N}$. Since we have $\hat{V}(\mu\chi_{K_1}) = (\mu \circ \sigma^{-1})\chi_{K_2}$, the proof is complete. □

Example 1. Set $L = [\frac{-8\pi}{3}, \frac{-4\pi}{3}) \cup [\frac{2\pi}{3}, \frac{4\pi}{3})$ and $F = [\frac{-4\pi}{3}, \frac{-2\pi}{3}) \cup [\frac{4\pi}{3}, \frac{8\pi}{3})$. Set $\psi_1 = (2\pi)^{-1/2}$ $\mathcal{F}^{-1}(\chi_L)$ and $\psi_2 = (2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_F)$. Dai and Larson [4] give an example of a unitary operator mapping wavelet ψ_1 to the Littlewood-Paley orthonormal wavelet (otherwise called Meyer wavelet).

This unitary belongs to $C_{\psi_1}(D,S)$. Details on this method can be found in [4, Ch. 5]. Let ν be a real-valued function satisfying $\nu(\gamma)+\nu(1-\gamma)=1$, if $0\le\gamma\le 1$, $\nu(\gamma)=0$ if $\gamma\le 0$, and $\nu(\gamma)=1$ if $\gamma\ge 1$. Applying Proposition 1 we have that wavelets ψ_1 and ψ_2 are associated with MRAs. Set

$$h_1(\gamma) = \begin{cases} e^{i\gamma/2}\cos\left[\frac{\pi}{2}\nu\left(-\frac{3\gamma}{4\pi}-1\right)\right], & \gamma\in\left[\frac{-8\pi}{3},\frac{-4\pi}{3}\right) \\ e^{i\gamma/2}\sin\left[\frac{\pi}{2}\nu\left(\frac{3\gamma}{2\pi}-1\right)\right], & \gamma\in\left[\frac{2\pi}{3},\frac{4\pi}{3}\right) \end{cases}$$

$$h_2(\gamma) = \begin{cases} e^{i\gamma/2}\sin\left[\frac{\pi}{2}\nu\left(-\frac{3\gamma}{2\pi}-1\right)\right], & \gamma\in\left[\frac{-4\pi}{3},\frac{-2\pi}{3}\right) \\ e^{i\gamma/2}\cos\left[\frac{\pi}{2}\nu\left(\frac{3\gamma}{4\pi}-1\right)\right], & \gamma\in\left[\frac{4\pi}{3},\frac{8\pi}{3}\right) \end{cases}$$

Extend h_i $(i=1,2)$ such that for every $\gamma\in\mathbf{R}$ we have $h_i(\gamma)=h_i(2\gamma)$. Set $h_i(0)=0$. Therefore the multiplicative operators M_{h_i} acting on $L^2(\mathbf{R})$ defined by $M_{h_i}g=h_i g$ are well-defined and commute with M and D. Define σ by

$$\sigma(s)=\begin{cases} s+2\pi & s\in\left[\frac{-4\pi}{3},\frac{-2\pi}{3}\right) \\ s-4\pi & s\in\left[\frac{4\pi}{3},\frac{8\pi}{3}\right) \end{cases}$$

Notice $\sigma(F)=L$. Now $\{2^nF : n\in\mathbf{Z}\}$, $\{2^nL : n\in\mathbf{Z}\}$ are measurable partitions of $\mathbf{R}$, thus following [4] we extend σ to a bijection of the whole real line onto itself by $\sigma(\gamma)=2^n\sigma(2^{-n}\gamma)$ for $\gamma\in 2^nF$. Notice also $\sigma^2=Id_{\mathbf{R}}$. It can be proved that the operator U such that $\hat{U}g=g\circ\sigma$ belongs to the local-commutant $C_{\psi_1}(D,S)$ and maps ψ_1 to ψ_2. By Corollary 2 U maps every scaling function for the MRA corresponding to ψ_1 to a scaling function for the MRA corresponding to ψ_2. On the other hand, the operator V defined by

$$\hat{V}=M_{h_1}I+M_{h_2}\hat{U} \tag{2.7}$$

is unitary, belongs to $C_{\psi_1}(D,S)$, and maps this wavelet to the Littlewood-Paley wavelet (refer to [4] for the details). Next we prove that V *does not map any* scaling function of the MRA corresponding to the wavelet ψ_1 to a scaling function of the Littlewood-Paley MRA. Set $K=[-4\pi/3,2\pi/3)$. By Proposition 1 every scaling function for the MRA corresponding to the wavelet ψ_1 is of the form $(2\pi)^{-1/2}\mathcal{F}^{-1}(\mu\chi_K)$ where μ is a 2π-periodic unimodular function.

