The Journal of Fourier Analysis and Applications

Volume 4, Issue 1, 1998

Discrete Gabor Transforms: the Gabor-Gram Matrix Approach

Sigang Qiu

Communicated by Joshua Zeevi

ABSTRACT. The fundamental problem of discrete Gabor transforms *is to compute a set of* Gabor coefficients *in efficient ways. Recent study on the subject is an indirect approach: in order to compute the Gabor coefficients, one needs to find an auxiliary bi-orthogonal window function γ.*

We are seeking a direct approach in this paper. We introduce concepts of Gabor-Gram matrices *and investigate their structural properties. We propose iterative methods to compute the* Gabor coefficients. *Simple solutions for critical sampling, certain oversampling, and undersampling cases are developed.*

1. Introduction

The *Gabor transform* was originally formulated by Gabor [9] in 1946. The idea is to *represent* or *best-approximate* a signal by a set of *coefficients* over a set of *TF-translated copies* of a *window function* $\mathbf{g}$. In the discrete case, given a Gabor window $\mathbf{g} \in \mathbb{C}^N$ and TF-lattice constants pair (a, b), the discrete Gabor representation of a signal $\mathbf{x} \in \mathbb{C}^N$ is of the form:

$$\mathbf{x} = \sum_{n=0}^{\tilde{a}-1} \sum_{m=0}^{\tilde{b}-1} c_{n,m} \mathbf{g}_{n,m} . \tag{1.1}$$

For $n = 0, 1, \ldots, \tilde{a} - 1$ and $m = 0, 1, \ldots, \tilde{b} - 1$, $\mathbf{g}_{n,m} := M_{mb} T_{na} \mathbf{g}$ are the discrete time-frequency shifted copies of $\mathbf{g}$, $c_{n,m}$ are the *Gabor coefficients.* $(\mathbf{g}, a, b)$ is called a Gabor triple. We call $(\tilde{a}, \tilde{b})$ with $\tilde{a} = \frac{N}{a}$ and $\tilde{b} = \frac{N}{b}$ the *dual lattice constants.* We say $(\mathbf{g}, a, b)$ generates a Gabor frame if $\{\mathbf{g}_{n,m}\}_{n,m}$ is a frame [3, 4].

The advantage of the *Gabor transform* is based on TF-localizations of the Gabor family. The subject has been studied by mathematicians and engineers [1, 3, 4, 5, 12, 23, 25]. Since the Gabor family $\{\mathbf{g}_{n,m}\}$ need not be orthogonal, difficulties arise from the determination of the Gabor coefficients.

Math Subject Classifications. 65T99, 42c15, 65T20.

Keywords and Phrases. discrete Gabor transform (DGT), Gabor-Gram matrix, unitarily equivalent Gabor-Gram matrix (UEGM), Gabor window, dual Gabor window.

Acknowledgements and Notes. The author was with University of Connecticut, Storrs, CT 06269-3009.

ISSN 1069-5869

In 1980, Bastianns [2] introduced a bi-orthogonal function γ and turned the representation (1.1) to an *orthogonal-like* form:

$$\mathbf{x} = \sum_{n=0}^{\bar{a}-1}\sum_{m=0}^{\bar{b}-1} \langle \mathbf{x}, \gamma_{n,m}\rangle \mathbf{g}_{n,m} \,. \tag{1.2}$$

The active studies on the subject turn to seek γ in efficient ways. Many interesting approaches have been developed [7, 14, 16, 17, 18, 19, 20, 22, 24]. However, this γ is an auxiliary function. We only need to use γ to determine the coefficients $c_{n,m}$.

Can we have an efficient way of computing $c_{n,m}$ without using γ ? In this article, we present a conjugate-gradient (CG) approach to compute the Gabor coefficients without using γ. We consider general cases: critical sampling, oversampling, and undersampling cases. We do not assume $(\mathbf{g}, a, b)$ generates a Gabor frame.

In Section 2, we fix notation and present preliminaries. The concept of Gabor-Gram matrices is introduced. In Section 3, the structural properties of Gabor-Gram matrices are investigated. Then we present fast algorithms for computing the *Gabor coefficients* and the *dual Gabor windows* [18, 19, 20] in Section 4. Special cases including critical sampling cases are considered. Simple algorithms are derived to compute both the *Gabor coefficients* and the *dual Gabor windows.* Numerical results are illustrated in Section 5.

2. Notation and Preliminaries

Throughout this article, we use notation introduced in [18, 19, 20]. Bold letters (e.g., $\mathbf{g}$, $\mathbf{A}$) denote row vectors and matrices. We use $\mathbf{A} = \left(A_{k,l}\right)_{p\times q}$ to denote a matrix in $\mathbb{C}^{p\times q}$, where $A_{k,l}$ denotes the (k, l)th entry of $\mathbf{A}$. $\mathbf{x} = \left(x_j\right)_{j=0}^{N-1}$ (or $(x(j))_{j=0}^{N-1}$) denotes the row vector in $\mathbb{C}^N$, where x_j or $x(j)$ is the $(j+1)$th entry of $\mathbf{g}$. Superscripted uppercase bold letters, such as $\mathbf{A}^{(p,q)}$, refer to submatrices of $\mathbf{A}$. A' denotes the *conjugate and transpose* of $\mathbf{A}$, while $\mathbf{A}^t$ is the *transpose* of $\mathbf{A}$. If $\mathbf{A} = \left(A_{k,l}\right)_{p_1\times q_1}$ and $\mathbf{B} = \left(B_{k,l}\right)_{p_2\times q_2}$, the tensor product $\mathbf{A}\otimes\mathbf{B}$ is defined to be the partitioned matrix [13, p.407]: $\mathbf{A}\otimes\mathbf{B} = \left(A_{k,l}\mathbf{B}\right) \in \mathbb{C}^{p_1p_2\times q_1q_2}$.

