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The Calderón Reproducing Formula, Windowed X-Ray Transforms, and Radon Transforms in L^p-Spaces

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ABSTRACT. The generalized Calderón reproducing formula involving "wavelet measure" is established for functions $f \in L^p(\mathbb{R}^n)$. The special choice of the wavelet measure in the reproducing formula gives rise to the continuous decomposition of f into wavelets, and enables one to obtain inversion formulae for generalized windowed X-ray transforms, the Radon transform, and k-plane transforms. The admissibility conditions for the wavelet measure μ are presented in terms of μ itself and in terms of the Fourier transform of μ.

1. Introduction

The classical Calderón reproducing formula reads

$$f = \int_0^\infty \frac{f * u_t * v_t}{t} dt , \tag{1.1}$$

where $u_t(x) = t^{-n}u(x/t)$, $v_t(x) = t^{-n}v(x/t)$, u and v are sufficiently nice normalized radial wavelet functions on $\mathbb{R}^n$ (see, e.g., [5]). The generalization of (1.1) involving nonradial wavelets u and v was given in [12] and can be written in the form

$$f = \int_{SO(n)} d\gamma \int_0^\infty \frac{f * u_{\gamma,t} * v_{\gamma,t}}{t} dt , \tag{1.2}$$

where $u_{\gamma,t}$ and $v_{\gamma,t}$ are rotated versions of u_t and v_t. In [5] and [12] it was assumed that $f \in L^2(\mathbb{R}^n)$.

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Since the properties of the operator in the right-hand side of (1.2) depend on the combination $u * v$ (not on u and v separately), it makes sense to investigate the formal integral

$$\int_{SO(n)} d\gamma \int_0^\infty \frac{f * \mu_{\gamma,t}}{t}\, dt \tag{1.3}$$

generated by the arbitrary Borel measure μ on $\mathbb{R}^n$. For $\mu = u * v$ this gives the integral in (1.2). In the case $n = 1$, such an investigation was carried out in [17, 18] for $f \in L^p(\mathbb{R})$. We also mention the papers [16, 22] devoted to (1.1) for $f \in L^p$. Holschneider [8] investigated the formula (1.2) in the case $n = 2$, $f \in L^p(\mathbb{R}^2)$ for certain distributions u and v such that their convolution $u * v$ is a regular function. Also, he has shown that if one of the distributions, say $u(x)$, has the form $u(x) = \delta(x_1) \times 1(x_2)$ with the delta function in the x_1-variable and v sufficiently nice, then (1.2) leads to the inversion formula for the Radon transform Rf on $\mathbb{R}^2$. The formula of Holschneider can be written in the form

$$f = \int_0^\infty R^\# \left(Rf * v_t^{(1)}\right) \frac{dt}{t^2}, \quad v^{(1)}(s) = \int_{-\infty}^{\infty} v\,(s, x_2)\, dx_2\,, \tag{1.4}$$

where $R^\#$ is the backprojection operator and the integral is interpreted as the limit of the corresponding truncated integral in the L^p-norm and in the a.e. sense.

Our goal is to generalize (1.4) to the n-dimensional case for k-plane transforms, $1 \le k \le n-1$. A natural generalization of (1.4) reads

$$f = \int_0^\infty P^\# A_{\alpha,t}\, \varphi \frac{dt}{t^{1+k}}, \quad \varphi\left(\zeta, x''\right) = \left(P_\zeta f\right)\left(x''\right)\,, \tag{1.5}$$

where

$$\left(A_{\alpha,t}\, \varphi\right)\left(\zeta, x''\right) = \int_{SO_\zeta(n-k)} \left(\varphi * \left(P_\zeta \alpha\right)_{\sigma,t}\right)\left(x''\right) d\sigma\,, \tag{1.6}$$

$\zeta \in G_{k,n}$ (the Grassmann manifold of k-dimensional subspaces in $\mathbb{R}^n$), $x'' \in \zeta^\perp$ (the orthogonal complement of ζ), $(P_\zeta f)(x'') \equiv (Pf)(\zeta, x'')$ is the k-plane transform of f, $P^\#$ is the dual of P; $SO_\zeta(n-k)$ is the subgroup of $SO(n)$ which consists of rotations of the subspace $\zeta^\perp$. Here α is an arbitrary finite measure, satisfying certain cancellation and growth conditions; $(P_\zeta \alpha)_{\sigma,t}$ denotes the rotated and dilated version of the k-plane transform of α. For α radial, the $SO_\zeta(n-k)$-component in (1.5) may be omitted. Precise definitions and statements are given below.

The convolution $\varphi * (P_\zeta \alpha)_{\sigma,t}$ can be regarded as the continuous wavelet transform of $\varphi(\zeta, x'')$ in the x''-variable. The structure of the formula (1.5) is *a priori* transparent if we take into account the classical inversion formula (see, e.g., [10])

$$f = c_{k,n} P^\# (-\Delta)^{k/2} \varphi, \quad \varphi = Pf\,, \tag{1.7}$$

involving the Laplacian Δ in the x''-variable and realizing the so-called convolution-backprojection algorithm used in modern CT-scanners. By making use of the general wavelet type representation of $(-\Delta)^\lambda$, $\lambda \in \mathbb{C}$, in $\mathbb{R}^n$, given by

$$(-\Delta)^\lambda \psi = \int_{SO(n)} d\gamma \int_0^\infty \frac{\psi * \mu_{\gamma,t}}{t^{1+\lambda}}\, dt \tag{1.8}$$

with a normalized "wavelet measure" μ, one can readily get (1.5) from (1.7) and (1.8). The equality (1.8) can be checked easily by the formal application of the Fourier transform. The details related to the inversion of P via (1.7) and (1.8) can be found in [20, 21] (concerning wavelet type representations of the operator $(-\Delta)^\lambda$, see [18, 19]).

Thus, two approaches can be applied to the inversion of Radon transforms in terms of continuous wavelet transforms. The first one is that of Holschneider which is based on the reproducing formula. The second one employs the wavelet type representation of positive powers of the Laplacian. The advantage of the first approach is that we do not use (1.7). This observation may be helpful in more general situations related to Radon transforms on some manifolds rather than planes.

Our investigation was also motivated by the following reasons:

(a) In order to recover f from $g = Pf$ it suffices to know the wavelet transforms of g.

(b) If α in (1.5) is well localized, then (1.5) gives a local (more precisely, "quasi-local") reconstruction of f (up to the dilation by t). This may be important for the case of k odd when $(-\Delta)^{k/2}$ is nonlocal in principle.

(c) If the Fourier transform $\hat{\alpha}$ is well localized, then the continuous wavelet transform in (1.5) and (1.6)) serves as a filter in the frequency domain.

(d) By making use of (1.3) one can invert the generalized windowed X-ray transforms

$$(X_\nu f)(x, v) = \int_{-\infty}^{\infty} f(x + tv) d\nu(t); \quad x, v \in \mathbb{R}^n . \tag{1.9}$$

In the case of an absolutely continuous measure ν with the density $g(t)$ such transforms were studied by Kaizer and Streater [9] in connection with applications in physics. Here g serves as a window function in the time variable t.

The article is organized as follows. In Section 2 the integral (1.3) is examined for $f \in L^p$. It is shown that this integral coincides with f for the wide class of admissible measures μ, and convergence of the integral can be interpreted in the L^p-norm and in the a.e. sense. In Section 3 we reformulate the results of Section 2 for the case of two measures when $\mu = \mu^{(1)} * \mu^{(2)}$. Such a reformulation gives a decomposition of f into the integral of wavelet functions (or measures). Section 4 is devoted to the explicit inversion of the windowed X-ray transform (1.9) of $f \in L^p(\mathbb{R}^n)$. Sections 5 and 6 contain a generalization of Holschneider's method for the usual Radon transform $(k = n - 1)$ and for k-plane transforms, respectively.

One should mention the papers by Berenstein and Walnut [1] and Walnut [28], which are also devoted to studying the Radon transform by using wavelets. Our approach and technique differ from those in these papers.

Notation. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ we write $\langle x, y\rangle = x_1 y_1 + \dots + x_n y_n$. Let $e_1 = (1, 0 \dots, 0)$, $\Sigma_{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$, $|\Sigma_{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$; $[a]$ is the integer part of the real number a. Given a function $k(x)$ on $\mathbb{R}^n$ and $\varepsilon > 0$ (instead of ε there may be t, ρ or another letter) we denote $k_\varepsilon(x) = \varepsilon^{-n} k(x/\varepsilon)$. The notation $C(\mathbb{R}^n)$, $C^\infty(\mathbb{R}^n)$, $L^p(\mathbb{R}^n)$ is standard; $C_0(\mathbb{R}^n) = \{f \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} f(x) = 0\}$; $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of rapidly decreasing C^∞-functions with a standard topology; $\mathcal{S}'(\mathbb{R}^n)$ is the dual of $\mathcal{S}(\mathbb{R}^n)$; $\Phi = \Phi(\mathbb{R}^n)$ is the subspace of $\mathcal{S}$, which consists of functions orthogonal to all polynomials. Given a finite Borel measure μ on $\mathbb{R}^n$, we denote by $\|\mu\|$ the total variation of $|\mu|$. The Fourier transform and its inverse are defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{i\langle x, \xi\rangle} dx, \quad \overset{\vee}{g}(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} g(\xi) e^{-i\langle x, \xi\rangle} d\xi .$$

Sometimes we use the abbreviations "$\lesssim$" and "$\simeq$" instead of "$\le$" and "$=$", respectively, if the corresponding relations hold up to a constant factor. □

2. Reproducing Formula with One Measure

Given a locally finite measure μ on $\mathbb{R}^n$, we denote by $\mu_{\gamma,t}$ the rotated and dilated version of μ such that

$$(\mu_{\gamma,t}, f) = \int_{\mathbb{R}^n} f(t\gamma y) d\mu(y), \qquad t > 0, \qquad \gamma \in SO(n),$$

for sufficiently nice f. Let

$$\left(W_\mu f\right)(x, \gamma, t) = \int_{\mathbb{R}^n} f(x - t\gamma y) d\mu(y) = f * \mu_{\gamma,t}. \tag{2.1}$$

In the case $d\mu(x) = g(x)dx, \quad g \in L^1_{loc}$, we also write

$$\left(W_g f\right)(x, \gamma, t) = \int_{\mathbb{R}^n} f(x - t\gamma y) g(y) dy = t^{-n} \int_{\mathbb{R}^n} f(y) g\left(\frac{\gamma^{-1}(x-y)}{t}\right) dy. \tag{2.2}$$

If $\mu(\mathbb{R}^n) = 0$, the integrals (2.1) and (2.2) will be called the wavelet transforms. Denote formally

$$I(\mu, f) = \int_{SO(n)} d\gamma \int_0^\infty \frac{f * \mu_{\gamma,t}}{t} dt = \int_{SO(n)} d\gamma \int_0^\infty \left(W_\mu f\right)(x, \gamma, t) \frac{dt}{t}. \tag{2.3}$$

The following statement is rather standard (cf. [5]).

