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Partial Sums of Orthonormal Bases Preserving Positivity – and Martingales

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ABSTRACT. *We characterize, for finite measure spaces, those orthonormal bases with the following positivity property: if f is a non-negative function, then the partial sums in the expansion of f are non-negative. The bases are necessarily generalized Haar functions and the partial sums are a martingale closed on the right by f.*

1. Introduction

Suppose that $\mathcal{X}$ is a set in a Euclidean space, e.g., the unit interval, and that $\{\phi_m : m = 1, \ldots\}$ is an orthonormal basis, $\phi_1 \equiv 1$, with respect to Lebesgue measure. If $f \in L_2(\mathcal{X})$ is a non-negative function, then the partial sums,

$$\sum_{m=1}^{M} \phi_m(x) \int dt\, \phi_m(t) f(t),\ x \in \mathcal{X}\,, \tag{1.1}$$

are not necessarily non-negative, as is well known. We begin by pointing out that a generalized version of the Haar functions does preserve the positivity of the partial sums (1.1), for positive $f \in L_p(\mathcal{X}),\ p \geq 1$. We prove a converse without the Euclidean assumptions: assume that for each $M \geq 1$, Eq. (1.1) is non-negative $a.e.$ whenever f is non-negative; then the ϕ_m are a generalized version of the Haar functions Definition 1, and (1.1) is a martingale closed on the right by f (Theorem 3).

This article was motivated by issues in medical image reconstruction, in particular [1], but relates to other imaging areas, for example, astronomy. Images are typically defined by non-negative functions f and it happens that one reconstructs or estimates the image by computing the first M inner products (the "Fourier" coefficients) of the image relative to some orthonormal system, taking (1.1) as the image. Although one can always "adjust" any negative values appearing in (1.1), an alternate approach is to employ an orthonormal system which ensures that positivity is preserved.

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We turn to the assumptions. Let $\mathcal{X}$ be a set, let $\mathcal{A}$ be sigma-algebra of subsets of $\mathcal{X}$, and let μ be a measure on $\mathcal{A}$, $0 < \mu(\mathcal{X}) < \infty$. We write $\mu_x \equiv \mu(\mathcal{X})$, and when μ is a probability we write P. We assume the measure μ is non-atomic (continuous): if $B \in \mathcal{A}$ and $\mu(B) = \gamma > 0$, then $\{\mu(C) : C \subset B\} = [0, \gamma]$, an interval.

2. Martingales and Generalized Haar Functions

We assume that the reader is familiar with martingales and the Haar functions. For the definition of martingales we refer to [5], and among other sources for martingales, we mention [3, 4], and [6]. For the Haar functions defined on the unit interval, we follow the notation in [4] and also cite [6], among others. A discussion of the Haar functions as a wavelet basis on the real line can be found, for example, in [2].

For $n = 1, 2, \ldots$, let $\mathcal{B}_n = \{B_{n,k} : k = 1, \ldots, 2^{n-1}\} \subset \mathcal{A}$ satisfy

$$B_{1,1} = \mathcal{X} \; ; \tag{2.1}$$

assuming that $\mathcal{B}_n$ is defined, let $\mathcal{B}_{n+1}$ satisfy

$$\begin{aligned} B_{n+1,k} \cap B_{n+1,j} &= \emptyset, \; j \neq k \; ; & (2.2)\\ B_{n+1,2k-1} \cup B_{n+1,2k} &= B_{n,k}, \; k = 1, \ldots, 2^{n-1} \; ; & (2.3)\\ P\left(B_{n+1,k}\right) &> 0, \; k = 1, \ldots, 2^n \; . & (2.4) \end{aligned}$$

Following the paradigm of the Haar functions, we define a sequence of functions $\{h_{n,k} : k = 1, \ldots, 2^{n-2}\}$ by

$$h_{1,1}(x) \equiv 1 \, ,$$

and for $n = 2, \ldots$ by

$$h_{n,k}(x) = \begin{cases} \alpha_{n,2k-1} & x \in B_{n,2k-1} \\ -\alpha_{n,2k} & x \in B_{n,2k} \\ 0 & \text{elsewhere} \end{cases} \, , \tag{2.5}$$

choosing the $\{\alpha_{n,k}\}$ so that $\{h_{n,k}\}$ is an orthonormal sequence; $\{h_{n,k}\}$ is a basis if and only if the class of sets $\cup\mathcal{B}_n$ is a set of generators for $\mathcal{A}$.