We view $\mathbb{C}^N \equiv L^2(\mathbb{Z}_N)$. Signals are considered as N-periodic row vectors in $\mathbb{C}^N$. The inner product of two signals $\mathbf{x} = (x_k)_{k=0}^{N-1} \in \mathbb{C}^N$ and $\mathbf{y} = (y_k)_{k=0}^{N-1} \in \mathbb{C}^N$ is given by $\langle \mathbf{x}, \mathbf{y}\rangle = sum_{k=0}^{N-1} x_k\overline{y_k}$. $||\mathbf{x}|| = \sqrt{\sum_{k=0}^{N-1}|x_k|^2}$ is the norm of $\mathbf{x}$. We use "*" to denote the usual matrix multiplication. $\mathbf{x} * \mathbf{A}$ is the matrix-vector multiplication of $\mathbf{x}$ and $\mathbf{A}$.

In addition, $\mathcal{F}_r$ denotes the *discrete Fourier transform* (DFT) and $\mathbf{F}_r$ denotes the *Fourier matrix* [6, p.32] of order r. $\mathcal{F}_r(\mathbf{x}) = \mathbf{x} * \mathbf{F}_r$ is the DFT of $\mathbf{x} \in \mathbb{C}^r$. $\text{diag}(d_1, d_2, \ldots, d_r)$ denotes the $r\times r$ *diagonal matrix* with diagonal elements d_k for $k = 1, 2, \ldots, r$. $\mathbf{I}_r$ is the $r\times r$ identity matrix and $\mathbf{e}_r$ is the r-dimensional unit row vector whose first entry is 1. We use $\text{circ}(c_1, c_2, \ldots, c_r)$ to denote the circulant matrix [6, p.66] and $\text{bcirc}(\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_r)$ to denote the block circulant matrix [6, p.176].

2.1 Gabor-Gram Matrices

Given $(\mathbf{g}, a, b)$, we define $\mathrm{GAB}_{(\mathbf{g},a,b)}$ as an $\tilde{N} \times N$ ($\tilde{N} = \frac{N^2}{ab}$) matrix of the following block form:

$$\mathrm{GAB}_{(\mathbf{g},a,b)} = \begin{pmatrix} \mathbf{g} \\ M_b\mathbf{g} \\ \vdots \\ M_{(\tilde{b}-1)b}\mathbf{g} \\ \vdots \\ T_{(\tilde{a}-1)a}\mathbf{g} \\ M_b T_{(\tilde{a}-1)a}\mathbf{g} \\ \vdots \\ M_{(\tilde{b}-1)b} T_{(\tilde{a}-1)a}\mathbf{g} \end{pmatrix}. \tag{2.1}$$

We call $\mathrm{GAB}_{(\mathbf{g},a,b)}$ the *Gabor-Gram basic matrix* associated with $(\mathbf{g}, a, b)$.

Remark. Obviously, $\mathrm{GAB}_{(\mathbf{g},a,b)}$ and the Gabor basic matrix $\mathrm{GAB}(\mathbf{g}, a, b)$ introduced in [18, 19] are the same, except for their row vectors being arranged in different order. Both the row vectors of $\mathrm{GAB}_{(\mathbf{g},a,b)}$ and $\mathrm{GAB}(\mathbf{g}, a, b)$ form the Gabor family $\{\mathbf{g}_{n,m}\}_{n,m}$. It is not difficult to check that

$$\left[\mathrm{GAB}_{(\mathbf{g},a,b)}\right]' * \left[\mathrm{GAB}_{(\mathbf{g},a,b)}\right] = [\mathrm{GAB}(\mathbf{g}, a, b)]' * [\mathrm{GAB}(\mathbf{g}, a, b)]\,. \quad \square \tag{2.2}$$

Definition 1. ***(Gabor-Gram Matrix)***

For Gabor triples $(\mathbf{g}_1, a, b)$ and $(\mathbf{g}_2, a, b)$, we call the $\tilde{N} \times \tilde{N}$ matrix

$$\mathbf{GM}_{(\mathbf{g}_1,\mathbf{g}_2,a,b)} = \left[GAB_{(\mathbf{g}_1,a,b)}\right] * \left[GAB_{(\mathbf{g}_2,a,b}\right]' \tag{2.3}$$

the Gabor-Gram matrix. *In particular, if $\mathbf{g}_1 = \mathbf{g}_2 = \mathbf{g}$, we write $\mathbf{GM}_{(\mathbf{g},a,b)} = \mathbf{GM}_{(\mathbf{g},\mathbf{g},a,b)}$ and call $\mathbf{GM}_{(\mathbf{g},a,b)}$ the* Gabor-Gram matrix *associated with $(\mathbf{g}, a, b)$. If no confusion occurs, we simply use* **GM** *to denote the Gabor-Gram matrix.*

It is easy to verify the following *bi-orthogonality* [23] in matrix forms.

Proposition 1.