Theorem 1.
Let μ be a finite Borel measure on $\mathbb{R}^n$ such that the integral

$$c_\mu = \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} \frac{\hat{\mu}(\eta)}{|\eta|^n} d\eta = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon < |\eta| < \rho} \frac{\hat{\mu}(\eta)}{|\eta|^n} d\eta \tag{2.4}$$

is finite. Then for $f \in L^2$,

$$I(\mu, f) = \overset{(L^2)}{\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}}} \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{f * \mu_{\gamma,t}}{t} dt = c_\mu f. \tag{2.5}$$

Proof. Let

$$I_{\varepsilon,\rho}(\mu, f) = \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{f * \mu_{\gamma,t}}{t} dt, \qquad 0 < \varepsilon < \rho < \infty, \tag{2.6}$$

and assume that $f \in L^1 \cap L^2$. Then $I_{\varepsilon,\rho}(\mu, f) \in L^1 \cap L^2$ and $(I_{\varepsilon,\rho}(\mu, f))^\wedge(\xi) = \hat{k}_{\varepsilon,\rho}(\xi) \hat{f}(\xi)$, where

$$\hat{k}_{\varepsilon,\rho}(\xi) = \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{\hat{\mu}(t\gamma\xi)}{t} dt = \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon|\xi| < |\eta| < \rho|\xi|} \frac{\hat{\mu}(\eta)}{|\eta|^n} d\eta, \qquad \xi \neq 0.$$

If c_μ is finite, then the function $\psi(r) = \int_{|y|<r} \hat{\mu}(\eta) d\eta / |\eta|^n$ is continuous on $[0, \infty]$, and there is a constant $A = \sup_{r>0} |\psi(r)|$ such that $|\hat{k}_{\varepsilon,\rho}(\xi)| \leq 2A/|\Sigma_{n-1}|$ for all $\rho > \varepsilon > 0$ and all $\xi \in \mathbb{R}^n$. By the Plancherel formula, this gives

$$\left\| I_{\varepsilon,\rho}(\mu, f) \right\|_2 = \left\| \hat{k}_{\varepsilon,\rho} \hat{f} \right\|_2 \leq 2A |\Sigma_{n-1}|^{-1} \|f\|_2 \tag{2.7}$$

and

$$\left\| I_{\varepsilon,\rho}(\mu, f) - c_\mu f \right\|_2 = \left\| \left(\hat{k}_{\varepsilon,\rho} - c_\mu \right) \hat{f} \right\|_2 \to 0 \qquad as \quad \varepsilon \to 0, \quad \rho \to \infty . \tag{2.8}$$

The result for arbitrary $f \in L^2$ follows from (2.7) and (2.8) by taking into account that $\| I_{\varepsilon,\rho}(f, \mu) \|_2 \le \|\mu\| \, \|f\|_2 \log(\rho/\varepsilon)$. □

Our next goal is to extend Theorem 1 to $f \in L^p$ and to present conditions for μ without using the Fourier transform.

For convenience of the reader we recall the following auxiliary lemma.

Lemma 1.

Let $k(x) \in L^1$, $k_\rho(x) = \rho^{-n}k(x/\rho)$. If $f \in L^p$, $1 < p < \infty$, then $\|f * k_\rho\|_p \to 0$ as $\rho \to \infty$. If $f \in C_0$, then $\sup_x |(f * k_\rho)(x)| \to 0$ as $\rho \to \infty$. If $f \in L^p$, $1 \le p < \infty$, and $k(x)$ has a decreasing integrable radial majorant, then $(f * k_\rho)(x) \to 0$ as $\rho \to \infty$ almost everywhere on $\mathbb{R}^n$.

Proof. The proof of the first two statements can be found in [18, Theorem 1.15]. The last statement follows in the usual way from the estimate $\sup_\rho |(f * k_\rho)(x)| \le c(Mf)(x)$ where $(Mf)(x)$ is the Hardy-Littlewood maximal function (cf. [25]). □

Definition 1.

A locally finite measure μ is called admissible if

$$k(x) \overset{def}{=} \frac{1}{|\Sigma_{n-1}|\,|x|^n} \int_{|y|<|x|} d\mu(y) \in L^1 . \tag{2.9}$$

Theorem 2.

Assume that f and μ are such that μ is admissible and the function $(\gamma, t) \to (|f| * |\mu|_{\gamma,t})(x)$ belongs to $L^1(SO(n) \times [\varepsilon, \rho])$ for all $0 < \varepsilon < \rho < \infty$ and almost all x.

(i) *If $f \in L^p$, $1 < p < \infty$, then*

$$I(\mu, f) = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{f * \mu_{\gamma,t}}{t} dt = k_0 f \tag{2.10}$$

where $\lim = \overset{(L^p)}{\lim}$,

$$k_0 = \int_{\mathbb{R}^n} k(x)dx , \tag{2.11}$$

$k(x)$ being defined by (2.9).

(ii) *If $f \in C_0$, then (2.10) holds with the limit interpreted in the C-norm.*

(iii) *If $f \in L^p$, $1 \le p < \infty$, and $k(x)$ has a decreasing integrable majorant, then (2.10) holds a.e. on $\mathbb{R}^n$.*

Proof. The truncated integral $I_{\varepsilon,\rho}(\mu, f)$ [see (2.6)] can be represented in the form

$$I_{\varepsilon,\rho}(\mu, f) = k_\varepsilon * f - k_\rho * f \tag{2.12}$$

where $k_\varepsilon(x) = \varepsilon^{-n}k(x/\varepsilon)$, $\quad k_\rho(x) = \rho^{-n}k(x/\rho)$. Indeed,

$$\begin{aligned} I_{\varepsilon,\rho}(\mu, f) &= \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{dt}{t} \int_{\mathbb{R}^n} f(x - t\gamma y) d\mu(y) = \int_{\mathbb{R}^n} d\mu(y) \int_\varepsilon^\rho \frac{dt}{t} \int_{SO(n)} f(x - t\gamma y) d\gamma \\ &= \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} d\mu(y) \int_{\varepsilon|y|<|z|<\rho|y|} f(x - z) \frac{dz}{|z|^n} \end{aligned}$$

$$= \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} f(x-z) \frac{dz}{|z|^n} \int_{|z|/\rho<|y|<|z|/\varepsilon} d\mu(y) = k_\varepsilon * f - k_\rho * f . \tag{2.13}$$

Now the statements of the theorem become obvious in view of the usual machinery of the approximation to the identity [23] and Lemma 1. □

The following statement gives examples of classes of measures that satisfy the conditions of Theorem 2.

Theorem 3.

Let μ be a finite Borel measure such that $\mu(\mathbb{R}^n) = 0$. Assume that

$$(a) \qquad \int_{\mathbb{R}^n} |log|x||d|\mu|(x) < \infty \tag{2.14}$$

or

$$(b) \qquad d\mu(x) = g(x)dx, \qquad g \in H^1(\text{the real Hardy space on } \mathbb{R}^n) . \tag{2.15}$$

Then μ is admissible and the constant k_0 in (2.11) can be evaluated as follows:

$$k_0 = \int_{\mathbb{R}^n} \log \frac{1}{|x|} d\mu(x) \tag{2.16}$$

in the case (a) and

$$k_0 = A_n \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{x_j}{|x|} (\mathcal{R}_j g)(x)dx, \qquad A_n = \frac{i\pi^{(n+1)/2}}{|\Sigma_{n-1}| \Gamma((n+1)/2)}, \tag{2.17}$$

($\mathcal{R}_j g$ being the Riesz transformations of g) in the case (b). Under these assumptions statements (i) and (ii) of Theorem 2 hold. If moreover,

$$\int_{|x|<1} |x|^{-\delta} d|\mu|(x) < \infty \quad \text{for some } \delta > 0 , \tag{2.18}$$

then, given $f \in L^p$, $1 \le p < \infty$, the relation (2.10) is valid for almost all x.

Proof. We first note that since μ is finite, then $I_{\varepsilon,\rho}(|\mu|, |f|) \in L^p$ for all $0 < \varepsilon < \rho < \infty$. If $\mu(\mathbb{R}^n) = 0$ and (2.14) holds, then μ is admissible because $k(x) \in L^1$. Indeed,

$$\begin{aligned} |\Sigma_{n-1}| \int_{\mathbb{R}^n} |k(x)|dx &\le \int_{|x|<1} \frac{dx}{|x|^n} \int_{|y|<|x|} d|\mu|(y) \\ &+ \int_{|x|>1} \frac{dx}{|x|^n} \int_{|y|>|x|} d|\mu|(y) \\ &= |\Sigma_{n-1}| \int_{\mathbb{R}^n} \big|\log|y|\big| d|\mu|(y) < \infty . \end{aligned}$$

Similarly one can show that $k_0 = \int_{\mathbb{R}^n} k(x)dx = \int_{\mathbb{R}^n} \log(1/|y|)d\mu(y)$. Furthermore, if μ satisfies (2.18), then $|k(x)| \le c_1|x|^{\delta-n}$ for $|x| \le 1$ and $k(x)$ has a decreasing radial summable majorant. Thus, the part related to (a) is proved.

In order to handle the case $d\mu(x) = g(x)dx$, $g \in H^1$ we need some facts from the theory of Hardy spaces (see, e.g., [3, 26]). We recall that a function $a(x)$ is called *an atom* if $a(x)$ is supported in a ball B, $|a(x)| \le |B|^{-1}$ and $\int a(x)dx = 0$.

Theorem 4.

A summable function g belongs to H^1 if and only if

$$g = \sum_{j=0}^{\infty} \lambda_j a_j \tag{2.19}$$

where a_j is an atom and $\Sigma_{j=0}^{\infty}|\lambda_j| < \infty$. If $g \in H^1$, then $\|g\|_{H^1}$ is equivalent to inf $\Sigma_{j=0}^{\infty}|\lambda_j|$ *where the infimum is taken over all decompositions (2.19).*

First we show that the operator $(Kg)(x) = |x|^{-n} \int_{|y|<|x|} g(y)dy$ is bounded from H^1 into L^1. Let $g(x) \equiv a(x)$ be an atom supported by the ball B of radius r. Then $|(Ka)(x)| \le r^{-n}$. Moreover, $\|Ka\|_1 \le c_n$, where c_n is a constant depending on n and independent of B. Indeed, if $0 \in \overline{B}$, then

$$\|Ka\|_1 = \int_{|x|<2r} \left| \int_{|y|<|x|} a(y)dy \right| \frac{dx}{|x|^n} \le \frac{1}{r^n} \int_{|x|<2r} dx = 2^n |B_1| \ .$$

If $0 \notin \overline{B}$ and r_0 $(> r)$ is the distance between 0 and the center of B, then

$$\|Ka\|_1 = \int_{r_0-r}^{r_0+r} \frac{d\rho}{\rho} \left| \int_{|y|<\rho,\ y\in B} a(y)dy \right| \le \frac{1}{|B|} \int_{r_0-r}^{r_0+r} \frac{|C_{\rho,r}|}{\rho} d\rho = c_n$$

[here $C_{\rho,r}$ is the cylinder of the height ρ with the base $B_r^{(n-1)}$ (the $(n-1)$-dimensional ball of radius r)].