Notice that if γ is in $[-2\pi/3,-\pi/3)\cup[\pi/3,2\pi/3)$ then $\gamma-\pi$ is in $[-2\pi/3,-\pi/3)\cup[\pi/3,2\pi/3)$ modulo 2π-translations. Notice that $e^{i\gamma}\overline{\mu(\gamma)}\mu(\gamma-\pi)+e^{-i\gamma}\overline{\mu(\gamma-\pi)}\mu(\gamma)$ does not vanish identically in $[-\frac{2\pi}{3},-\frac{\pi}{3})\cup[\frac{\pi}{3},\frac{2\pi}{3})$. Set $\phi=(2\pi)^{-1/2}V\mathcal{F}^{-1}(\mu\chi_K)$. Assume that γ satisfies $\pi/3\le|\gamma|\le(2\pi)/3$. Then it is not hard to verify $\mu(\sigma(\gamma))=\mu(\gamma-\pi)$, therefore we have

$$\sqrt{2\pi}\hat{\phi}(\gamma)=h_1(\gamma)\mu(\gamma)+h_2(\gamma)\mu(\gamma-\pi)$$

Since ϕ is a scaling function, by Poisson Summation formula, we have

$$|h_1(\gamma)\mu(\gamma)+h_2(\gamma)\mu(\gamma-\pi)|\le 1$$

For $\gamma\in[\frac{\pi}{3},\frac{2\pi}{3})$ we get

$$h_1(\gamma)=e^{i\gamma}\sin\left[\frac{\pi}{2}\nu\left(\frac{3\gamma}{\pi}-1\right)\right] \quad , \quad h_2(\gamma)=e^{2i\gamma}\cos\left[\frac{\pi}{2}\nu\left(\frac{3\gamma}{\pi}-1\right)\right]$$

while for $\gamma\in[\frac{-2\pi}{3},\frac{-\pi}{3})$

$$h_1(\gamma)=e^{2i\gamma}\cos\left[\frac{\pi}{2}\nu\left(-\frac{3\gamma}{\pi}-1\right)\right] \quad , \quad h_2(\gamma)=e^{i\gamma}\sin\left[\frac{\pi}{2}\nu\left(-\frac{3\gamma}{\pi}-1\right)\right]$$

In both cases for all such γ we have $|h_1(\gamma)|^2 + |h_2(\gamma)|^2 = 1$. For $\gamma \in [\frac{\pi}{3}, \frac{2\pi}{3})$ we have

$$|h_1(\gamma)\mu(\gamma) + h_2(\gamma)\mu(\gamma - \pi)|^2 = 1 + \frac{1}{2}\sin\left[\pi\nu\left(\frac{3\gamma}{\pi} - 1\right)\right]\left[e^{i\gamma}\overline{\mu(\gamma)}\mu(\gamma - \pi) + e^{-i\gamma}\overline{\mu(\gamma - \pi)}\mu(\gamma)\right]$$

while for $\gamma \in [-\frac{2\pi}{3}, -\frac{\pi}{3})$

$$|h_1(\gamma)\mu(\gamma) + h_2(\gamma)\mu(\gamma - \pi)|^2 = 1 + \frac{1}{2}\sin\left[\pi\nu\left(-\frac{3\gamma}{\pi} - 1\right)\right]\left[e^{-i\gamma}\overline{\mu(\gamma)}\mu(\gamma - \pi) + e^{i\gamma}\overline{\mu(\gamma - \pi)}\mu(\gamma)\right]$$