Given $(\mathbf{g}, a, b)$ and a signal $\mathbf{h}$ and set $\hat{\mathbf{g}} = \mathcal{F}_N(\mathbf{g})$ and $\hat{\mathbf{h}} = \mathcal{F}(\mathbf{h})$. The following statements are equivalent:

1. *$(\mathbf{g}, a, b)$ generates a Gabor frame.*
2. $\mathbf{GM}_{(\mathbf{g},\mathbf{h},\tilde{b},\tilde{a})} = \mathbf{GM}_{(\mathbf{h},\mathbf{g},\tilde{b},\tilde{a})} = \frac{ab}{N}\mathbf{I}_{ab}$.
3. $\mathbf{h} * [GAB_{(\mathbf{g},\tilde{b},\tilde{a})}]' = \frac{ab}{N}\mathbf{e}_{ab}$.
4. $\mathbf{g} * [GAB_{(\mathbf{h},\tilde{b},\tilde{a})}]' = \frac{ab}{N}\mathbf{e}_{ab}$.
5. *$(\hat{\mathbf{g}}, b, a)$ generates a Gabor frame.*
6. $\mathbf{GM}_{(\hat{\mathbf{h}},\hat{\mathbf{g}},\tilde{a},\tilde{b})} = \mathbf{GM}_{(\hat{\mathbf{g}},\hat{\mathbf{h}},\tilde{a},\tilde{b})} = \mathbf{I}_{ab}$.
7. $\hat{\mathbf{h}} * [GAB_{(\hat{\mathbf{g}},\tilde{a},\tilde{b})}]' = \mathbf{e}_{ab}$.
8. $\hat{\mathbf{g}} * [GAB_{(\hat{\mathbf{h}},\tilde{a},\tilde{b})}]' = \mathbf{e}_{ab}$.

The last four conditions are based on the *commutative relation* [18, p.2262]. The fact is that $(\mathbf{g}, a, b)$ generates a Gabor frame if and only if $(\hat{\mathbf{g}}, b, a)$ does.

2.2 Tensor Discrete Fourier Transform (TDFT)

Definition 2. *[Tensor Discrete Fourier Transform (TDFT)]*

Let r_1 and r_2 be integers and $r = r_1 r_2$, we call

$$\begin{aligned} \mathbf{F}_{r_1,r_2,r} &= \mathbf{F}_{r_1} \otimes \mathbf{F}_{r_2} , \\ \mathbf{F}_{0,r_2,r} &= \mathbf{I}_{r_1} \otimes \mathbf{F}_{r_2} , \\ \mathbf{F}_{r_1,0,r} &= \mathbf{F}_{r_1} \otimes \mathbf{I}_{r_2} \end{aligned}$$

the tensor Fourier matrices *of order (r_1, r_2, r), $(0, r_2, r)$ and $(r_1, 0, r)$, respectively.*

The linear mapping:

$$\mathcal{F}_{r_1,r_2,r} : \mathbb{C}^r \mapsto \mathbb{C}^r, \quad \mathcal{F}_{r_1,r_2,r}(\mathbf{x}) = \mathbf{x} * \mathbf{F}_{r_1,r_2,r}$$

is called the tensor discrete Fourier transform (TDFT) *of order (r_1, r_2, r). Similarly, $\mathcal{F}_{0,r_2,r}$ and $\mathcal{F}_{r_1,0,r}$ can be defined.*

Proposition 2.

Tensor Fourier matrices $\mathbf{F}_{r_1,r_2,r}$, $\mathbf{F}_{r_1,0,r}$ *and* $\mathbf{F}_{0,r_2,r}$ *are $r \times r$ unitary matrices.*

Proof. By the properties of tensor products of matrices [13, pp.406–411],

$$\begin{aligned} \mathbf{F}'_{r_1,r_2,r} * \mathbf{F}_{r_1,r_2,r} &= \left(\mathbf{F}_{r_1} \otimes \mathbf{F}_{r_2}\right)' * \left(\mathbf{F}_{r_1} \otimes \mathbf{F}_{r_2}\right) = \left(\mathbf{F}'_{r_1} \otimes \mathbf{F}'_{r_2}\right) * \left(\mathbf{F}_{r_1} \otimes \mathbf{F}_{r_2}\right) \\ &= \left(\mathbf{F}'_{r_1} * \mathbf{F}_{r_1}\right) \otimes \left(\mathbf{F}'_{r_2} * \mathbf{F}_{r_2}\right) = \mathbf{I}_{r_1} \otimes \mathbf{I}_{r_2} = \mathbf{I} . \end{aligned}$$

Hence, $\mathbf{F}_{r_1,r_2,r}$ is a unitary matrix. With the same arguments, we show that $\mathbf{F}_{0,r_2,r}$ and $\mathbf{F}_{r_1,0,r}$ are unitary matrices. □

In practice, the *discrete Fourier transform* (DFT) is performed by the *fast Fourier transform* (FFT). For a signal $\mathbf{x} \in \mathbb{C}^r$, we show that the TDFT of $\mathbf{x}$ can be computed by FFT with the total complexity no more than $\mathcal{O}(r \log r)$.

Proposition 3.

For a signal $\mathbf{x} = (x_k)_{k=0}^{r-1} \in \mathbb{C}^r$ $(r = r_1 r_2)$, the TDFT of $\mathbf{x}$ are determined via FFT.