Now let g have a general form (2.19). Since $\|a_j\|_1 \le 1$ and $\Sigma_{j=0}^{\infty}|\lambda_j| < \infty$, the series (2.19) converges in the L^1-norm. Since $\|Ka_j\|_1 \le c_n$, the same is true for the series $K_1 g = \Sigma_{j=0}^{\infty}\lambda_j Ka_j$, and $\|K_1 g\|_1 \le c_n \Sigma_{j=0}^{\infty}|\lambda_j| < const\ \|g\|_{H^1}$. It remains to show that $Kg = K_1 g$. Let B_r be a ball of radius r centered in the origin,

$$L_{r,\delta} = \left\{ f : \quad \|f\|_{r,\delta} = \int_{B_r} |f(x)||x|^{\delta} dx < \infty \right\}, \qquad \delta > 0\,.$$

Since K is bounded from L^1 into $L_{r,\delta}$, then

$$Kg = K \left(\overset{(L^1)}{\lim_{N\to\infty}} \sum_{j=1}^{N} \lambda_j a_j \right) = \overset{L_{r,\delta}}{\lim_{N\to\infty}} \sum_{j=1}^{N} \lambda_j K a_j = \sum_{j=1}^{\infty} \lambda_k K a_j = K_1 g$$

because the L^1-convergence is stronger than that in the $L_{r,\delta}$-norm. Thus,

$$\|k\|_1 = \|Kg\|_1 \le c\|g\|_{H^1}, \qquad g \in H^1\,, \tag{2.20}$$

and therefore $d\mu(x) = g(x)dx$ is an admissible measure.

Let us show that

$$k_0 \overset{\text{def}}{=} \tfrac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} (Kg)(x)dx = A_n \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{x_j}{|x|} \left(\mathcal{R}_j g\right)(x)dx\,, \tag{2.21}$$

$$A_n = \frac{i\pi^{(n+1)/2}}{|\Sigma_{n-1}|\,\Gamma((n+1)/2)}\,.$$

Since K and $\mathcal{R}_j$ are bounded from H^1 into L^1, it suffices to check (2.22) for functions g belonging to the space Φ (see Notation) which is dense in H^1 (see, [24, p. 128]). According to Theorem 1, for such g we have

$$k_0 = \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} \frac{\hat{g}(\xi)}{|\xi|^n} d\xi = \sum_{j=1}^{n} \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} \left(\mathcal{R}_j \mathcal{R}_j g\right)^\wedge(\xi) \frac{d\xi}{|\xi|^n} = \sum_{j=1}^{n} \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} (\mathcal{R}_j g_j)^\wedge(\xi) \frac{d\xi}{|\xi|^n}\,,$$

where

$$g_j(x) = \left(\mathcal{R}_j g\right)(x) = \frac{\Gamma((n+1)/2)}{i\pi^{(n+1)/2}} \text{ v.p.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} g(y)dy$$

are the Riesz transformations of g such that $(\mathcal{R}_j g)^\wedge(\xi) = \frac{\xi_j}{|\xi|}\hat{g}(\xi)$. By the Parseval equality it follows that

$$k_0 = \sum_{j=1}^{n} \frac{1}{|\Sigma_{n-1}|} \int_{\mathbb{R}^n} \frac{\xi_j}{|\xi|^{n+1}} \hat{g}_j(\xi)d\xi = \frac{i\pi^{(n+1)/2}}{|\Sigma_{n-1}|\,\Gamma((n+1)/2)} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{x_j}{|x|} \left(\mathcal{R}_j g\right)(x)dx$$

which was required. □

3. Reproducing Formula with Two Measures

In the previous section we exhibited inversion formulae for the transformation

$$\left(W_\mu f\right)(x, \gamma, t) = \int_{\mathbb{R}^n} f(x - t\gamma y)d\mu(y), \qquad t > 0, \quad \gamma \in SO(n)\,, \tag{3.1}$$

provided that μ is admissible. In practice, one often looks for the wavelet expansion of f or inverts $W_\mu f$ with the non-admissible μ. In these cases the results of Section 2 may be used if we put $\mu = \mu^{(1)} * \mu^{(2)}$, where $\mu^{(1)}$ is a wavelet measure (or function) with respect to which the expansion of f is needed, and $\mu^{(2)}$ is the original measure (or function).

Definition 2.

A pair of measures $\mu^{(1)}$, $\mu^{(2)}$ is called admissible if their convolution $\mu = \mu^{(1)} * \mu^{(2)}$ is admissible, i.e.,

$$k(x) = \frac{1}{|\Sigma_{n-1}|\,|x|^n} \int_{|y|<|x|} d\left(\mu^{(1)} * \mu^{(2)}\right)(y) \in L^1\,. \tag{3.2}$$

The following statement is a direct consequence of Theorem 2.

Theorem 5.

Let $\mu^{(1)}$, $\mu^{(2)}$ be an admissible pair of measures. Assume that f and $\mu = \mu^{(1)} * \mu^{(2)}$ are such that $\mu(\{0\}) = 0$ and the function $(\gamma, t) \to (|f| * |\mu|_{\gamma,t})(x)$ belongs to $L^1(SO(n) \times [\varepsilon, \rho])$ for all $0 < \varepsilon < \rho < \infty$ and almost all x.

(i) *If $f \in L^p$, $1 < p < \infty$, then*

$$f(x) = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \frac{1}{k_0} \int_{\varepsilon}^{\rho} \frac{dt}{t} \int_{SO(n)} d\gamma \int_{\mathbb{R}^n} \left(W_{\mu^{(2)}} f\right)(x - t\gamma y, \gamma, t)d\mu^{(1)}(y)\,, \tag{3.3}$$

$\lim = \overset{(L^p)}{\lim}$, *provided that $k_0 = \int_{\mathbb{R}^n} k(x)dx \neq 0$.*

(ii) *If $f \in C_0$, then (3.3) holds with the uniform convergence.*

(iii) *If f and $\mu = \mu^{(1)} * \mu^{(2)}$ satisfy* (iii) *in Theorem 2, then (3.3) holds for almost all $x \in \mathbb{R}^n$.*

If $d\mu^{(1)}(y) = g(y)dy$, then (3.3) can be written in the usual form as the wavelet expansion of f. In order to see this let $\lambda = (y, t, \gamma) \in G = \mathbb{R}^n \times \mathbb{R}^+ \times SO(n)$, $d\lambda = dy\frac{dt}{t}d\gamma$, $g_\lambda(x) = t^{-n}g(\frac{\gamma^{-1}(x-y)}{t})$. Then (3.3) reads

$$f(x) = \frac{1}{k_0} \int_G \left(W_{\mu^{(2)}} f\right)(\lambda)g_\lambda(x)d\lambda\,. \tag{3.4}$$

Note that if $\mu = \mu^{(1)} * \mu^{(2)}$ is radial (i.e., μ is invariant under rotation), then the $SO(n)$-component in all formulae in Sections 2 and 3 may be omitted. In this case, (3.3) and (3.4) coincide with the classical Calderón reproducing formula (cf. [5]).

4. Windowed X-Ray Transforms

The results of Sections 2 and 3 can be applied to generalization of the notion of the windowed X-ray transform (see [9]) and enable us to obtain explicit inversion formulae involving continuous wavelet transforms.

Let $y = (y_1, y') \in \mathbb{R}^n$, $y' = (y_2, \ldots, y_n) \in \mathbb{R}^{n-1}$. We apply the consideration of preceding sections to the measure $\mu(y) = \nu(y_1) \times \delta(y')$ where ν is a certain measure on $\mathbb{R}^1$ and $\delta(y')$ designates the usual delta function on $\mathbb{R}^{n-1}$. According to Definition 1, μ is admissible if

$$k(x) = \frac{1}{|\Sigma_{n-1}||x|^n} \int_{|y|<|x|} d\mu(y) = \frac{1}{|\Sigma_{n-1}||x|^n} \int_{(-|x|,|x|)} d\nu \in L^1\left(\mathbb{R}^n\right)$$

or

$$\tilde{k}(r) = \frac{1}{4} \int_{(-r,r)} d\nu\,(y_1) \in L^1\left(\mathbb{R}^+\right) . \tag{4.1}$$

Clearly, $\int_{\mathbb{R}^n} k(x)dx = \int_{\mathbb{R}^+} \tilde{k}(r)dr$. Furthermore,

$$\begin{aligned} I_{\varepsilon,\rho}(\mu, f) &= \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{f * \mu_{\gamma,\tau}}{\tau} d\tau = \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{d\tau}{\tau} \int_{-\infty}^{\infty} f\left(x - \tau y_1 \gamma e_1\right) d\nu\,(y_1) \\ &= \qquad\qquad\qquad\qquad (\, e_1(1, 0, \ldots, 0)\,) \\ &= \frac{1}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} d\sigma \int_\varepsilon^\rho \frac{d\tau}{\tau} \int_{-\infty}^{\infty} f\left(x + \tau y_1 \sigma\right) d\nu\,(y_1) \;. \end{aligned}$$

Put $v = \tau\sigma \in \mathbb{R}^n$, $\quad t = y_1 \in \mathbb{R}$ and denote

$$\left(X_\nu f\right)(x, v) = \int_{-\infty}^{\infty} f(x + tv) d\nu(t) \,. \tag{4.2}$$

Then

$$I_{\varepsilon,\rho}(\mu, f) = \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon<|v|<\rho} \left(X_\nu f\right)(x, v) \frac{dv}{|v|^n} \,. \tag{4.3}$$

In the case when $v \in \Sigma_{n-1}$ and ν is the Lebesgue measure, the integral (4.2) coincides with the usual X-ray transform (see, e.g., [13]). If ν is absolutely continuous with the compactly supported density g, then $(X_g f)(x, v) = \int_{-\infty}^{\infty} f(x + tv)g(t)dt$ is known as the windowed X-ray transform (see [9]). Theorems 2 and 3 imply the following inversion statements.

Theorem 6.