We write $\{h_{n,k}\}$ as $\{\phi_m : m = 1, \ldots\}$ with the natural ordering and let $\mathcal{A}_M$ denote the sigma-algebra generated by $\{\phi_m : m = 1, \ldots, M\}$. Fix $p \geq 1$ and let $f \in L_p(\mathcal{X})$. Let $E(f|\mathcal{A}_M)$ denote the conditional expectation of f given the sigma-algebra $\mathcal{A}_M$, defined up to a set (in $\mathcal{A}_M$) of probability zero (see, e.g., [5, p. 341]; but here, the only set of probability zero is the null set). Then, as is known, at least for the Haar functions (see, e.g., [6, p. 482]),

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\, \phi_m(t) f(t) = E\left(f \,|\mathcal{A}_M\right) \, . \tag{2.6}$$

If moreover $f \geq 0 \, a.e.$, then $E(f|\mathcal{A}_M) \geq 0$ so that the left side of (2.6) is non-negative everywhere. In addition, $\{E(f|\mathcal{A}_M) : M = 1, \ldots\}$ is a martingale closed on the right by f, and if $\cup\mathcal{B}_n (= \cup\mathcal{A}_M)$ is set of generators for $\mathcal{A}$ then the well-known theorems of Lévy and Doob imply that

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\, \phi_m(t) f(t) \to f, \; a.e. \text{ and in } L_p(\mathcal{X})$$

(see, e.g., [5, p. 396]).

3. Non-Negative Partial Sums

We need an elementary result, whose proof follows the lemma.

Theorem 1.
Fix $\alpha > 0$. Let $\phi : \mathcal{X} \to [-1/\alpha, \alpha]$ satisfy

$$\int \mu(dx)\, \phi(x) = 0\,. \tag{3.1}$$

Then

$$\int \mu(dx)\, \phi^2(x) \leq \mu_x\,. \tag{3.2}$$

Equality holds in (3.2) if and only if for an $A \in \mathcal{A}$, $\mu(A) = \mu_x/(1+\alpha^2)$,

$$\phi(x) = \begin{cases} \alpha, & x \in A \\ -1/\alpha, & x \in A^c \end{cases} \; a.e.\,. \tag{3.3}$$

Corollary 1.
Let $\phi : \mathcal{X} \to R$ be a bounded function satisfying (3.1). If

$$(ess\ sup\ \phi)|ess\ inf\,\phi| < 1\,,$$

then

$$\int \mu(dx)\, \phi^2(x) < \mu_x\,.$$

We omit the easy proof of the next lemma.

Lemma 1.
Let $\phi : \mathcal{X} \to [0, \alpha]$, let $B = \{x : \phi(x) > 0\}$, and let

$$c = \int \mu(dx)\, \phi(x)\,.$$

Then $c \leq \alpha\mu(B)$, and equality holds if and only if $\phi = \alpha$ on B a.e.. Let $C \subset B$ satisfy $\mu(C) = c/\alpha$, and let $\psi = \alpha I_C$. Then

$$\int \mu(dx)\, \phi(dx) \;=\; \int \mu(dx)\, \psi(x)\,, \tag{3.4}$$

$$\int \mu(dx)\, \phi^2(x) \;\leq\; \int \mu(dx)\, \psi^2(x) = \alpha^2 \mu(C)\,, \tag{3.5}$$

and equality holds in (3.5) if and only if $\phi = \alpha$ on B a.e..

Proof of Theorem 1. Write $\phi = \phi^+ - \phi^-$, where $\phi^+ = \max(\phi, 0)$, $\phi^- = -\min(\phi, 0)$, and associate with ϕ^+ and ϕ^- the functions ψ^+ and ψ^- of Lemma 1. Then, letting $\psi = \psi^+ - \psi^-$, we have

$$\begin{aligned} \int \mu(dx)\, \phi^2(x) &\leq \int \mu(dx)\, \psi^2(x) \\ &= \alpha^2 \mu\left(C^+\right) + (1/\alpha)^2 \left(C^-\right)\,, \end{aligned} \tag{3.6}$$

where C^+ and C^- correspond to the sets C of the lemma for ϕ^+ and ϕ^-, the inequality following from (3.5). A calculation shows that (3.6) equals $\mu(C^+ \cup C^-) \leq \mu_x$, proving (3.2), and this inequality is an equality if and only if $\mu(C^+) = \mu(B^+)$, $\mu(C^-) = \mu(B^-)$, and $\mu(B^+ \cup B^-) = \mu_x$, which is true if and only if ϕ satisfies (3.3). □

Theorem 2.