The assumptions on ν imply $\sin[\pi\nu(\pm\frac{3\gamma}{\pi} - 1)] \geq 0$. Now let $\gamma \in [-\frac{2\pi}{3}, -\frac{\pi}{3}) \cup [\frac{\pi}{3}, \frac{2\pi}{3})$. Then for $\gamma_1 := \gamma + \pi \in [\frac{\pi}{3}, \frac{2\pi}{3}) \cup [-\frac{2\pi}{3}, -\frac{\pi}{3})$ we get

$$e^{-i\gamma}\overline{\mu(\gamma)}\mu(\gamma - \pi) + e^{i\gamma}\overline{\mu(\gamma - \pi)}\mu(\gamma) = -\left[e^{i\gamma_1}\overline{\mu(\gamma_1)}\mu(\gamma_1 - \pi) + e^{-i\gamma_1}\overline{\mu(\gamma_1 - \pi)}\mu(\gamma_1)\right]$$

The last equation shows that there exists a non-null subset of $[-\frac{2\pi}{3}, -\frac{\pi}{3}) \cup [\frac{\pi}{3}, \frac{2\pi}{3})$ such that for every γ in this set we have $|h_1(\gamma)\mu(\gamma) + h_2(\gamma)\mu(\gamma - \pi)| > 1$, which is absurd because we assumed that ϕ is a scaling function.

The unitary operator V of the previous example was derived by a simple operator-theoretic interpolation between U and the identity operator [recall (2.7)]. However, V exhibits a significant change in its behavior concerning scaling functions, although both U and the identity operator map every scaling function of the MRA defined by $(2\pi)^{-1/2}\mathcal{F}^{-1}(\chi_K)$ to a scaling function. Combining the previous example, Corollary 2, and the discussion following Corollary 1, we conclude that the local-commutant techniques do not apply in the study of scaling functions.

Question. Does the conclusion of the previous example extend to arbitrary interpolation pairs? For the definition of interpolation pair see [4, Ch. 5]. □

3. Scaling Functions and Low-Pass Filters

As it was mentioned in the introduction our operator-theoretic characterization of scaling functions does not lead to a general construction method of scaling functions. Thus we have to employ classical analytic methods for their construction. Next we deal with this problem, we present the difficulties and related open questions. Scaling functions determine uniquely their corresponding low-pass filter by the two-scale relation (1.1). As it was shown in [16] this is not an one-to-one correspondence. Scaling functions having the same low-pass filters produce MRAs, which were called equivalent and a characterization of all such scaling functions was given in [16]. Set $\mu = \frac{1}{\sqrt{2}}m$. If

$$\frac{1}{\sqrt{2\pi}}\prod_{n=1}^{\infty}\mu\left(\frac{\gamma}{2^n}\right) \tag{3.1}$$

converges almost everywhere in **R**, then Mallat [15] proved that this infinite product converges to a square-integrable function defined on **R**. The a.e. convergence of the infinite product (3.1) is not an immediate consequence of the low-pass filter definition. In fact up to the best of our knowledge there exist only sufficient conditions (nevertheless mild) for the a.e. convergence of the infinite product (3.1). For more details see [11]. Such a sufficient condition is the Hölder continuity of the low-pass filter at the origin: If there exist δ and C positive constants such that for every $|\gamma| < \delta$ we have $|m(\gamma) - m(0)| < C|\gamma|^\alpha \ \alpha > 0$. Then the infinite product of (3.1) converges uniformly on compact subsets of **R**. On the other hand if ϕ is a scaling function such that $\hat{\phi}$ is continuous at the origin, then $\hat{\phi}$ is given by the infinite product (3.1). But this continuity property is not always the case.

Example 2. Equations $\hat{\phi}(\gamma) = (2\pi)^{-1/2}\frac{1-e^{-i\gamma}}{i\gamma}$ and $m = (1+e^{-i\cdot})/\sqrt{2}$ give a scaling function and the corresponding low-pass filter for the Haar MRA respectively. We can recover $\hat{\phi}$ from (3.1), but if we set

$$\hat{\omega}(\gamma) = \begin{cases} \hat{\phi}(\gamma) & \text{if } \gamma \geq 0 \\ -\hat{\phi}(\gamma) & \text{if } \gamma < 0 \end{cases}$$

then ω is scaling function for an MRA different from the Haar MRA and the corresponding low-pass filter for ω is again m [16, Cor. 2.12]. Notice that $\hat{\omega}$ cannot be given by (3.1) because of its discontinuity at the origin.