Let ν be an admissible measure (i.e., $\tilde{k}(r) = r^{-1}\nu((-r, r)) \in L^1(\mathbb{R}^+)$) and let $(X_{|\nu|}|f|)(x, v)$ be locally summable in the v-variable away from the origin for almost all $x \in \mathbb{R}^n$.

(i) *If $f \in L^p$, $1 < p < \infty$, then*

$$f(x) = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \frac{1}{|\Sigma_{n-1}|\,k_0} \int_{\varepsilon<|v|<\rho} \left(X_\nu f\right)(x, v) \frac{dv}{|v|^n} \,, \tag{4.4}$$

$\lim = \overset{(L^p)}{\lim}$, *provided that*

$$k_0 = \int_{\mathbb{R}^+} \tilde{k}(r)dr \neq 0 \,. \tag{4.5}$$

(ii) *If $f \in C_0$, then (4.4) holds with limit interpreted in the C-norm.*

(iii) *If $f \in L^p$, $1 \le p < \infty$, and $\tilde{k}(r)$ has a decreasing majorant belonging to $L^1(\mathbb{R}^+)$, then (4.4) holds a.e. on $\mathbb{R}^n$.*

Theorem 7.

Let ν be a finite Borel measure on $\mathbb{R}$ such that $\nu(\mathbb{R}) = 0$. Assume that

$$(a) \qquad \int_{-\infty}^{\infty} |\log|t||d|\nu|(t) < \infty \tag{4.6}$$

or

$$(b) \qquad d\nu(t) = g(t)dt, \quad g \in H^1(\mathbb{R})\,. \tag{4.7}$$

Then ν is admissible and the constant k_0 in (4.4) has the form

$$k_0 = \begin{cases} \int_{-\infty}^{\infty} \log(1/t)d\mu(t) & \text{in the case } (a)\,, \\ \frac{\pi i}{2}\int_{-\infty}^{\infty}(Hg)(t)\, sgn\, t\, dt & \text{in the case } (b)\,, \end{cases}$$

$(Hg)(t) = (\pi i)^{-1}\int_{-\infty}^{\infty} g(\tau)\frac{d\tau}{t-\tau}$ being the Hilbert transform of g. Under these assumptions the statements (i) *and* (ii) *of Theorem 6 hold. If moreover,*

$$\int_{|t|<1} |t|^{-\delta}d|\nu|(t) < \infty \quad \textit{for some } \delta > 0\,, \tag{4.8}$$

then for $f \in L^p$, $1 \le p < \infty$, the relation (4.4) is valid for almost all x.

In order to invert $X_\nu f$ with the non-admissible measure ν one may use the argument of Section 3. Let $\mu = \mu^{(1)} * \mu^{(2)}$, $\mu^{(i)}(y) = \nu^{(i)}(y_1) \times \delta(y')$; $i = 1, 2$. As above we have

$$\begin{aligned} I_{\varepsilon,\rho}(\mu, f) &= \int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{d\eta}{\eta} \int_{-\infty}^{\infty} d\nu^{(1)}(y_1) \int_{-\infty}^{\infty} f(x - \eta y_1 \gamma e_1 - \eta z_1 \gamma e_1)\, d\nu^{(2)}(z_1) \\ &= \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon<|v|<\rho} \frac{dv}{|v|^n} \int_{-\infty}^{\infty} d\nu^{(1)}(t) \int_{-\infty}^{\infty} f(x + tv + \tau v) d\nu^{(2)}(\tau) \\ &= \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon<|v|<\rho} \frac{dv}{|v|^n} \int_{-\infty}^{\infty} f(x + tv) d\left(\nu^{(1)} * \nu^{(2)}\right)(t) \\ &= \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon<|v|<\rho} \frac{dv}{|v|^n} \int_{-\infty}^{\infty} \left(X_{\nu^{(2)}} f\right)(x + tv, v) d\nu^{(1)}(t)\,. \end{aligned}$$

We say that the pair of measures $\nu^{(1)}$, $\nu^{(2)}$ is admissible if $\nu = \nu^{(1)} * \nu^{(2)}$ is admissible, i.e.,

$$\tilde{k}(r) = \frac{1}{r} \int_{(-r,r)} d\left(\nu^{(1)} * \nu^{(2)}\right)(t) \in L^1\left(\mathbb{R}^+\right)\,. \tag{4.9}$$

The corresponding inversion formula, which is similar to (3.3), reads

$$f(x) = \frac{1}{|\Sigma_{n-1}|\, k_0} \int_{\mathbb{R}^n} \frac{dv}{|v|^n} \int_{-\infty}^{\infty} \left(X_{\nu^{(2)}} f\right)(x + tv, v) d\nu^{(1)}(t)\,. \tag{4.10}$$

We leave to the reader to state the analog of Theorem 5 which justifies this formula.

Let us give a simple example. Assume that we want to invert $(X_{\nu^{(2)}} f)(x, v)$, where $\nu^{(2)}$ is an arbitrary finite measure supported by $[0, \infty]$. Choose $\nu^{(1)} = \delta_a - \delta_b$ where δ_a and δ_b are unit Dirac masses at the points a and b, respectively. Let $0 < a < b < \infty$. Clearly, $\nu = \nu^{(1)} * \nu^{(2)}$ is a finite Borel measure because the linear functional

$$\varphi \to (\nu, \varphi) = \int_{-\infty}^{\infty} d\nu^{(1)}(x) \int_{-\infty}^{\infty} \overline{\varphi}(x+y) d\nu^{(2)}(y) = \int_{-\infty}^{\infty} [\overline{\varphi}(a+y) - \overline{\varphi}(b+y)] \, d\nu^{(2)}(y)$$

is bounded on C_0. Moreover, since $\hat{\nu}(\xi) = \hat{\nu}_2(\xi)(e^{ia\xi} - e^{ib\xi})$, then $\nu(\mathbb{R}) = \hat{\nu}(0) = 0$. One can readily check that

$$\tilde{k}(r) = \frac{1}{r} \int_{(-r,r)} d\nu(t) = \frac{1}{r} \int_{[r-b, r-a)} d\nu^{(2)}(t) \text{ and } k_0 = \int_0^{\infty} \tilde{k}(r) dr = \int_0^{\infty} \log \frac{b+t}{a+t} d\nu^{(2)}(t)$$

(the last integral is absolutely convergent).

Thus, Theorem 7 and (4.10) lead to the following statement (for the sake of convenience we change the notation for measures).

Theorem 8.

Let ν be a finite Borel measure supported by the positive half-line, and assume that $a < b$ are the arbitrary positive numbers.

(i) *If $f \in L^p$, $1 < p < \infty$, then*

$$f(x) = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \frac{1}{|\Sigma_{n-1}| \, k_0} \int_{\varepsilon < |v| < \rho} \frac{(X_v f)(x + av, v) - (X_v f)(x + bv, v)}{|v|^n} dv \,, \tag{4.11}$$

$\lim = \overset{(L^p)}{\lim}$, *provided that* $k_0 = \int_0^\infty \log \frac{b+t}{a+t} d\nu(t) \neq 0$.

(ii) *If $f \in C_0$, then the limit in (4.11) may be understood in the C-norm.*

(iii) *If, moreover, ν is compactly supported, then (4.11) holds in the a.e. sense for $f \in L^p$, $1 \le p < \infty$.*

We conclude this section by exhibiting an analog of Theorem 1 for windowed X-ray transforms.

Theorem 9.

Let ν be a finite Borel measure on $\mathbb{R}$ such that the integral

$$c_\nu = \int_{-\infty}^{\infty} \frac{\hat{\nu}(\xi)}{|\xi|} d\xi = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \int_{\varepsilon < |\eta| < \rho} \frac{\hat{\nu}(\xi)}{|\xi|} d\xi$$

is finite and different from zero. Then

$$f = \overset{(L^2)}{\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}}} \frac{1}{c_\nu \, |\Sigma_{n-1}|} \int_{\varepsilon < |\eta| < \rho} (X_v f)(x, v) \frac{dv}{|v|^n} \,.$$

Proof. According to (4.3) and Theorem 1 it suffices to show that for $\mu(y) = \nu(y_1) \times \delta(y')$ the limit

$$c_\mu = \lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}} \frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon < |\eta| < \rho} \frac{\hat{\mu}(\eta)}{|\eta|^n} d\eta$$

exists and is equal to c_ν. By taking into account that $\hat{\mu}(\eta) = \hat{\nu}(\eta_1)$ we have

$$\frac{1}{|\Sigma_{n-1}|} \int_{\varepsilon < |\eta| < \rho} \frac{\hat{\mu}(\eta)}{|\eta|^n} d\eta = \frac{1}{|\Sigma_{n-1}|} \int_\varepsilon^\rho \frac{dr}{r} \int_{\Sigma_{n-1}} \hat{\nu}\left(r\langle \sigma, e_1 \rangle\right) d\sigma$$

$$= \frac{|\Sigma_{n-2}|}{|\Sigma_{n-1}|} \int_\varepsilon^\rho \frac{dr}{r} \int_{-1}^1 \hat{\nu}(rt) \left(1-t^2\right)^{(n-3)/2} dt$$
$$= \frac{|\Sigma_{n-2}|}{|\Sigma_{n-1}|} \int_{-1}^1 \left(1-t^2\right)^{(n-3)/2} dt \int_{\varepsilon|t|<|\xi|<\rho|t|} \frac{\hat{\nu}(\xi)}{|\xi|} \to A \int_{-\infty}^{\infty} \frac{\hat{\nu}(\xi)}{|\xi|},$$

where

$$A = \frac{|\Sigma_{n-2}|}{|\Sigma_{n-1}|} \int_{-1}^1 \left(1-t^2\right)^{(n-3)/2} dt = 1 \qquad \left(\text{we recall that } |\Sigma_{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}\right). \quad \square$$

Corollary 1. ***(for two measures)***
Let $\nu = \nu^{(1)} * \nu^{(2)}$, be a finite Borel measure on $\mathbb{R}$ such that the integral

$$c_\nu = \int_{-\infty}^{\infty} \frac{\widehat{\nu^{(1)}}(\xi)\widehat{\nu^{(2)}}(\xi)}{|\xi|} d\xi = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_{\varepsilon<|\eta|<\rho} \frac{\hat{\nu}^{(1)}(\xi)\hat{\nu}^{(2)}(\xi)}{|\xi|} d\xi$$

is finite and different from zero. Then

$$f = \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}^{(L^2)} \frac{1}{c_\nu \left|\sum_{n-1}\right|} \int_{\varepsilon<|v|<\rho} \frac{dv}{|v|^n} \int_{-\infty}^{\infty} \left(X_{\nu^{(2)}} f\right)(x+tv, v) d\nu^{(1)} dt\,.$$

The last formula leads to the wavelet expansion of f [cf. (3.4)].