Let ϕ satisfy

$$\begin{aligned} \int \mu(dt)\,\phi(t) &= 0\,, \\ \int \mu(dt)\,\phi^2(t) &= \mu_x\,, \end{aligned} \tag{3.7}$$

and so that if $f \in L_2(\mathcal{X})$ and $f \geq 0$ then

$$\int \mu(dt)\, f(t) + \phi(x) \int \mu(dt)\, f(t)\phi(t) \geq 0\, a.e. \tag{3.8}$$

Then there exists $A \in \mathcal{A}$ so that a.e.

$$\phi(x) = \begin{cases} \alpha, & x \in A \\ -1/\alpha, & x \in A^c \end{cases}, \tag{3.9}$$

where $\alpha = (\mu(A^c)/\mu(A))^{1/2}$. Consequently, (3.8) holds for $f \in L_1(\mathcal{X})$, $f \geq 0$.

Proof. For $B \in \mathcal{A}$, let $f = I_B$, so that (3.8) becomes

$$\mu(B) + \phi(x) \int_B \mu(dt)\,\phi(t) \geq 0\, a.e.$$

Define a measure ν on $\mathcal{X} \times \mathcal{X}$ by

$$\begin{aligned} \nu(A \times B) &= \int_{A \times B} \mu(dx) \times \mu(dt)\ (1 + \phi(x)\phi(t)) \\ &= \int_A \mu(dx) \left(\mu(B) + \phi(x) \int_B \mu(dt)\,\phi(t) \right) . \end{aligned}$$

Since $\nu \geq 0$, we have $\phi(x)\phi(t) \geq -1\, a.e.\, (\mu \times \mu)$, that is,

$$(\text{ess sup}\ \phi)|\text{ess inf}\ \phi| \leq 1\,. \tag{3.10}$$

Because of Corollary 1 and (3.7), the inequality in (3.10) must be an equality, and the theorem follows from Theorem 1. □

Although an orthonormal sequence remains an orthonormal sequence after a re-arrangement of its terms, a re-arrangement of an orthonormal sequence preserving positivity of its partial sums may lose the positivity property. For example, Theorem 2 shows that if we define the new orthonormal sequence $\{h_{1,1}(x), h_{3,1}(x), h_{2,1}(x), h_{2,2}(x), h_{3,2}(x), \ldots\}$ (the ellipsis means the natural order), then positivity of the partial sums is not preserved. This becomes transparent after a calculation with the Haar functions on the unit interval. Clearly, re-arranging the terms $h_{n,k}$ by changing the order of the k for a fixed n preserves positivity. But consider the following sequence:

$$\left\{h_{1,1}(x), h_{2,1}(x), h_{3,1}(x), h_{3,2}(x), h_{2,2}(x), h_{3,3}(x), h_{3,4}(x), \ldots\right\}\,.$$

Because the associated sets $B_{n,k}$ are ordered by an inclusion relation (see Definition 1), the partial sums of this sequence form a martingale, implying that positivity is preserved. Or consider the following sequence:

$$\left\{h_{1,1}(x), h_{2,1}(x), \ldots, h_{n,1}, \ldots\right\}\,.$$

Positivity of partial sums is preserved, but the sequence is not a basis. In other words, if an orthonormal sequence preserves positivity, some of the re-arrangements and deletions of the sequence do the same, and some do not.

Recall that $\mathcal{A}_M$ denotes the smallest sigma-algebra generated by given functions $\{\phi_m : m = 1, \ldots, M\}$. The following definition describes the orthonormal sequences of interest.

Definition 1.