Therefore, it is natural to ask whether in every equivalence class of MRAs there exists an MRA with a scaling function which can be derived by (3.1). Based on the characterization of equivalent MRAs given in [16] this question is equivalent to the following: Given an arbitrary scaling function ϕ is it possible to find unimodular functions defined on **R** m_1 2π-periodic and h satisfying $h(2t) = h(t)$ a.e. such that $\mathcal{F}^{-1}(m_1 h\hat{\phi})$ is a scaling function which can be given by the infinite product (3.1)? If the answer is affirmative the infinite product (3.1) essentially leads to the construction of all scaling functions.

If the function ϕ is defined from the infinite product (3.1) it obviously satisfies the two-scale relation (1.1). But it is not obvious and it is not always true that $\{S^n\phi : n \in \mathbf{Z}\}$ is an orthonormal set. The discussion on this issue is beyond the purpose of this article, but we will return to it after Corollary 4 for a brief comment. So let us assume that the integer translates of ϕ form an orthonormal set. Define $V_0 := [S^n\phi : n \in \mathbf{Z}]^-$ and $V_j = D^j(V_0)$, $j \in \mathbf{Z}$. Then $\{V_j\}_j$ is an increasing sequence of subspaces of $L^2(\mathbf{R})$. It can be proved that $\cap_j V_j$ is trivial, but $\overline{\cup_j V_j} = L^2(\mathbf{R})$ may fail. The last equality is true if and only if

$$\lim_{j\to\infty} \left|\hat{\phi}\left(\frac{\gamma}{2^j}\right)\right| = \frac{1}{\sqrt{2\pi}} \qquad \text{a.e.}$$

For proofs of the last two results see [11, Th. 7.5.2]. Thus we come naturally to our next issue: How to find low-pass filters associated with scaling functions. Our next main result will show the connection the unitary operators which map a scaling function to every other function produce between their corresponding low-pass filters.

Let $\{e_n\}_n$ be the usual orthonormal basis of $l^2(\mathbf{Z})$. Set T for the bilateral shift of multiplicity one on $l^2(\mathbf{Z})$, i.e., $Te_n = e_{n+1}$ for every integer n. Let M be the multiplicative operator defined by $Mf(t) = e^{it}f(t)$, $f \in L^2([-\pi,\pi))$.

Definition 3.

The operator $\mathcal{D}$ defined by

$$\mathcal{D}c(n) = c(2n)\,, \quad c \in l^2(\mathbf{Z})$$

is called downsampling operator.

Remark 3.

Set F for the isomorphism between $l^2(\mathbf{Z})$ and $L^2([-\pi,\pi))$ defined by $Fe_n = e^{-in\cdot}$ for every $n \in \mathbf{Z}$. The adjoint of $\mathcal{D}$ is often called *upsampling* operator, is an isometry, and is denoted by $\mathcal{U}$. □

We now recall a few facts that we use in the rest of this section which are now folk-wisdom in wavelet literature. We present them with a rigorous operator-theoretic formulation since we intent to use them in the rest of our article. Assume that m is a low-pass filter and $m = \sum_n h_n e^{-in\cdot}$. It is straightforward that $(h_n)_n$ belongs to $l^2(\mathbf{Z})$. Since m is essentially bounded, it yields the multiplicative operator $\mathbf{H}_0$ defined by

$$\mathbf{H}_0 f = \overline{m} f \quad , \quad f \in L^2([-\pi, \pi))$$

Then $F^{-1}\mathbf{H}_0 F$ is a bounded operator acting on $l^2(\mathbf{Z})$. Now set $H_0 = F^{-1}\mathbf{H}_0 F$ and $H = \mathcal{D}H_0$. Similarly we define $\mathbf{G}_0$ by

$$\mathbf{G}_0 f = e^{i\cdot}\overline{m(.+\pi)} f \quad , \quad f \in L^2([-\pi, \pi))$$