5. Radon Transforms

5.1 Preliminaries

Let $\mathbb{P}^n$ be the manifold of all hyperplanes in $\mathbb{R}^n$. The Radon transform of sufficiently nice function f on $\mathbb{R}^n$ is defined by

$$(Rf)(\tau) = \int_\tau f(x) dm_\tau(x), \qquad \tau \in \mathbb{P}^n, \tag{5.1}$$

where m_τ is the euclidean measure on τ. Each hyperplane τ may be parameterized by $(\theta, s) \in \tilde{\mathbb{R}}^n = \Sigma_{n-1} \times \mathbb{R}$ so that $\tau = \{x \in \mathbb{R}^n : \langle x, \theta\rangle = s\}$. Since (θ, s) and $(-\theta, -s)$ define the same hyperplane, the correspondence between $\tilde{\mathbb{R}}^n$ and $\mathbb{P}^n$ is not one-to-one. The mapping $(\theta, s) \to \tau$ is a double covering on $\mathbb{P}^n$, and each function on $\mathbb{P}^n$ can be identified with the even function on $\tilde{\mathbb{R}}^n$. Under the (θ, s)-parameterization, the Radon transform (5.1) reads

$$(R_\theta f)(s) \equiv (Rf)(\theta, s) = \int_{\theta^\perp} f(s\theta + u) du\,, \tag{5.2}$$

where $\theta \in \Sigma_{n-1}$, $s \in \mathbb{R}$; du stands for the euclidean measure on the subspace $\theta^\perp$ orthogonal to θ, such that $dx = ds du$ for $x = s\theta + u$. We consider $\tilde{\mathbb{R}}^n$ as the measure space with the product measure $d\theta ds$ where $d\theta$ is the rotation invariant measure on Σ_{n-1} and ds designates the Lebesgue measure on $\mathbb{R}$.

The Radon transform R represents a linear continuous operator from $\mathcal{S}(\mathbb{R}^n)$ into the similar space $\mathcal{S}(\tilde{\mathbb{R}}^n)$ (see, [7]). For locally integrable functions f the following statements are known.

Theorem 10. [24]

If f is nonnegative, then Rf is defined almost everywhere on $\tilde{\mathbb{R}}^n$ and is locally integrable on $\tilde{\mathbb{R}}^n$ if and only if

$$\int\limits_{\mathbb{R}^n} f(x)\frac{dx}{1+|x|} < \infty .$$

For each $\delta > 0$ and any measurable function f on $\mathbb{R}^n$ the following estimate holds:

$$\int_{\tilde{\mathbb{R}}^n} |(Rf)(\theta, s)| \frac{d\theta ds}{(1+|s|)^{1+\delta}} \le c \int\limits_{\mathbb{R}^n} |f(x)| \frac{dx}{1+|x|} .$$

Corollary 2.

For $1 \le p < n/(n-1)$ and $\delta > 0$,

$$\int_{\tilde{\mathbb{R}}^n} |(Rf)(\theta, s)| \frac{d\theta ds}{(1+|s|)^{1+\delta}} \le c\|f\|_p .$$

We note that in the case $p \ge n/(n-1)$ the function $f(x) = (2+|x|)^{-n/p} (\log(2+|x|))^{-1}$ belongs to $L^p(\mathbb{R}^n)$ and is not integrable over any hyperplane.

Given $1 \le q, r \le \infty$, we define the space

$$L^{q,r}\left(\tilde{\mathbb{R}}^n\right) = \left\{ \varphi(\theta, s) : \|\varphi\|_{q,r} = \left(\int_{\Sigma_{n-1}} \left[\int\limits_{-\infty}^{\infty} |\varphi(\theta, s)|^r ds \right]^{q/r} d\theta \right)^{1/q} < \infty \right\} .$$

Theorem 11. [14]

For $n \ge 2$ an a priori *inequality*

$$\|Rf\|_{q,r} \le c_{p,q,r}\|f\|_p$$

holds if and only if $1 \le p < n/(n-1)$, $q \le p'$ $(p^{-1} + p'^{-1} = 1)$, and $r^{-1} = np^{-1} - n + 1$.

Let us define the *Radon transform of Borel measures on* $\mathbb{R}^n$. Given $\theta \in \Sigma_{n-1}$, let E_θ be the orthogonal projection onto the line $\ell_\theta = \{s\theta : -\infty < s < \infty\}$ and let r_θ be an arbitrary rotation such that $r_\theta e_1 = \theta$. We define the Radon transform $R_\theta \mu$ of the Borel measure μ on $\mathbb{R}^n$ as the image of μ under the mapping $r_\theta^{-1} E_\theta$. This means that for each $\theta \in \Sigma_{n-1}$, $R_\theta \mu$ is the Borel measure on $\mathbb{R}$ such that

$$(R_\theta \mu)(\Omega) = \mu\left(E_\theta^{-1} r_\theta \Omega\right) = \mu\left(\theta^\perp \times r_\theta \Omega\right), \quad \Omega \subset \mathbb{R} .$$

The above definition does not depend on the choice of r_θ and corresponds to (5.2). Indeed, if $d\mu(x) = f(x)dx$, $f \in L^1(\mathbb{R}^n)$, and $\tau \in \mathbb{P}^n$ is given by the equation $\langle x, \theta \rangle = s$, then $R_\theta \mu$ is absolutely continuous on $\mathbb{R}$ with the density $(R_\theta f)(s)$ because

$$(R_\theta \mu)(\Omega) = \int_{r_\theta \Omega \times \theta^\perp} f(x)dx = \int_\Omega ds \int_{\theta^\perp} f(s\theta + u)du = \int_\Omega (R_\theta f)(s) ds .$$

Clearly, if μ is finite on $\mathbb{R}^n$, then $R_\theta \mu$ is finite on $\mathbb{R}$ for each $\theta \in \Sigma_{n-1}$. The idea of the above definition was borrowed from the more general consideration in [11].

Lemma 2.

Suppose that μ is a Borel measure on $\mathbb{R}^n$ and φ is a Borel function on $\mathbb{R}$. Then

$$\int_{\mathbb{R}} \varphi(s) d(R_\theta \mu)(s) = \int\limits_{\mathbb{R}^n} \varphi\left(r_\theta^{-1} E_\theta x\right) d\mu(x) = \int\limits_{\mathbb{R}^n} \varphi(\langle x, \theta \rangle) d\mu(x) .$$

This statement follows from Theorem 1.19 of [11]. Assuming $\varphi(s) \equiv \varphi(\theta, s)$ and integrating the above equality over Σ_{n-1} we get

$$\int_{\Sigma_{n-1}} d\theta \int_{\mathbb{R}} \varphi(\theta, s) d\left(R_\theta \mu\right)(s) = \int_{\mathbb{R}^n} d\mu(x) \int_{\Sigma_{n-1}} \varphi(\theta, \langle x, \theta\rangle) d\mu(x)$$

or

$$\int_{\tilde{\mathbb{R}}^n} \varphi(\theta, s) d\theta d\left(R_\theta \mu\right)(s) = \int_{\mathbb{R}^n} \left(R^{\#}\varphi\right)(x) d\mu(x) \tag{5.3}$$

where the integral

$$\left(R^{\#}\varphi\right)(x) = \int_{\Sigma_{n-1}} \varphi(\theta, \langle x, \theta\rangle) d\theta$$

is known as the dual Radon transform (see, [7]). In particular, if $d\mu(x) = f(x)dx$, then (5.3) yields the well-known duality relation

$$(Rf, \varphi)^{\sim} = \left(f, R^{\#}\varphi\right) \tag{5.4}$$

in which the following notation is used:

$$(f, g) = \int_{\mathbb{R}^n} f(x)\overline{g(x)}dx, \quad (\psi, \varphi)^{\sim} = \int_{\tilde{\mathbb{R}}^n} \psi(\theta, s)\overline{\varphi(\theta, s)} d\theta ds \ . \tag{5.5}$$

The equality (5.4) also can be obtained directly with the aid of the Fubini theorem provided that one of the integrals $(R|f|, |\varphi|)$, $(|f|, R^{\#}|\varphi|)$ is finite.

Lemma 3.

If φ is a locally integrable tempered function on $\tilde{\mathbb{R}}^n$, then

$$R^{\#}\varphi \in L^1_{loc}\left(\mathbb{R}^n\right) \cap \mathcal{S}'\left(\mathbb{R}^n\right) \ .$$

Proof. According to (5.4) for the arbitrary $\omega \in \mathcal{S}(\mathbb{R}^n)$ we have $(R^{\#}\varphi, \omega) = (\varphi, R\omega)^{\sim}$, and the result follows from the continuity of R from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}(\tilde{\mathbb{R}}^n)$. □

5.2 Inversion of Radon Transforms

Assume that $y = (y_1, \dots, y_n) = (y_1, y')$, $\nu(y) = \delta(y_1) \times \mathcal{L}(y')$ where $\delta(y_1)$ is the unit Dirac measure on $\mathbb{R}$ and $\mathcal{L}(y')$ is the Lebesgue measure on $\mathbb{R}^{n-1}$.

Lemma 4.

Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/(n-1)$. Then for the arbitrary finite measure α on $\mathbb{R}^n$,

$$\int_{SO(n)} \left(f * \alpha_{\gamma,t} * \nu_{\gamma,t}\right) d\gamma = \frac{t^{1-n}}{|\Sigma_{n-1}|} R^{\#}\left(Rf * \alpha_t^{(1)}\right) \tag{5.6}$$

where $\alpha_t^{(1)}$ is the dilated version of the measure

$$\alpha^{(1)} = \alpha \circ E_\theta^{-1} = R_{e_1}\alpha \tag{5.7}$$

(the "projection" of α onto the first coordinate axis).