Fix $M \geq 2$. A (finite) orthonormal sequence $\{\phi_m : m = 1, \ldots, M\}$ defined up to a set of probability zero by

$$\phi_1 \equiv 1, \tag{3.11}$$

$$\phi_2 = \begin{cases} \alpha_{2,1}, & x \in B_{2,1} \\ -\alpha_{2,2}, & x \in B_{2,2} \left(= B_{2,1}^c\right) \end{cases}, \tag{3.12}$$

$$\phi_m = \begin{cases} \alpha_{n,j(n)}, & x \in B_{n,j(n)} \\ -\alpha_{n,j(n)+1}, & x \in B_{n,j(n)+1} \\ 0 & \text{elsewhere} \end{cases}, \quad m = 3, \ldots, M, \tag{3.13}$$

where

$$\alpha_{n,j(n)} = \left(\frac{1}{P\left(B_{n,j(n)}\right) + P\left(B_{n,j(n)+1}\right)}\right)^{1/2} \left(\frac{P\left(B_{n,j(n)+1}\right)}{P\left(B_{n,j(n)}\right)}\right)^{1/2},$$

$$\alpha_{n,j(n)+1} = \left(\frac{1}{P\left(B_{n,j(n)}\right) + P\left(B_{n,j(n)+1}\right)}\right)^{1/2} \left(\frac{P\left(B_{n,j(n)}\right)}{P\left(B_{n,j(n)+1}\right)}\right)^{1/2}, \tag{3.14}$$

is said to be an M-pyramid if for $2 \leq m \leq M$, the $\{B_{n,j(n)}, B_{n,j(n)+1}\}$ associated with ϕ_m through (3.12) and (3.13) satisfy: there is an atom $A_{m-1} \in \mathcal{A}_{m-1}$ so that $B_{n,j(n)} \cup B_{n,j(n)+1} \subset A_{m-1}$ and $P(B_{n,j(n)} \cup B_{n,j(n)+1}) = P(A_{m-1})$; by definition, $\{\phi_1\}$ is a 1-pyramid. And $\{\phi_m : m = 1, \ldots\}$ is a pyramid if it is an M-pyramid for each $M \geq 1$.

We omit the elementary proof of the next lemma.

Lemma 2.

Let $f \in L_1(\mathcal{X})$. If $\{\phi_m\}$ is a pyramid, then for $M \geq 1$,

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\, \phi_m(t) f(t) = E\left(f \mid \mathcal{A}_M\right) \text{ a.e.},$$

a martingale closed on the right by f.

Corollary 2.

If $x \in B_{n,j(n)}$, an atom of $\mathcal{A}_M$, and if $P(B_{n,j(n)}) > 0$ then

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\, \phi_m(t) f(t) = \frac{1}{P\left(B_{n,j(n)}\right)} \int_{B_{n,j(n)}} P(dt)\, f(t).$$

Lemma 3.

Suppose for $B \in \mathcal{A}$, $0 < P(B) < 1$, $\phi \in L_2(\mathcal{X})$ is a function which does not vanish on B, satisfies

$$\int P(dt)\, \phi(t) = 0, \tag{3.15}$$

and for a.e. $x \in B$,

$$\frac{1}{P(B)} \int_B P(dt)\, f(t) + \phi(x) \int P(dt)\, \phi(t) f(t) \geq 0\,,$$

for $f \in L_2(\mathcal{X})$, $f \geq 0$. *Then* $\phi = 0\, a.e.$ *on* B^c.

Proof. Consider $f \geq 0$ which vanish on B. Then for $a.e.\, x \in B$,

$$\phi(x) \int_{B^c} P(dt)\, \phi(t) f(t) \geq 0\,.$$

If it is false that $\phi = 0$ $a.e.$ on B^c, then either $\phi \geq 0$ $a.e.$ or $\phi \leq 0$ $a.e.$ on B. But if, e.g., $\phi \geq 0$ on B, then necessarily $\phi \geq 0$ on B^c, contradicting (3.15). □

Lemma 4.