(i). *Let $\mathbf{y} = \mathcal{F}_{0,r_2,r}(\mathbf{x}) := (y_k)_{k=0}^{r-1}$. For $j = 0, 1, \ldots, r_1 - 1$, set $\mathbf{y}^{(j)} := (y_k)_{k=jr_2}^{(j+1)r_2-1}$ and $\mathbf{x}^{(j)} := (x_k)_{k=jr_2}^{(j+1)r_2-1}$. Then*

$$\mathbf{y}^{(j)} = \mathcal{F}_{r_2}\left(\mathbf{x}^{(j)}\right) . \tag{2.4}$$

The total complexity is $\mathcal{O}(r \log r_2)$.

(ii). *Let $\mathbf{y} = \mathcal{F}_{r_1,0,r}(\mathbf{x}) := (y_k)_{k=0}^{r-1}$. For $j = 0, 1, \ldots, r_2 - 1$, set $\mathbf{y}^{(j)} := \left(y_{j+sr_2}\right)_{s=0}^{r_1-1}$ and $\mathbf{x}^{(j)} := \left(x_{j+sr_2}\right)_{s=0}^{r_1-1}$. Then*

$$\mathbf{y}^{(j)} = \mathcal{F}_{r_1}\left(\mathbf{x}^{(j)}\right) . \tag{2.5}$$

The total complexity is $\mathcal{O}(r \log r_1)$.

(iii). *Let $\mathbf{y} = \mathcal{F}_{r_1,r_2,r}(\mathbf{x})$, then*

$$\mathbf{y} = \mathcal{F}_{0,r_2,r}\left(\mathcal{F}_{r_1,0,r}(\mathbf{x})\right) = \mathcal{F}_{r_1,0,r}\left(\mathcal{F}_{0,r_2,r}(\mathbf{x})\right) . \tag{2.6}$$

The total complexity is $\mathcal{O}(r \log r)$.

Proof. (i). For $j = 0, 1, \ldots, r_1 - 1$, let $\mathbf{e}^{(j+1)}$ be the r_1-dimensional unit vector whose $(j+1)$th entry is 1. Then $\mathbf{x} = \sum_{j=0}^{r_1-1} \mathbf{e}^{(j+1)} \otimes \mathbf{x}^{(j)}$. Thus,

$$\mathbf{y} = \mathcal{F}_{0,r_2,r}(\mathbf{x}) = \mathbf{x} * \left(\mathbf{I}_{r_1} \otimes \mathbf{F}_{r_2}\right)$$

$$
\begin{aligned}
&= \sum_{j=0}^{r_1-1} \left\{ \left(\mathbf{e}^{(j+1)} \otimes \mathbf{x}^{(j)} \right) * \left(\mathbf{I}_{r_1} \otimes \mathbf{F}_{r_2} \right) \right\} \\
&= \sum_{j=0}^{r_1-1} \left\{ \left(\mathbf{e}^{(j+1)} * \mathbf{I}_{r_1} \right) \otimes \left(\mathbf{x}^{(j)} * \mathbf{F}_{r_2} \right) \right\} \\
&= \sum_{j=0}^{r_1-1} \left\{ \mathbf{e}^{(j+1)} \otimes \mathcal{F}_{r_2} \left(\mathbf{x}^{(j)} \right) \right\} .
\end{aligned}
$$

Therefore,

$$
\sum_{j=0}^{r_1-1} \mathbf{e}^{(j+1)} \otimes \mathbf{y}^{(j)} = \mathbf{y} = \sum_{j=0}^{r_1-1} \left\{ \mathbf{e}^{(j+1)} \otimes \mathcal{F}_{r_2} \left(\mathbf{x}^{(j)} \right) \right\} .
$$

This leads to Eq. (2.4). The complexity is $r_1 \mathcal{O}(r_2 \log r_2) \leq \mathcal{O}(r \log r_2)$.

Equation (2.5) can be shown similarly. The complexity is $r_2 \mathcal{O}(r_1 \log r_1) \leq \mathcal{O}(r \log r_1)$.

Since $\mathbf{F}_{r_1} \otimes \mathbf{F}_{r_2} = \left(\mathbf{F}_{r_1} \otimes \mathbf{I}_{r_2}\right) * \left(\mathbf{I}_{r_1} \otimes \mathbf{F}_{r_2}\right) = \left(\mathbf{I}_{r_1} \otimes \mathbf{F}_{r_2}\right) * \left(\mathbf{F}_{r_1} \otimes \mathbf{I}_{r_2}\right)$, Equation (2.6) follows and the complexity is $r_1 \mathcal{O}(r_2 \log r_2) + r_2 \mathcal{O}(r_1 \log r_1) \leq \mathcal{O}(r \log r_2) + \mathcal{O}(r \log r_1) = \mathcal{O}(r \log r)$.
□

Similarly, the inverse tensor discrete Fourier transforms (ITDFT) $\mathcal{F}^{-1}_{0,r_2,r}$, $\mathcal{F}^{-1}_{r_1,0,r}$, and $\mathcal{F}^{-1}_{r_1,r_2,r}$ can be defined. They are performed with the total complexity $\mathcal{O}(r \log r_2)$, $\mathcal{O}(r \log r_1)$, and $\mathcal{O}(r \log r)$, respectively.

2.3 Rotation Operator σ_a

Definition 3. *(σ_a)*

Given a matrix $\mathbf{A} = \left(A_{k,l}\right)_{p \times q}$ and a positive integer a, we define $\mathbf{B} = \sigma_a(\mathbf{A}) := \left(B_{k,l}\right)_{p \times q}$, where $B_{k,l}$ is given by

$$
B_{k,l} = A_{1+\mathrm{mod}(k+p-2,p),1+\mathrm{mod}(l+a-1,q)} ,
$$

for $k = 1, 2, \ldots, p$ and $l = 1, 2, \ldots, q$.