Proof. We observe that

$$\left(f * \nu_{\gamma,t}\right)(x) = t^{1-n}\left(R_\theta f\right)(\langle x, \theta\rangle), \quad \theta = \gamma e_1 \ .$$

Indeed,

$$\begin{aligned}(f * \nu_{\gamma,t})(x) &= \int\limits_{\mathbb{R}^{n-1}} f(x - t\gamma y')\,dy' = t^{1-n} \int\limits_{\mathbb{R}^{n-1}} f(\langle x, \gamma e_1\rangle \gamma e_1 - \gamma z')\,dz' \\ &= t^{1-n} \int_{\theta^\perp} f(\langle x, \theta\rangle\theta - u)du = t^{1-n} (R_\theta f)(\langle x, \theta\rangle)\,.\end{aligned}$$

It follows that

$$\int_{SO(n)} (f * \alpha_{\gamma,t} * \nu_{\gamma,t})(x)d\gamma = \int_{SO(n)} d\gamma \int\limits_{\mathbb{R}^n} (f * \nu_{\gamma,t})(x - t\gamma y)d\alpha(y)$$

$$= t^{1-n} \int_{SO(n)} d\gamma \int\limits_{\mathbb{R}^n} (Rf)(\langle x - t\gamma y, \gamma e_1\rangle, \gamma e_1)\,d\alpha(y)$$

$$= \frac{t^{1-n}}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} d\theta \int\limits_{\mathbb{R}^n} (R_\theta f)(\langle x, \theta\rangle - ty_1)\,d\alpha(y) =^{\text{(Lemma 2)}}$$

$$= \frac{t^{1-n}}{|\Sigma_{n-1}|} \int_{\Sigma_{n-1}} d\theta \int\limits_{-\infty}^{\infty} (R_\theta f)(\langle x, \theta\rangle - ty_1)\,d\alpha^{(1)}(y_1)$$

which coincides with the right-hand side of (5.6). The application of the Fubini theorem in the above calculations is possible because for nonnegative f and α by Theorem 11 we have $R|f| * |\alpha|_t^{(1)} \in L^{1,r}(\tilde{\mathbb{R}}^n)$ and therefore by Lemma 3, $R^\#(R|f| * |\alpha|_t^{(1)})(x)$ is finite for almost all x. □

Corollary 3.

Assume that $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/(n-1)$; α is a finite Borel measure on $\mathbb{R}^n$. Then

$$\int_\varepsilon^\infty R^\#\left(Rf * \alpha_t^{(1)}\right)\frac{dt}{t^n} = h_\varepsilon * f, \quad \varepsilon > 0\,, \tag{5.8}$$

where

$$h(x) = \frac{\left|B_1^{(n-1)}\right|}{|x|^n} \int_{|s|<|x|} (|x|^2 - s^2)^{(n-1)/2} d\alpha^{(1)}(s)\,, \tag{5.9}$$

$|B_1^{(n-1)}| = \pi^{(n-1)/2}/\,\Gamma((n+1)/2)$ *being the volume of the $(n-1)$-dimensional unit ball.*

Proof. Put $\mu = \alpha * \nu$, $\nu = \delta(y_1) \times \mathcal{L}(y')$, in (5.6). One can readily check that $\mu(y) = \alpha^{(1)}(y_1) \times \mathcal{L}(y')$. Moreover, the function $(\gamma, t) \to (|f| * |\mu|_{\gamma,t})(x)$ belongs to $L^1(SO(n) \times [\varepsilon, \rho])$ for almost all x because by (5.6),

$$\int_{SO(n)} d\gamma \int_\varepsilon^\rho \frac{|f| * |\mu|_{\gamma,t}}{t}\,dt = \frac{1}{|\Sigma_{n-1}|}\int_\varepsilon^\rho R^\#\left(R|f| * |\alpha|_t^{(1)}\right)\frac{dt}{t^n} \overset{\text{a.e.}}{<} \infty$$

in accordance with Theorem 11 and Lemma 3. By (2.12) and (5.6) it follows that

$$\int_\varepsilon^\rho R^\#\left(Rf * \alpha_t^{(1)}\right)\frac{dt}{t^n} = |\Sigma_{n-1}| \int_\varepsilon^\rho \frac{f * \mu_{\gamma,t}}{t}\,dt = h_\varepsilon * f - h_\rho * f \tag{5.10}$$

where

$$\begin{aligned}h(x) &= \frac{1}{|x|^n}\int_{|y|<|x|} d\mu(y) = \frac{1}{|x|^n}\int_{|y_1|<|x|} d\alpha^{(1)}(y_1) \int_{|y'|<\sqrt{|x|^2-y_1^2}} dy' \\ &= \frac{\left|B_1^{(n-1)}\right|}{|x|^n}\int_{|y_1|<|x|} \left(|x|^2 - y_1^2\right)^{(n-1)/2} d\alpha^{(1)}(y_1)\,.\end{aligned}$$

In order to obtain (5.8) we first note that the integral in the left-hand side of this equality belongs to $L^1_{loc}(\mathbb{R}^n) \cap \mathcal{S}'(\mathbb{R}^n)$. Indeed, since

$$\left\| Rf * \alpha_t^{(1)} \right\|_{1,r} \leq c\|Rf\|_{1,r} \left\|\alpha^{(1)}\right\| \leq c\|f\|_p \left\|\alpha^{(1)}\right\| ,$$

then by (5.4) for any $\omega \in \mathcal{S}(\mathbb{R}^n)$ we get

$$\left(\int_\varepsilon^\infty R^\# \left(Rf * \alpha_t^{(1)}\right) \frac{dt}{t^n}, \omega\right) = \int_\varepsilon^\infty \left(Rf * \alpha_t^{(1)}, R\omega\right)^\sim \frac{dt}{t^n} \leq c(\omega)\|f\|_p \left\|\alpha^{(1)}\right\| \int_\varepsilon^\infty \frac{dt}{t^n} < \infty .$$

Thus, it suffices to check the validity of (5.8) in the $\mathcal{S}'$-sense. We have

$$\begin{aligned}
\left(\int_\varepsilon^\infty R^\# \left(Rf * \alpha_t^{(1)}\right) \frac{dt}{t}, \omega\right) &= \lim_{\rho\to\infty} \int_\varepsilon^\rho \frac{dt}{t^n} \int_{\mathbb{R}^n} \overline{\omega}(x) R^\# \left(Rf * \alpha_t^{(1)}\right)(x)dx \\
&= \lim_{\rho\to\infty} \left(h_\varepsilon * f - h_\rho * f, \omega\right) = (h_\varepsilon * f, \omega) \\
&\quad - \lim_{\rho\to\infty} \int_{\mathbb{R}^n} f(y)dy \int_{\mathbb{R}^n} h(z)\omega(y+\rho z)dz = (h_\varepsilon * f, \omega)
\end{aligned}$$

and the result follows. □

Our next goal is to show that the kernel $h(x)$ in (5.9) belongs to $L^1(\mathbb{R}^n)$ under certain natural conditions. For this purpose the following general statement will be useful.

Lemma 5.

Assume that $1 \leq k \leq n-1$, m is a finite Borel measure on $\mathbb{R}^{n-k}$,

$$h_{(m)}(x) = \frac{\left|B_1^{(k)}\right|}{|x|^n} \int_{|y''|<|x|} \left(|x|^2 - |y''|^2\right)^{k/2} dm\left(y''\right) .$$

If

$$\int_{|y''|>1} |y''|^\beta \, d|m|\left(y''\right) < \infty \quad \text{for some } \beta > k \tag{5.11}$$

and

$$\int_{\mathbb{R}^{n-k}} (y'')^j \, dm\left(y''\right) = 0 \ \text{ for } |j| = 0, 2, \ldots, 2[k/2] , \tag{5.12}$$

then $h_{(m)}(x)$ enjoys the following properties:

$$(i) \qquad h_{(m)}(x) \in L^1\left(\mathbb{R}^n\right), \quad h_{(m)}(x) = \begin{cases} O\left(|x|^{k-n}\right) & \text{if } |x| < 1 , \\ O\left(|x|^{k-n-\gamma}\right) & \text{if } |x| > 1 , \end{cases} \tag{5.13}$$

where $\gamma = min(\beta, 2[k/2]+2)$;

$$(ii) \int_{\mathbb{R}^n} h_{(m)}(x)dx = \begin{cases} \frac{\pi^{1+k/2}(-1)^{(k+1)/2}|\Sigma_{n-1}|}{2\Gamma(1+k/2)} \int_{\mathbb{R}^{n-k}} |y''|^k \, dm\left(y''\right) & \text{if } k \text{ is odd} , \\ \frac{\pi^{k/2}(-1)^{1+k/2}|\Sigma_{n-1}|}{(k/2)!} \int_{\mathbb{R}^{n-k}} |y''|^k \log|y''| \, dm\left(y''\right) & \text{if } k \text{ is even} . \end{cases} \tag{5.14}$$

Proof. Denote

$$\lambda_{m,k}(\eta) = [\Gamma(1+k/2)\eta]^{-1} \int_{|y''|^2<\eta} \left(\eta - |y''|^2\right)^{k/2} dm\,(y'') \; .$$

Clearly,

$$h_{(m)}(x) = \frac{\pi^{k/2}\,|\Sigma_{n-1}|}{|x|^{n-2}} \lambda_{m,k}\left(|x|^2\right) ,$$

and the relations $h_{(m)}(x) \in L^1(\mathbb{R}^n)$ and $\lambda_{m,k}(\eta) \in L^1(0,\infty)$ are equivalent. Moreover, $\int_{R^n} h_{(m)}(x)dx = (\pi^{k/2}|\Sigma_{n-1}|/2) \int_0^\infty \lambda_{m,k}(\eta)d\eta$. It remains to apply Lemma 17.1 from [18]. □

Theorem 12.

Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/(n-1)$. Let α be an arbitrary "wavelet measure" satisfying the following conditions:

$$\int_{\mathbb{R}^n} |y_1|^\beta \, d|\alpha|(y) < \infty \quad \text{for some} \quad \beta > n-1 \; , \tag{5.15}$$

$$\int_{\mathbb{R}^n} y_1^j \, d\alpha(y) = 0 \quad \text{for} \quad j = 0, 2, \ldots, 2\,[(n-1)/2] \; . \tag{5.16}$$

Then

$$\overset{(L^p)}{\lim_{\varepsilon\to 0}} \int_\varepsilon^\infty R^\# \left(Rf * \alpha_t^{(1)}\right) \frac{dt}{t^n} = h_0 f, \quad \alpha^{(1)} = R_{e_1}\alpha \; , \tag{5.17}$$

where

$$h_0 = \begin{cases} \frac{\pi^{n+1/2}(-1)^{n/2}}{\Gamma(n/2)\Gamma((n+1)/2)} \int_{\mathbb{R}^n} |y_1|^{n-1} \, d\alpha(y) & \text{if } n \text{ is even} \; , \\ \frac{2\pi^{n-1/2}(-1)^{(n+1)/2}}{\Gamma(n/2)\Gamma((n+1)/2)} \int_{\mathbb{R}^n} |y_1|^{n-1} \log|y_1| \, d\alpha(y) & \text{if } n \text{ is odd} \; . \end{cases} \tag{5.18}$$

The limit in (5.17) can be understood also in the a.e.-sense. If $f \in L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for $1 \le q < n/(n-1)$ and $1 \le p \le \infty$ (L^∞ should be understood as C_0), then (5.17) is also true.