Set $G_0 = F^{-1}\mathbf{G}_0 F$ and $G = \mathcal{D}G_0$. It is easy to verify

$$HH^* = GG^* = H^*H + G^*G = I \qquad GH^* = HG^* = 0 \tag{3.2}$$

It can also easily be verified that H_0 and G_0 commute with T and $T\mathcal{D} = \mathcal{D}T^2$. Similarly we can prove $T^{-1}\mathcal{D} = \mathcal{D}T^{-2}$. Therefore, we get

$$TH = HT^2 \qquad TG = GT^2 \tag{3.3}$$

and

$$T^{-1}H = HT^{-2} \qquad T^{-1}G = GT^{-2} \tag{3.4}$$

Equation (3.3) shows that the k-column of H^* is of the zero-column of H^* shifted by $2k$. This situation can be generalized in the case of low-pass filters for MRA in $L^2(\mathbf{R}^n)$ in the sense of [9]. We proceed now to the low-pass filter paramatrization. For an arbitrary bounded operator A acting on $l^2(\mathbf{Z})$ and the usual orthonormal basis of $l^2(\mathbf{Z})$ $\{e_n : n \in \mathbf{Z}\}$ we adopt the notation $A_{k,n} = \langle Ae_n, e_k \rangle$.

We have already mentioned in the previous section that it is difficult to construct the unitary operators of Theorem 1. On the other hand, these operators yield an elegant relation between the low-pass filters corresponding to MRAs as shown by the following theorem.

Theorem 2.

Let m_0 be a low-pass filter associated with a given MRA. If m is another low-pass filter associated with an MRA, then there exists a unique unitary operator W acting on $L^2([-\pi, \pi))$, commuting with M^2 such that $m = Wm_0$ and $W(e^{i\cdot}\overline{m_0(.+\pi)}) = e^{i\cdot}\overline{m(.+\pi)}$.

Proof. Assume that m_0 and m are associated with the MRAs $\mathcal{M}$ and $\mathcal{N}$, respectively. Assume that ϕ, ψ and $\tilde{\phi}$, $\tilde{\psi}$ are scaling function and orthonormal wavelet for $\mathcal{M}$ and $\mathcal{N}$, respectively. Let U be a unitary operator satisfying properties (2.1) and (2.2) of Theorem 1. Let H, G be the contractive mappings corresponding to m_0 and $\tilde{H}$, $\tilde{G}$ be those corresponding to m. For every integer k we have

$$\phi_{1,k} = \sum_n H_{n,k}\phi_{0,n} + \sum_n G_{n,k}\psi_{0,n}$$

By the definition of $\tilde{H}$ and $\tilde{G}$ one can verify

$$\tilde{\phi}_{0,n} = \sum_m \left(\tilde{H}^*\right)_{m,n} \tilde{\phi}_{1,m} \qquad \text{and} \qquad \tilde{\psi}_{0,n} = \sum_m \left(\tilde{G}^*\right)_{m,n} \tilde{\phi}_{1,m}$$

Combining the previous equations we get

$$U\phi_{1,k} = \sum_m \left(\tilde{H}^*H + \tilde{G}^*G\right)_{m,k} \tilde{\phi}_{1,m} \tag{3.5}$$

But U maps V_1 onto $\tilde{V}_1$ isometrically, thus $\tilde{H}^*H + \tilde{G}^*G$ is a unitary operator acting on $l^2(\mathbf{Z})$, which we denote by $\mathcal{I}$. Using (3.2), (3.3), and (3.4) one can easily see $\mathcal{I}T^2 = T^2\mathcal{I}$ and

$$\tilde{H}^* = \mathcal{I}H^* \qquad \tilde{G}^* = \mathcal{I}G^*$$
$$\tilde{H}^*e_0 = \mathcal{I}H^*e_0 \quad \tilde{G}^*e_0 = \mathcal{I}G^*e_0$$

Set $W = F\mathcal{I}F^{-1}$. Since $\mathcal{I} \in \{T^2\}'$ we get that W commutes with M^2 and

$$m = Wm_0 \qquad W\left(e^{i\cdot}\overline{m_0(.+\pi)}\right) = e^{i\cdot}\overline{m(.+\pi)}$$

since $m = F\tilde{H}^*e_0$ and $m_0 = FH^*e_0$.