Inductively, we define $\sigma^r(\mathbf{B}) = \sigma^{r-1}(\sigma(\mathbf{B}))$ for $r > 1$.

Proposition 4.

For a matrix $\mathbf{B} = \left(B_{k,l}\right)_{\tilde{a} \times \tilde{b}} \in \mathbb{C}^{\tilde{a},\tilde{b}}$, $\sigma_a^{\tilde{a}}(\mathbf{B}) = \mathbf{B}$.

Proof. By the definition of σ_a, we derive inductively that

$$
\begin{aligned}
\sigma_a^{\tilde{a}}(\mathbf{B}) &= \left(B_{1+\mathrm{mod}(k+\tilde{a}-(\tilde{a}+1),\tilde{a}),1+\mathrm{mod}(l+\tilde{a}a-1,\tilde{b})}\right)_{\tilde{a} \times \tilde{b}} \\
&= \left(B_{1+(k-1),1+(l-1)}\right)_{\tilde{a} \times \tilde{b}} \\
&= \left(B_{k,l}\right)_{\tilde{a} \times \tilde{b}} .
\end{aligned}
$$

Thus, $\sigma_a^{\tilde{a}}(\mathbf{B}) = \mathbf{B}$. □

3. Characterizations of Gabor-Gram Matrices

In this section, we show that the Gabor-Gram matrices are determined by the *unitarily equivalent Gabor-Gram matrices* (UEGM) via TDFT. We characterize the banded and block structures of UEGM matrices. If N divides ab, we show that the Gabor-Gram matrices are block-circulant matrices. The structural observations on these matrices are the key to derive fast algorithms.

3.1 Gabor-Gram Matrix Structures

Theorem 1.

Let $\mathbf{GM} = \mathbf{GM}_{(\mathbf{g}_1,\mathbf{g}_2,a,b)} := \left(G_{k,l}\right)_{\tilde{N}\times\tilde{N}}$.

(1) *For any* $k = j_1\tilde{b}+r_1$ *and* $l = j_2\tilde{b}+r_2$ *with* $j_1, j_2 = 0, 1, \ldots, \tilde{a}-1$ *and* $r_1, r_2 = 1, 2, \ldots, \tilde{b}$,

$$G_{k,l} = \sum_{s=0}^{N-1} \left\{\omega^{s(r_1-r_2)b} T_{j_1a}g_1(s)\overline{T_{j_2a}g_2(s)}\right\} , \tag{3.1}$$

where $\omega = e^{-2\pi i/N}$.

(2) *For* $p, q = 1, 2, \ldots, \tilde{a}$, *define* $\mathbf{D}^{(p,q)} = \left(D_{k,l}^{(p,q)}\right)_{\tilde{b}\times\tilde{b}}$ *with* $D_{k,l}^{(p,q)} = G_{(p-1)\tilde{b}+k,(q-1)\tilde{b}+l}$ *for* $k, l = 1, 2, \ldots, \tilde{b}$. *Then* $\mathbf{D}^{(p,q)}$ *are* $\tilde{b}\times\tilde{b}$ circulant matrices *and* **GM** *is of the following block form.*

$$\mathbf{GM} = \begin{pmatrix} \mathbf{D}^{(1,1)} & \mathbf{D}^{(1,2)} & \ldots & \mathbf{D}^{(1,\tilde{a})} \\ \mathbf{D}^{(2,1)} & \mathbf{D}^{(2,2)} & \ldots & \mathbf{D}^{(2,\tilde{a})} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}^{(\tilde{a},1)} & \mathbf{D}^{(\tilde{a},2)} & \ldots & \mathbf{D}^{(\tilde{a},\tilde{a})} \end{pmatrix} . \tag{3.2}$$

(3) *For* $p, q = 1, 2, \ldots, \tilde{a}-1$,

$$D_{k,l}^{(p+1,q+1)} = D_{k,l}^{(p,q)} \cdot \varpi^{k-l} \text{ for } k, l = 1, 2, \ldots, \tilde{b} , \tag{3.3}$$

where $\varpi = e^{-2a\pi i/\tilde{b}}$.

Proof. (1) The (k, l)th entry of **GM** is given as

$$\begin{aligned} G_{k,l} &= \left(M_{(r_1-1)b}T_{j_1a}\mathbf{g}_1\right) * \left(M_{(r_2-1)b}T_{j_2a}\mathbf{g}_2\right)' \\ &= \sum_{s=0}^{N-1} \left(\omega^{s(r_1-1)b}T_{j_1a}g_1(s)\right) * \left(\overline{\omega^{s(r_2-1)b}T_{j_2a}g_2(s)}\right) \\ &= \sum_{s=0}^{N-1} \left\{\omega^{s(r_1-r_2)b}T_{j_1a}g_1(s)\overline{T_{j_2a}g_2(s)}\right\} . \end{aligned}$$

We obtain Eq. (3.1).