Proof. We make use of Corollary 3 and Lemma 5 with $k = n-1$ and $dm(y'') = d\alpha^{(1)}(y_1)$. The required statements then follow by the usual machinery of the approximation to the identity [25]. □

6. k-Plane Transforms

6.1 Basic Definitions and Auxiliary Facts

Let $\mathcal{G}_{k,n}$ be the manifold of all non-oriented k-planes in $\mathbb{R}^n$. For sufficiently nice function f on $\mathbb{R}^n$, the k-plane transform is defined by

$$(Pf)(\tau) = \int_\tau f(x)dm_\tau(x), \quad \tau \in \mathcal{G}_{k,n} \; , \tag{6.1}$$

m_τ being the euclidean measure on τ. In order to parameterize $\mathcal{G}_{k,n}$ we introduce the Grassmann manifold $G_{k,n}$ of non-oriented k-dimensional subspaces of $\mathbb{R}^n$. Under the identification $G_{k,n} = O(n)/O(k) \times O(n-k)$, the set $G_{k,n}$ can be regarded as the $k(n-k)$-dimensional compact manifold on which the group $O(n)$ acts transitively. We denote by $d\zeta$ the invariant measure on $G_{k,n}$ with the total mass 1. Another parameterization of $\mathcal{G}_{k,n}$ which is similar to that in Section 5 and employs the

Stiefel manifold of orthonormal frames is also possible (see, [15]). We will not use it in this article. Given a fixed $\zeta \in G_{k,n}$, each vector $x \in \mathbb{R}^n$ can be written as $x = (x', x'') = x'+x''$ where $x' \in \zeta$ and $x'' \in \zeta^{\perp}$, $\zeta^{\perp}$ is the orthogonal complement to ζ in $\mathbb{R}^n$. Each k-plane $\tau \in \mathcal{G}_{k,n}$ can be parameterized by the pair (ζ, x'') where $\zeta \in G_{k,n}$ and $x'' \in \zeta^{\perp}$. The correspondence $\tau \to (\zeta, x'')$ is one-to-one in contrast to that in Section 5 ($k = n-1$) where we had a double covering. The manifold $\mathcal{G}_{k,n}$ will be endowed with the product measure $d\zeta dx''$, where dx'' denotes the usual euclidean measure on $\zeta^{\perp}$.

Under this parameterization the k-plane transform (6.1) reads

$$(Pf)\left(\zeta, x''\right) \equiv \left(P_{\zeta} f\right)\left(x''\right) = \int_{\zeta} f\left(x' + x''\right) dx', \quad \zeta \in G_{k,n}, \ x'' \in \zeta^{\perp}. \tag{6.2}$$

Let $(e_1, \ldots, e_n)$ be the natural orthonormal basis in $\mathbb{R}^n$, and denote by $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, $1 \le k \le n-1$, the subspaces of $\mathbb{R}^n$, generated by the sets $(e_1, \ldots, e_k)$ and $(e_{k+1}, \ldots, e_n)$, respectively. For $y \in \mathbb{R}^n$ we write $y = y' + y''$ where $y' \in \mathbb{R}^k$, $y'' \in \mathbb{R}^{n-k}$. Every $\zeta \in G_{k,n}$ can be written in the form $\zeta = \gamma \mathbb{R}^k$ for some $\gamma \in SO(n)$. Given $\gamma \in SO(n)$ and f on $\mathbb{R}^n$, we denote $f_{(\gamma)}(x) = f(\gamma x)$. Then

$$(Pf)\left(\zeta, x''\right) = \left(Pf_{(\gamma)}\right)\left(\mathbb{R}^k; y''\right) \quad \text{where } \zeta = \gamma \mathbb{R}^k, \ x'' = \gamma y''. \tag{6.3}$$

We denote by $\mathcal{S}(\mathcal{G}_{k,n})$ the Schwartz type space of infinitely differentiable functions $\varphi(\zeta, x'')$ on $\mathcal{G}_{k,n}$ rapidly decreasing in the x''-variable. The k-plane transform is a linear continuous map from $\mathcal{S}(\mathbb{R}^n)$ into $\mathcal{S}_{k,n}$ (concerning this fact and the precise definition of the space $\mathcal{S}(\mathcal{G}_{k,n})$ see, [6]).

Clearly, if $f \in L^1(\mathbb{R}^n)$, then $\int_{\zeta^{\perp}} (P_{\zeta} f)(x'') dx'' = \int_{\mathbb{R}^n} f(x) dx$ for each $\zeta \in G_{k,n}$, and therefore $\|Pf\|_{L^1(\mathcal{G}_{k,n})} \le \|f\|_{L^1(\mathbb{R}^n)}$.

Theorem 13. ***[24]***

Let f be a nonnegative measurable function on $\mathbb{R}^n$. Then $(Pf)(\zeta, x'')$ is defined almost everywhere on $\mathcal{G}_{k,n}$ if and only if

$$\int\limits_{\mathbb{R}^n} \frac{f(x)dx}{(1+|x|)^{n-k}} < \infty .$$

For each $\delta > 0$ and a measurable function f,

$$\int_{\mathcal{G}_{k,n}} \left|(Pf)\left(\zeta, x''\right)\right| \frac{d\zeta dx''}{(1+|x''|)^{n-k+\delta}} \le c \int\limits_{\mathbb{R}^n} \frac{|f(x)|dx}{(1+|x|)^{n-k}} . \tag{6.4}$$

Corollary 4.

For $1 \le p < n/k$ and $\delta > 0$,

$$\int_{\mathcal{G}_{k,n}} \left|(Pf)\left(\zeta, x''\right)\right| \frac{d\zeta dx''}{(1+|x''|)^{n-k+\delta}} \le c\|f\|_p . \tag{6.5}$$

By using (6.3) and the obvious formula

$$\int_{G_{k,n}} g(\zeta) d\zeta = \int_{SO(n)} g\left(\gamma \mathbb{R}^k\right) d\gamma \tag{6.6}$$

one can write (6.5) in the form

$$\int_{SO(n)} d\gamma \int\limits_{\mathbb{R}^{n-k}} \left|\left(Pf_{(\gamma)}\right)\left(\mathbb{R}^k, y''\right)\right| \frac{dy''}{(1+|y''|)^{n-k+\delta}} \le c\|f\|_p, \quad \delta > 0 . \tag{6.7}$$

The relation (6.5) shows that for $1 \leq p < n/k$, the function $f \in L^p(\mathbb{R}^n)$ is integrable over almost every translate of almost every subspace $\zeta \in G_{k,n}$. For $p \geq n/k$ this is not true as is shown by the function $f(x) = (2 + |x|)^{-n/p}(\log(2 + |x|))^{-1}$ (see, [22]).

The k-plane transforms of Borel measures on $\mathbb{R}^n$ can be defined as in Section 5 by using the orthogonal projection $E_{\zeta^\perp} : \mathbb{R}^n \to \zeta^\perp$ along the "direction" ζ. Given a measure μ on $\mathbb{R}^n$, we define its k-plane transform $P_\zeta \mu$ as a measure on $\zeta^\perp$ such that

$$\left(P_\zeta \mu\right)(\Omega) = \mu\left(E_{\zeta^\perp}^{-1}\Omega\right) = \int_{\Omega\times\zeta} d\mu(y), \quad \Omega \subset \zeta^\perp .$$

Clearly, if μ is finite on $\mathbb{R}^n$, then $P_\zeta \mu$ is finite on $\zeta^\perp$ for each $\zeta \in G_{k,n}$.

Lemma 6. ***(cf. [11], p. 16)***

Let μ be a Borel measure on $\mathbb{R}^n$ and let g be a Borel function on $\zeta^\perp$. Then

$$\int_{\zeta^\perp} \varphi\left(x''\right) d\left(P_\zeta \mu\right)\left(x''\right) = \int_{\mathbb{R}^n} \varphi\left(E_{\zeta^\perp} x\right) d\mu(x) .$$

As in Section 5 (put $\varphi \equiv \varphi(\zeta, x')$ and integrate over $\zeta \in G_{k,n}$), one can arrive at the duality relation involving the dual k-plane transform

$$\left(P^\# \varphi\right)(x) = \int_{G_{k,n}} \varphi\left(\zeta, E_{\zeta^\perp} x\right) d\zeta \tag{6.8}$$

and having the form

$$(Pf, \varphi)^\sim = \left(f, P^\# \varphi\right) , \tag{6.9}$$

where

$$(Pf, \varphi)^\sim = \int_{\mathcal{G}_{k,n}} (Pf)\left(\zeta, x''\right) \overline{\varphi\left(\zeta, x''\right)} d\zeta dx'' .$$

The following statement is analogous to Lemma 3.

Lemma 7.

If φ is a locally integrable tempered function on $\mathcal{G}_{k,n}$, then

$$P^\# \varphi \in L^1_{loc}\left(\mathbb{R}^n\right) \cap S'\left(\mathbb{R}^n\right) .$$

Remark 1.

The reader may be disappointed by not finding an analogue of the Oberlin-Stein theorem for k-plane transforms in this section. Such an analogue would be very helpful, but unfortunately (as far as I know) it represents an open problem. Concerning this problem the reader is referred to [2, 4, 27]. We shall see that one can also use Corollary 4 for our purposes, which covers the whole range of k $(1 \leq k \leq n-1)$ and p $(1 \leq p < n/k)$. □

6.2 Inversion of k-Plane Transforms

In order to obtain inversion formulae involving continuous wavelet transforms, we use the same method as in Section 5. Put $\nu(y) = \mathcal{L}(y') \times \delta(y'')$ where $\mathcal{L}(y')$ is the Lebesgue measure on $\mathbb{R}^k$ and $\delta(y'')$ is the delta function on $\mathbb{R}^{n-k}$.

Lemma 8.

Let $f \in L^p(\mathbb{R}^n)$, $1 \leq p < n/k$. Then for the arbitrary finite measure α on $\mathbb{R}^n$ and $t > 0$,

$$\int_{SO(n)} \left(f * \nu_{\gamma,t} * \alpha_{\gamma,t}\right) d\gamma = t^{-k} P^\# M_{\alpha,t} f \tag{6.10}$$

where

$$\left(M_{\alpha,t} f\right)(\zeta, x'') = \int_{SO_\zeta(n-k)} \left(P_\zeta f * \left(P_\zeta \alpha\right)_{\sigma,t}\right)(x'')\, d\sigma\,, \tag{6.11}$$

$SO_\zeta(n-k)$ being the subgroup of $SO(n)$ consisting of rotations in the $\zeta^\perp$-plane.