Let $f \in L_2(\mathcal{X})$. *Let* $M \geq 1$. *Let* $\{\phi_m : m = 1, \ldots, M\}$ *be an orthonormal sequence which is an M-pyramid. Let* ϕ_{M+1} *be orthonormal to* $\{\phi_m : m = 1 \ldots M\}$. *Assume that* $f \geq 0$ *implies*

$$\sum_{m=1}^{M+1} \phi_m(x) \int P(dt)\, \phi_m(t) f(t) \geq 0\, a.e.\,. \tag{3.16}$$

Then $\{\phi : m = 1, \ldots, M+1\}$ *is an M+1-pyramid. Also, (3.16) holds for* $f \in L_1(X)$, $f \geq 0$.

Proof. Theorem 2 proves the case $M = 1$. We argue inductively and assume the lemma is proved for an $M \geq 1$. Let $B_{n,j(n)}$ be an atom of $\mathcal{A}_M$ on which ϕ_{M+1} does not vanish, $P(B_{n,j(n)}) > 0$. Using Corollary 2, (3.16) becomes for $a.e.\, x \in B_{n,j(n)}$,

$$\frac{1}{P\left(B_{n,j(n)}\right)} \int_{B_{n,j(n)}} P(dt)\, f(t) + \phi_{M+1}(x) \int P(dt)\, \phi_{M+1}(t) f(t) \geq 0\,. \tag{3.17}$$

By Lemma 3, ϕ_{M+1} vanishes $a.e.$ off $B_{n,j(n)}$ and (3.17) becomes for $a.e.\, x \in B_{n,j(n)}$,

$$\frac{1}{P\left(B_{n,j(n)}\right)} \int_{B_{n,j(n)}} P(dt)\, f(t) + \phi_{M+1}(x) \int_{B_{n,j(n)}} P(dt)\, \phi_{M+1}(t) f(t) \geq 0\,. \tag{3.18}$$

If we set

$$\psi(x) = \left(P\left(B_{n,j(n)}\right)\right)^{1/2} \phi_{M+1}(x)\,,$$

then (3.18) becomes for $a.e.\, x \in B_{n,j(n)}$,

$$\int_{B_{n,j(n)}} P(dt)\, f(t) + \psi(x) \int_{B_{n,j(n)}} P(dt)\, \psi(t) f(t) \geq 0\,.$$

Since

$$\int_{B_{n,j(n)}} P(dt)\, \psi^2(t) = P\left(B_{n,j(n)}\right)\,,$$

ψ satisfies the hypotheses of Theorem 2, upon setting $\mathcal{X} = B_{n,j(n)}$. In other words, ϕ_{M+1} has the form (3.13), where ϕ_{M+1} vanishes $a.e.$ off $B_{n,j(n)}$, $B_{n+1,j(n+1)} \cup B_{n+1,j(n+1)+1} \subset B_{n,j(n)}$ and $P(B_{n+1,j(n+1)} \cup B_{n+1,j(n+1)+1}) = P(B_{n,j(n)})$. The last assertion is immediate. □

The following theorem, stated for orthonormal bases, summarizes the preceding results; we omit the version of the theorem for orthonormal sequences, the only difference being the final assertion.

Theorem 3.

Let $(\mathcal{X}, \mathcal{A}, P)$ be a probability space and let $\{\phi_m : m = 1, \ldots\}$ be an orthonormal basis, $\phi_1 \equiv 1$. The following four assertions are equivalent:

(i) for each $f \in L_2(\mathcal{X})$ satisfying $f \geq 0$ and each $M \geq 1$,

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\,\phi_m(t) f(t) \geq 0\, a.e.\,;$$

(ii) $\{\phi_m : m = 1, \ldots\}$ is a pyramid;

(iii) for each $f \in L_1(\mathcal{X})$ satisfying $f \geq 0$ and each $M \geq 1$,

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\,\phi_m(t) f(t) \geq 0\, a.e.\,;$$

(iv) letting $\mathcal{A}_M$ be the sigma-algebra generated by $\{\phi_m : m = 1, \ldots, M\}$, for each $f \in L_1(\mathcal{X})$,

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\,\phi_m(t) f(t) = E\,(f\,|\mathcal{A}_M)\ a.e.\,,$$

a martingale closed on the right by f.

Under these conditions, for fixed $p \geq 1$ and $f \in L_p(\mathcal{X})$,

$$\sum_{m=1}^{M} \phi_m(x) \int P(dt)\,\phi_m(t) f(t) \to f,\ a.e.\ and\ L_p(\mathcal{X})\,.$$

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