(2) By Eq. (3.1),

$$D_{k,l}^{(p,q)} = G_{(p-1)\tilde{b}+k,(q-1)\tilde{b}+l} = \sum_{s=0}^{N-1} \left\{\omega^{s(k-l)b}T_{(p-1)a}g_1(s)\overline{T_{(q-1)a}g_2(s)}\right\} .$$

For any two pairs (k_1, l_1) and (k_2, l_2) with $k_1, l_1, k_2, l_2 = 1, 2, \ldots, \tilde{b}$, if $k_1 - l_1 = k_2 - l_2 \pmod{\tilde{b}}$, then $(k_1 - l_1)b = (k_2 - l_2)b \pmod{N}$. This yields $\omega^{s(k_1-l_1)b} = \omega^{s(k_2-l_2)b}$. Hence,

$$G_{(p-1)\tilde{b}+k_1,(q-1)\tilde{b}+l_1} = G_{(p-1)\tilde{b}+k_2,(q-1)\tilde{b}+l_2} .$$

Therefore, $D_{k_1,l_1}^{(p,q)} = D_{k_2,l_2}^{(p,q)}$. $\mathbf{D}_{p,q}$ is a circulant matrix.

(3) Using Eq. (3.1),

$$\begin{aligned} D_{k,l}^{(p+1,q+1)} &= G_{p\tilde{b}+k,q\tilde{b}+l} \\ &= \sum_{s=0}^{N-1} \left\{\omega^{s(k-l)b}T_{pa}g_1(s)\overline{T_{qa}g_2(s)}\right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{s=0}^{N-1} \left\{ \omega^{(s+a)(k-l)b} T_{(p-1)a} g_1(s) \overline{T_{(q-1)a} g_2(s)} \right\} \\
&= \sum_{s=0}^{N-1} \left\{ \omega^{a(k-l)b} \omega^{s(k-l)b} T_{(p-1)a} g_1(s) \overline{T_{(q-1)a} g_2(s)} \right\} \\
&= \omega^{a(k-l)b} \sum_{s=0}^{N-1} \left\{ \omega^{s(k-l)b} T_{(p-1)a} g_1(s) \overline{T_{(q-1)a} g_2(s)} \right\} \\
&= \omega^{a(k-l)b} D_{k,l}^{(p,q)} = \varpi^{(k-l)} D_{k,l}^{(p,q)} .
\end{aligned}
$$

Equation (3.3) is derived. □

Corollary 1.

Let $\mathbf{Q} = \left(Q_{k,l}\right)_{\tilde{b}\times\tilde{b}}$ *be the circulant matrix whose first row vector is* $\mathbf{q} = \left(\varpi^{-j}\right)_{j=0}^{\tilde{b}-1}$. *Then* $\mathbf{D}^{(p+1,q+1)}$ *is the Hadamard product of* $\mathbf{D}^{(p,q)}$ *and* $\mathbf{Q}$ *[10, 11]:*

$$
\mathbf{D}^{(p+1,q+1)} = \left(D_{k,l}^{(p,q)} \cdot Q_{k,l}\right)_{\tilde{b}\times\tilde{b}} \quad \text{for } p, q = 0, 1, \ldots, \tilde{a}-1 .
$$

Theorem 2. (Banded Structure)

Let $\widetilde{\mathbf{GM}} := \mathbf{U}' * \mathbf{GM} * \mathbf{U}$ *with* $\mathbf{U} = \mathbf{F}_{0,\tilde{b},\tilde{N}}$. *Then all the possible nonzero entries of* $\widetilde{\mathbf{GM}}$ *are in the kth diagonals for* $k = 0, \pm(\tilde{b}-1), \ldots, \pm(\tilde{b}-1)(\tilde{a}-1)$.

Proof. By the properties of *tensor products* of matrices [13, pp.406–411],

$$
\begin{aligned}
\widetilde{\mathbf{GM}} &= \mathbf{U}' * \mathbf{GM} * \mathbf{U} = \mathbf{F}'_{0,\tilde{b},\tilde{N}} * \mathbf{GM} * \mathbf{F}_{0,\tilde{b},\tilde{N}} \\
&= \left(\mathbf{I}_{\tilde{a}} \otimes \mathbf{F}'_{\tilde{b}}\right) * \begin{pmatrix} \mathbf{D}^{(1,1)} & \mathbf{D}^{(1,2)} & \ldots & \mathbf{D}^{(1,\tilde{a})} \\ \mathbf{D}^{(2,1)} & \mathbf{D}^{(2,2)} & \ldots & \mathbf{D}^{(2,\tilde{a})} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{D}^{(\tilde{a},1)} & \mathbf{D}^{(\tilde{a},2)} & \ldots & \mathbf{D}^{(\tilde{a},\tilde{a})} \end{pmatrix} * \left(\mathbf{I}_{\tilde{a}} \otimes \mathbf{F}_{\tilde{b}}\right) \\
&= \begin{pmatrix} \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,1)} * \mathbf{F}_{\tilde{b}} & \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,2)} * \mathbf{F}_{\tilde{b}} & \ldots & \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,\tilde{a})} * \mathbf{F}_{\tilde{b}} \\ \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(2,1)} * \mathbf{F}_{\tilde{b}} & \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(2,2)} * \mathbf{F}_{\tilde{b}} & \ldots & \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(2,\tilde{a})} * \mathbf{F}_{\tilde{b}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(\tilde{a},1)} * \mathbf{F}_{\tilde{b}} & \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(\tilde{a},2)} * \mathbf{F}_{\tilde{b}} & \ldots & \mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(\tilde{a},\tilde{a})} * \mathbf{F}_{\tilde{b}} \end{pmatrix} .
\end{aligned}
$$