Proof. The passage from the left-hand side of (6.10) to the right-hand side is based on the application of the Fubini theorem. In order to justify this application, we first show that for nonnegative f and α,

$$\left\| M_{\alpha,t} f \right\|_{L^1(\mathcal{G}_{k,n};(1+|x''|)^{-s})} \le c\|\alpha\|\, \|f\|_p, \quad s > n-k\,, \tag{6.12}$$

with c independent of t. This would imply that for complex-valued f and α,

$$\left|\left(P^\# M_{|\alpha|,t}|f|, \omega\right)\right| = \left|\left(M_{|\alpha|,t} f, P\omega\right)^{\sim}\right| \le c(\omega)\|\alpha\|\, \|f\|_p \tag{6.13}$$

uniformly in t for each $\omega \in \mathcal{S}(\mathbb{R}^n)$. In particular, the latter means that $(P^\# M_{|\alpha|,t}|f|)(x)$ is finite for almost all x and the application of the Fubini theorem below is possible.

In order to prove (6.12) we note that

$$P_\zeta \alpha_\sigma = \left(P_\zeta \alpha\right)_\sigma\,. \tag{6.14}$$

Indeed, for any Borel set $A \subset \zeta^\perp$ we have $(P_\zeta \alpha_\sigma)(A) = \alpha_\sigma(E_{\zeta^\perp}^{-1} A) = \alpha(E_{\zeta^\perp}^{-1}\sigma^{-1} A) = (P_\zeta\alpha)(\sigma^{-1}A) = (P_\zeta\alpha)_\sigma(A)$. Then

$$\begin{aligned}
&\int_{\mathcal{G}_{k,n}} \left(M_{\alpha,t} f\right)(\zeta, x'') \frac{d\zeta\, dx''}{(1+|x''|)^s} \\
= \;& \int_{G_{k,n}} d\zeta \int_{\zeta^\perp} \frac{dx''}{(1+|x''|)^s} \int_{SO_\zeta(n-k)} d\sigma \int_{\zeta^\perp} \left(P_\zeta f\right)(x'' - t\eta'')\, d\left(P_\zeta \alpha_\sigma\right)(\eta'') \\
& \left(\text{put } x'' = \gamma y'', \quad \zeta = \gamma\mathbb{R}^k, \quad \sigma = \gamma r, \quad f_{(\gamma)}(x) = f(\gamma x)\right) \\
= \;& \int_{SO(n)} d\gamma \int_{\mathbb{R}^{n-k}} \frac{dy''}{(1+|y''|)^s} \int_{SO(n-k)} dr \int_{\mathbb{R}^n} \left(P_{\gamma\mathbb{R}^k} f\right)\left(\gamma y'' - t E_{(\gamma\mathbb{R}^k)^\perp}\eta\right) d\alpha_{\gamma r}(\eta) \\
= \;& \int_{\mathbb{R}^n} d\alpha(\eta) \int_{SO(n-k)} dr \int_{SO(n)} d\gamma \int_{\mathbb{R}^{n-k}} \left(P_{\mathbb{R}^k} f_{(\gamma)}\right)(y'' - tr\eta'') \frac{dy''}{(1+|y''|)^s} \overset{(6.7)}{\le} \\
\le \;& c\int_{\mathbb{R}^n} d\alpha(\eta) \int_{SO(n-k)} \left\| f_{(\gamma)}\left(\cdot - tr\eta''\right)\right\|_p = c\|\alpha\|\, \|f\|_p\,.
\end{aligned}$$

Once (6.12) is established, we can prove (6.10). Given $\gamma \in SO(n)$, we write $x = x' + x''$, where $x' \in \gamma\mathbb{R}^k$ and $x'' \in (\gamma\mathbb{R}^k)^\perp$. Clearly,

$$\left(f * \nu_{\gamma,t}\right)(x) = \int_{\mathbb{R}^{n-k}} f\left(x - t\gamma y'\right) dy' = t^{-k}\left(P_{\gamma\mathbb{R}^k} f\right)(x'')\,.$$

It follows that

$$\begin{aligned}
A \;=\;& \int_{SO(n)} \left(f * \nu_{\gamma,t} * \alpha_{\gamma,t}\right) d\gamma = t^{-k} \int_{SO(n)} d\gamma \int_{\mathbb{R}^n} \left(P_{\gamma\mathbb{R}^k} f\right)\left((x - ty)''\right) d\alpha_\gamma(y) \\
=\;& t^{-k} \int_{SO(n)} d\gamma \int_{(\gamma\mathbb{R}^k)^\perp} \left(P_{\gamma\mathbb{R}^k} f\right)(x'' - ty'')\, d\left(P_{\gamma\mathbb{R}^k}\alpha_\gamma\right)(y'')\,.
\end{aligned}$$

Let us replace γ by γr, assuming $r \in SO(n-k)$ where $SO(n-k)$ is the subgroup of all rotations preserving $\mathbb{R}^k$. The integration over $SO(n-k)$ yields

$$A = \frac{1}{t^k} \int_{SO(n)} d\gamma \int_{SO(n-k)} \left(P_{\gamma \mathbb{R}^k} f * \left(P_{\gamma \mathbb{R}^k} \alpha_{\gamma r} \right)_t \right) dr \,. \tag{6.15}$$

Given $\gamma \in SO(n)$, there is a family of rotations which maps R^k onto $\zeta = \gamma \mathbb{R}^k$. We fix a rule according to which one can construct a certain concrete rotation $\gamma_\zeta : \mathbb{R}^k \to \zeta$ (for example, if γ is parameterized by the associated Euler angles, one may set the Euler angles, corresponding to the factor $SO(n-k)$ equal to zero). Once such a rule is fixed, for each $\gamma : \mathbb{R}^k \to \zeta$ the rotation $\gamma_\zeta : \mathbb{R}^k \to \zeta$ depends on $\zeta = \gamma \mathbb{R}^k \in G_{k,n}$ only. Put $\lambda = \gamma_\zeta^{-1} \gamma r \in SO(n-k)$ in (6.15). Then $\alpha_{\gamma r} = \alpha_{\gamma_\zeta \lambda}$, and by (6.6) we get

$$A = \frac{1}{t^k} \int_{G_{k,n}} d\zeta \int_{SO(n-k)} \left(P_\zeta f * (P_\zeta \alpha_{\gamma_\zeta \lambda})_t \, d\lambda = \int_{G_{k,n}} d\zeta \int_{SO_\zeta(n-k)} \left(P_\zeta f * (P_\zeta \alpha)_{\sigma,t} \right) d\sigma \,.$$

This coincides with (6.10). □

Corollary 5.

Assume that $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/k$, and α is a finite Borel measure on $\mathbb{R}^n$. Then for $\varepsilon > 0$,

$$\int_\varepsilon^\infty \frac{dt}{t^{1+k}} \int_{G_{k,n}} d\zeta \int_{SO_\zeta(n-k)} \left(P_\zeta f * (P_\zeta \alpha)_{\sigma,t} \right) (E_{\zeta^\perp} x) \, d\sigma = (h_\varepsilon * f)(x) \tag{6.16}$$

where

$$h(x) = \frac{c_{k,n}}{|x|^n} \int_{|y''|<|x|} \left(|x|^2 - |y''|^2 \right)^{k/2} d\alpha'(y''), \quad \alpha' = P_{\mathbb{R}^k} \alpha \,, \tag{6.17}$$

$c_{k,n} = |B_1^{(k)}|/|\Sigma_{n-1}|$, *$B_1^{(k)}$ being the unit ball in $\mathbb{R}^k$.*

Proof. The argument is similar to that in the proof of Corollary 3. We put $\mu = \alpha * \nu$, $\nu(y) = \mathcal{L}(y') \times \delta(y'')$. The $\mu(y) = \mathcal{L}(y') \times \alpha'(y'')$. By (6.10) and (6.13),

$$\int_\varepsilon^\infty \frac{dt}{t} \int_{SO(n)} \left(|f| * |\mu|_{\gamma,t} \right) d\gamma = \int_\varepsilon^\infty \left(P^\# M_{|\alpha|,t} |f| \right) \frac{dt}{t^{1+k}} \in L^1_{loc}(\mathbb{R}^n) \cap S'(\mathbb{R}^n) \,. \tag{6.18}$$

Thus, we can employ the representation (2.12) according to which

$$\int_\varepsilon^\rho P^\# M_{\alpha,t} f \frac{dt}{t^{1+k}} = h_\varepsilon * f - h_\rho * f, \quad 0 < \varepsilon < \rho < \infty \,, \tag{6.19}$$

where

$$\begin{aligned} h(x) &= \frac{|x|^{-n}}{|\Sigma_{n-1}|} \int_{|y|<|x|} d\mu(y) = \frac{|x|^{-n}}{|\Sigma_{n-1}|} \int_{|y''|<|x|} d\alpha'(y'') \int_{|y'|<\sqrt{|x|^2-|y''|^2}} dy' \\ &= \frac{|B_1^{(k)}|}{|\Sigma_{n-1}|} |x|^{-n} \int_{|y''|<|x|} \left(|x|^2 - |y''|^2 \right)^{k/2} d\alpha'(y'') \,. \end{aligned}$$

The relations (6.19) and (6.18) imply (6.16). Here one can use the same argument as in the proof of Corollary 3. □

Now we state the main result of this section.

Theorem 14.

Let $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/k$. Let α be an arbitrary "wavelet measure" satisfying the following conditions:

$$\int_{\mathbb{R}^n} |y''|^\beta \, d|\alpha|(y) < \infty \quad \text{for some } \beta > k \,, \tag{6.20}$$

$$\int_{\mathbb{R}^n} (y'')^j \, d\alpha(y) = 0 \quad \text{for } |j| = 0, 2, \ldots, 2\,[k/2] \,. \tag{6.21}$$

If $\varphi(\zeta, x'') = (Pf)(\zeta, x'')$ and

$$\left(A_{\alpha,t}\varphi\right)(\zeta, x'') = \int_{SO_\zeta(n-k)} \left(\varphi * \left(P_\zeta \alpha\right)_{\sigma,t}\right)(x'')\, d\sigma \,,$$

then

$$\overset{(L^p)}{\lim_{\varepsilon \to 0}} \int_\varepsilon^\infty P^\# A_{\alpha,t}\varphi \, \frac{dt}{t^{1+k}} = h_0 f \tag{6.22}$$

where

$$h_0 = \begin{cases} \frac{\pi^{1+k/2}(-1)^{(k+1)/2}}{2\Gamma(1+k/2)} \int_{\mathbb{R}^n} |y''|^k \, d\alpha(y) & \text{if } k \text{ is odd,} \\ \frac{\pi^{k/2}(-1)^{1+k/2}}{(k/2)!} \int_{\mathbb{R}^n} |y''|^k \log|y''| \, d\alpha(y) & \text{if } k \text{ is even.} \end{cases} \tag{6.23}$$

The limit in (6.22) can be understood also in the a.e.-sense.

If $f \in L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$ for some $q \in [1, n/k)$, and $1 \le p \le \infty$ (here $L^\infty = C_0$), then (6.23) is also true.

The proof is similar to that of Theorem 12 from Section 5, and is based on Corollary 5, Lemma 5, and Lemma 6.

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