Let W_1 be another unitary operator in $\{M^2\}'$ such that

$$m = W_1 m_0 \qquad W_1\left(e^{i\cdot}\overline{m_0(.+\pi)}\right) = e^{i\cdot}\overline{m(.+\pi)}$$

Equivalently $\tilde{H}^*e_0 = (F^{-1}W_1F)H^*e_0$ and $\tilde{G}^*e_0 = (F^{-1}W_1F)G^*e_0$, thus $\tilde{H}^* = (F^{-1}W_1F)H^*$ and $\tilde{G}^* = (F^{-1}W_1F)G^*$. Therefore, we have $F^{-1}W_1F = \tilde{H}^*H + \tilde{G}^*G$ which gives $W = W_1$. □

Equation (3.5) shows exactly how the unitary operators of Theorem 1 imply the celebrated in the introduction relation between low-pass filters of scaling functions. During the final preparations of this article, we discovered that a version of Theorem 2 concerning quadrature mirror filters (QMF) was independently obtained in [12] by Holschneider and Pincall. They use the concept of loop groups developed in [18] (see, also [13, Chapter 3]) to study QMF. Our motivation and techniques used in the proofs are entirely different although some of the underlying ideas are eventually the same. A QMF is a pair of functions a and b in $L^\infty([-\pi,\pi))$ such that $|a(\gamma)|^2 + |a(\gamma+\pi)|^2 = 1$, $|b(\gamma)|^2 + |b(\gamma+\pi)|^2 = 1$ and $a(\gamma)\overline{b(\gamma)} + a(\gamma+\pi)\overline{b(\gamma+\pi)} = 0$ a.e. QMF and low-pass filters are in one-to-one correspondence. To see this for every low-pass filter m, set $a = m$ and $b = e^{i\cdot}\overline{m(.+\pi)}$. For QMF-theory refer to [13]. It is implicit in the proof of Theorem 2 that it gives in fact a unitary parametrization of all low-pass filters not only of those associated with scaling functions.

Our next objective is to find the form of all unitary operators W commuting with M^2 using standard operator-theoretic techniques [1]. To accomplish this task set $L^2([-\pi,\pi),\mathbf{C}^2)$ for the Hilbert space of all functions defined on $[-\pi,\pi)$ taking values in $\mathbf{C}^2$ such that their coordinate functions are in $L^2([-\pi,\pi))$. Let $f = (f_1, f_2)$ and $g = (g_1, g_2)$ be two functions in $L^2([-\pi,\pi),\mathbf{C}^2)$. The inner product of $L^2([-\pi,\pi),\mathbf{C}^2)$ is given by

$$< f, g >= \int_{-\pi}^{\pi} (f_1\overline{g_1} + f_2\overline{g_2})\, d\lambda$$

Set $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$. We can identify $L^2([-\pi,\pi))$ and $L^2([-\pi,\pi),\mathbf{C}^2)$ via the isometric isomorphism V defined by the equations

$$\begin{aligned} Ve^{i2n\cdot} &= e^{in\cdot}\mathbf{i} \\ Ve^{i(2n+1)\cdot} &= e^{in\cdot}\mathbf{j} n \in \mathbf{Z} \end{aligned} \tag{3.6}$$

If M_z is defined by $M_zf = e^{i\cdot}f$, $f \in L^2([-\pi,\pi),\mathbf{C}^2)$, then it is easy to see $VM^2V^{-1} = M_z$. Also $\{M_z\}'$ (equivalently $\{M^2\}'$) is a von Neumann algebra. If Y is an arbitrary element of $\{M^2\}'$ then $VYV^{-1}$is of the form

$$VYV^{-1}f(t) = A(t)f(t) \qquad f \in L^2\left([-\pi,\pi),\mathbf{C}^2\right)$$

where $A(t)$ is a 2×2 matrix such that $t \to \|A(t)\|$ is essentially bounded. The 2×2 matrix representing the unitary operator VWV^{-1} must be unitary for almost every t in $[-\pi, \pi)$. We adopt the notation $\mathcal{U}(\{M^2\}')$ for the group of unitary operators in $\{M^2\}'$. It is also well known that this group is norm-connected (see [14]). When $A(t)$ is an a.e. unitary 2×2 matrix, there exist measurable functions A_1, A_2 and an a.e. unimodular function c defined on $[-\pi, \pi)$ satisfying $|A_1(t)|^2 + |A_2(t)|^2 = 1$ a.e. such that $A(t)$ is for almost every t in $[-\pi, \pi)$ of the form

$$\begin{pmatrix} A_1(t) & A_2(t) \\ -c(t)\overline{A_2(t)} & c(t)\overline{A_1(t)} \end{pmatrix}$$

If f is an arbitrary function in $L^2([-\pi, \pi))$ and f_e is the even part of f, i.e., $\frac{f(t)+f(t+\pi)}{2}$ and f_o is the odd part of f, i.e., $\frac{f(t)-f(t+\pi)}{2}$, (3.6) yields

$$Wf(t) = \left[A_1(2t) - e^{it}c(2t)\overline{A_2(2t)}\right] f_e(t) + \left[e^{-it}A_2(2t) + c(2t)\overline{A_1(2t)}\right] f_o(t) . \qquad (3.7)$$

The next corollary summarizes the previous argument.