By Theorem 1, $\mathbf{D}^{(p,q)}$ $(p, q = 1, 2, \ldots, \tilde{a})$ are circulant matrices. Thus [6, pp.72–80], $\mathbf{F}'_b * \mathbf{D}^{(p,q)} * \mathbf{F}_b$ $(p, q = 1, 2, \ldots, \tilde{b})$ are diagonal matrices. Therefore, $\widetilde{\mathbf{GM}}$ holds the *banded* structure as stated.
The proof is complete. □

We call $\widetilde{\mathbf{GM}}$ the *unitarily equivalent Gabor-Gram matrix* (UEGM) associated with Gabor triples $(\mathbf{g}_1, a, b)$ and $(\mathbf{g}_2, a, b)$. When $\mathbf{g}_1 = \mathbf{g}_2 := \mathbf{g}$, we say $\widetilde{\mathbf{GM}}$ is the UEGM matrix corresponding to $(\mathbf{g}, a, b)$. It is clear that $\widetilde{\mathbf{GM}}$ and $\mathbf{GM}$ are determined by each other via $\mathbf{F}_{0,\tilde{b},\tilde{N}}$. The total number of possible nonzero elements of $\widetilde{\mathbf{GM}}$ is $\tilde{a} \cdot (\tilde{a} \cdot \tilde{b}) = \tilde{N}\tilde{a}$.

Corollary 2.

If $a = N$, $\widetilde{\mathbf{GM}}$ *is a* $\tilde{b} \times \tilde{b}$ *diagonal matrix.*

Corollary 3.

If $\widetilde{\mathbf{GM}} = \left(\tilde{G}_{k,l}\right)_{\tilde{N}\times\tilde{N}}$, *then* $\tilde{G}_{k,l} = 0$ *if* $|k-l| \neq 0 \pmod{\tilde{b}}$.

Corollaries 2 and 3 are straightforward consequences of Theorem 2. Now let us do the following "zero-extracting" trick. Set $\mathbf{B}^{(1)} = \left(B_{k,l}^{(1)}\right)_{\tilde{a}\times\tilde{b}}$ with $B_{k,l}^{(1)} = \tilde{G}_{(k-1)\tilde{b}+l,l}$. Similarly, $\mathbf{B}^{(2)} = \left(B_{k,l}^{(2)}\right)_{\tilde{a}\times\tilde{b}}$ with $B_{k,l}^{(2)} = \tilde{G}_{(k-1)\tilde{b}+l,\tilde{b}+l}$. In general, for $j = 1, 2, \ldots, \tilde{a}$, $\mathbf{B}^{(j)} = \left(B_{k,l}^{(j)}\right)_{\tilde{a}\times\tilde{b}}$ with $B_{k,l}^{(j)} = \tilde{G}_{(k-1)\tilde{b}+l,(j-1)\tilde{b}+l}$ for $k = 1, 2, \ldots, \tilde{a}$ and $l = 1, 2, \ldots, \tilde{b}$.

Theorem 3. (Block Structure)
$\mathbf{B}^{(j+1)} = \sigma_a^j(\mathbf{B}^{(1)})$ *for* $j = 1, 2 \ldots, \tilde{a} - 1$.

Proof. The proof is based on Eq. (3.3). For $p, q = 1, 2, \ldots, \tilde{a}$, set $\mathbf{d}^{(p,q)} := \left(D_{1,l}^{(p,q)}\right)_{l=1}^{\tilde{b}}$, the first row vector of $\mathbf{D}^{(p,q)}$. Let $\mathbf{b}_p^{(q)}$ be the pth row vector of $\mathbf{B}^{(q)}$. By the construction of $\mathbf{B}^{(q)}$, $\mathbf{b}_p^{(q)}$ is the diagonal vector of $\mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(p,q)} * \mathbf{F}_{\tilde{b}}$. Thus,

$$\mathbf{b}_p^{(q)} = \mathcal{F}_{\tilde{b}}\left(\mathbf{d}^{(p,q)}\right) := \left(\hat{D}_{1,l}^{(p,q)}\right)_{l=1}^{\tilde{b}} .$$

By Eq. (3.3) we deduce that

$$\begin{aligned}
\left(\hat{D}_{1,l}^{(p+1,q+1)}\right)_{l=1}^{\tilde{b}} &= \mathbf{b}_{p+1}^{(q+1)} = \mathcal{F}_{\tilde{b}}\left(\mathbf{d}^{(p+1,q+1)}\right) \\
&= \mathcal{F}_{\tilde{b}}\left(\left(D_{1,l}^{(p+1,q+1)}\right)_{l=1}^{\tilde{b}}\right) \\
&= \mathcal{F}_{\tilde{b}}\left(\left(\varpi^{1-l}\cdot D_{1,l}^{(p,q)}\right)_{l=1}^{\tilde{b}}\right) \\
&= \mathcal{F}_{\tilde{b}}\left(\left(\omega^{a(1-l)}\cdot D_{1,l}^{(p,q)}\right)_{l=1}^{\tilde{b}}\right) \\
&= \mathcal{F}_{\tilde{b}}\left(\left(e^{2\pi i a(l-1)/\tilde{b}}\cdot D_{1,l}^{(p,q)}\right)_{l=1}^{\tilde{b}}\right) \\
&= \left(\hat{D}_{1,l+a}^{(p,q)}\right)_{l=1}^{\tilde{b}} .
\end{aligned}$$