Corollary 3.

Let A_1, A_2 be measurable functions defined on $[-\pi, \pi)$ such that $|A_1(t)|^2 + |A_2(t)|^2 = 1$ a.e. and c be an almost everywhere unimodular function defined on $[-\pi, \pi)$. Then the operator W defined by (3.7) is a unitary operator acting on $L^2([-\pi, \pi))$ commuting with M^2 and vice versa.

Let m_0 be a low-pass filter associated with an MRA. It can be proved by using the previous corollary that Wm_0 is again low-pass filter (not necessarily associated with an MRA). By the norm-connectedness of $\mathcal{U}(\{M^2\}')$ and the continuity of the map $U \to Um_0$ we have the following corollary.

Corollary 4.

The subset of $L^\infty([-\pi, \pi))$ consisted of low-pass filters is $L^2([-\pi, \pi))$-norm pathwise connected. In particular given any two low-pass filters associated with MRAs there exists a $L^2([-\pi, \pi))$-norm continuous path of low-pass filters (not necessarily associated with MRAs) connecting them.

Bonami, Durand and Weiss proved that the set of low-pass filters in $C^\infty([-\pi, \pi))$ and the set of low-pass filters in $C^\infty([-\pi, \pi))$ associated with scaling functions are pathwise-connected with respect to the topology of the Frechet space $C^\infty([-\pi, \pi))$ [2]. These are the first connectivity results for low-pass filters. After acceptance of the present paper we were informed that Garrigos generalized the previous Bonami, Durand and Weiss' results for the corresponding classes of low-pass filters contained in the Sobolev spaces $H_a([-\pi, \pi))$ for $a > 1/2$ [8].

One of the most interesting cases is when A_1 and A_2 are trigonometric polynomials, because this is exactly the case of low-pass filters which are trigonometric polynomials. In the sequel (Example 3) we show how we use Theorem 2 for an alternative construction of Daubechies' compactly supported scaling functions. If we set

$$A_1(t) = \frac{1 + e^{it}}{2} \quad A_2(t) = \frac{e^{it} - e^{2it}}{2} \quad c = 1 \quad t \in [-\pi, \pi)$$

we get a unitary operator W as in Corollary 3 such that if m_0 is the low-pass filter $m_0 = \frac{1}{\sqrt{2}}(1 + e^{-i\cdot})$ (for Haar MRA), then $Wm_0 = \frac{1}{\sqrt{2}}(1 + e^{-3i\cdot})$. In [3], Cohen studies the orthonormality of the set of the integer translations of the function defined as the Fourier transform of the infinite product (3.1). In the case $\mu = Wm_0/\sqrt{2}$ this set is not orthonormal. Conditions on the low-pass filters necessary and sufficient or simply sufficient for this set to be orthonormal are known as orthogonality conditions (see [3, 11, 15]).

Now let $m_0 = \sqrt{2}\chi_{[-\pi/2, \pi/2)}$ (this is a low-pass filter for Shannon MRA) and

$$A_1(t) = \frac{1 + \widehat{\mathcal{D}}\left(e^{it}r(t)\right)}{2} \qquad A_2(t) = \frac{e^{-it} + \widehat{\mathcal{D}}\left(e^{it}r(t)\right)}{2} \quad , \quad c = 1 \, , \quad t \in [-\pi, \pi)$$

where $\widehat{\mathcal{D}} = F\mathcal{D}F^{-1}$ and

$$r(t) = \begin{cases} 1 & \text{if } -\pi/2 \leq x < \pi/2 \\ -1 & \text{elsewhere} \end{cases}$$

Then Wm_0 is a low-pass filter for the Haar MRA. Of particular interest is to find the form of the unitary operators W which map $\sqrt{2}\chi_{[-\pi/2,\pi/2)}$ to a low-pass filter of the form $\sqrt{2}\chi_E$ (not necessarily associated with an MRA), where E is an appropriate Borel set. As in the proof of Proposition 1, E is such that $(E+\pi)$ is 2π-translation congruent to $[-\pi,\pi)\setminus E$ and vice-versa. Set

$$r_E(t) = \begin{cases} 1 & \text{if } x \in E \\ -1 & \text{elsewhere} \end{cases}$$

One can easily verify that these operators W correspond exactly to the 2×2 matrix form

$$A_{11}(t) = 1 \qquad A_{22}(t) = \frac{\widehat{\mathcal{D}}\left(e^{it} r_E(t)\right)}{\widehat{\mathcal{D}}\left(e^{it} r(t)\right)} \quad , \quad A_{21} = A_{12} = 0\ , \quad t \in [-\pi,\pi)$$

It is interesting to notice that for such unitaries W we always have $W^2 = I$.

Example 3. Once again set $m_0(t) = \frac{1}{\sqrt{2}}(1+e^{-it\cdot})$. The low-pass filters with four real coefficients can be obtained by the 2×2 unitary matrices $A(t)$ of the form

$$\begin{pmatrix} h_0 + h_1 e^{-it} & h_2 + h_3 e^{it} \\ -h_2 - h_3 e^{-it} & h_0 + h_1 e^{it} \end{pmatrix}$$

We also want $Wm_0(0) = \sqrt{2}$. The last equation combined with the fact that $A(t)$ is an a.e. unitary matrix lead to the following system of equations:

$$\begin{aligned} h_0 + h_1 &= 1 \\ h_0^2 + h_1^2 + h_2^2 + h_3^2 &= 1 \\ h_1 h_0 + h_2 h_3 &= 0 \end{aligned}$$

Let $h_0 = a \geq 0$. Then $h_1 = 1 - a$. We also get $h_2^2 = h_0 h_1$, thus $0 \leq a \leq 1$. An elementary calculation gives

$$\begin{aligned} h_2 &= \pm\sqrt{a(1-a)} \qquad\qquad h_3 = \mp\sqrt{a(1-a)} \\ Wm_0(t) &= \frac{1}{\sqrt{2}}\left[(h_1 - h_2)\,e^{it} + (h_0 + h_3) + (h_0 - h_3)\,e^{-it} + (h_1 + h_2)\,e^{-2it}\right]. \end{aligned}$$

If we want to obtain a scaling function with two vanishing moments, then we must have $(Wm_0)'(\pi) = 0$. This implies $a = 3/4$. Notice that with the particular low-pass filter m_0, if we want Wm_0 to be a trigonometric polynomial, (3.7) shows that it is enough to assume $c = 1$. Thus, the above form is a complete unitary paramatrization of all low-pass filters which are trigonometric polynomials with four real coefficients. Notice also that this procedure, which is generic, bypasses the spectral factorization process.

4. Open Problems

The proof of the fact that the unitary group in a von Neumann algebra is norm-connected relies on the construction of a norm-continuous path of unitary operators beginning from the identity and ending to each unitary operator in the algebra. Every such path lies in the algebra. Let m_0 and m be low-pass filters corresponding to scaling functions and $U \in \mathcal{U}(\{M^2\}')$ such that $m = Um_0$. Does every unitary operator in the path corresponding to U map m_0 to a low-pass filter associated with a scaling function?

Moreover, if both m_0 and m are trigonometric polynomials does this path contain unitary operators mapping m_0 to a trigonometric polynomial? If the answer is negative we may ask something stronger. Are the elements of $\mathcal{U}(\{M^2\}')$ which map a given trigonometric polynomial to a trigonometric polynomial extreme points of the unit sphere of the von Neumann algebra $\{M^2\}'$? Dai and Larson introduced operator-theoretic interpolation techniques in [4]. Since $\mathcal{U}(\{M^2\}')$ admits operator-theoretic interpolation, we expect to generate new filters, and thus new MRAs, by developing similar techniques.

Another perspective is an operator-theoretic formulation of orthonormality conditions. Can such a formulation yield a general solution for this problem? Can we state necessary and sufficient orthogonality conditions for more general classes of low-pass filters (even for simply continuous filters)?

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