Therefore, for $l = 1, 2, \ldots, \tilde{b}$,

$$\hat{D}_{1,l}^{(p+1,q+1)} = \hat{D}_{1,1+\mathrm{mod}(l+a-1,\tilde{b})}^{(p,q)} .$$

This yields

$$\mathbf{B}^{(q+1)} = \sigma_a(\mathbf{B}^{(q)}) \text{ for } q = 1, 2, \ldots, \tilde{a} - 1 .$$

Inductively, we obtain that

$$\mathbf{B}^{(q+1)} = \sigma_a^q(\mathbf{B}^{(1)}) \text{ for } j = 1, 2, \ldots, \tilde{a} - 1 . \qquad \square$$

Remark. Theorems 2 and 3 imply that $\widetilde{\mathbf{GM}}$ is determined completely by the $\tilde{a} \times \tilde{b}$ matrix $\mathbf{B}^{(1)}$. For simplicity, we write $\widetilde{\mathbf{B}} = \mathbf{B}^{(1)}$. We call $\widetilde{\mathbf{B}}$ the *unitarily equivalent Gabor-Gram nonzero-block matrix* (UEGNB) associated with **GM**. $\square$

3.2 Block-Circulant Gabor-Gram Matrices

In this section we consider the cases that N divides ab. In particular, if $N = ab$, it is critical sampling. We show that the Gabor-Gram matrices are unitarily diagonalized by TDFT.

Theorem 4.

If N divides ab, then **GM** *is a* block circulant *matrix and* **GM** *is* diagonalizable *by the unitary matrix* $\mathbf{V} = \mathbf{F}_{\tilde{a},\tilde{b},\tilde{N}}$. *Moreover, if*

$$\mathbf{d} = \mathcal{F}_{0,\tilde{b},\tilde{N}}\left(\mathcal{F}_{\tilde{a},0,\tilde{N}}\left(\mathbf{g}_1 * GAB_{(\mathbf{g}_2,a,b)}\right)\right) := \left(d_j\right)_{j=0}^{\tilde{N}-1} ,$$

then

$$\mathbf{V}' * \mathbf{GM} * \mathbf{V} = diag\left(d_0, d_1, \ldots, d_{\tilde{N}-1}\right) . \tag{3.4}$$

Proof. (i) Based on Eq. (3.1) in Theorem 1, we show that

$$\mathbf{GM} = \text{ bcirc }\left(\mathbf{D}^{(1,1)}, \mathbf{D}^{(1,2)}, \ldots, \mathbf{D}^{(1,\tilde{a})}\right) . \tag{3.5}$$

For any two pairs of indices (p_1, q_1) and (p_2, q_2) with $p_1, p_2, q_1, q_2 = 1, 2, \ldots, \tilde{a}$, we need to show that

$$\mathbf{D}^{(p_1,q_1)} = \mathbf{D}^{(p_2,q_2)}, \text{ if } p_1 - q_1 = p_2 - q_2 \pmod{\tilde{a}} .$$

In fact, by Theorem 1, $D_{k,l}^{(p_1,q_1)} = G_{(p_1-1)\tilde{b}+k,(q_1-1)\tilde{b}+l}$ and $D_{k,l}^{(p_2,q_2)} = G_{(p_2-1)\tilde{b}+k,(q_2-1)\tilde{b}+l}$, for $k, l = 1, 2, \ldots, \tilde{b}$. Since

$$\begin{aligned}
G_{(p_1-1)\tilde{b}+k,(q_1-1)\tilde{b}+l} &= \sum_{s=0}^{N-1}\left\{\omega^{s(k-l)b} T_{(p_1-1)a}g_1(s)\overline{T_{(q_1-1)a}g_2(s)}\right\} \\
&= \sum_{s=0}^{N-1}\left\{\omega^{s(k-l)b} T_{(p_2-1)a}g_1(s-(p_1-p_2)a)\overline{T_{(q_2-1)a}g_2(s-(p_1-p_2)a)}\right\} \\
&= \sum_{s=0}^{N-1}\left\{\omega^{(s-(p_1-p_2)a)(k-l)b} T_{(p_2-1)a}g_1(s)\overline{T_{(q_2-1)a}g_2(s)}\right\} \\
&= \sum_{s=0}^{N-1}\left\{\omega^{-(p_1-p_2)(k-l)N}\omega^{(s(k-l)b} T_{(p_2-1)a}g_1(s)\overline{T_{(q_2-1)a}g_2(s)}\right\} \\
&= G_{(p_2-1)\tilde{b}+k,(q_2-1)\tilde{b}+l} ,
\end{aligned}$$

$D_{k,l}^{(p_1,q_1)} = D_{k,l}^{(p_2,q_2)}$. Therefore, $\mathbf{D}_{k,l}^{(p_1,q_1)} = \mathbf{D}_{k,l}^{(p_2,q_2)}$.

(ii) Let π be the $\tilde{b}\times\tilde{b}$ basic circulant matrix [6, p.67]. Then **GM** can be written as the following form [6, p.178]:

$$\mathbf{GM} = \mathbf{I}_{\tilde{a}} \otimes \mathbf{D}^{(1,1)} + \pi \otimes \mathbf{D}^{(1,2)} + \ldots + \pi^{\tilde{a}-1} \otimes \mathbf{D}^{(1,\tilde{a})} . \tag{3.6}$$

By the properties of matrix-tensor-products [13, p.408],

$$\begin{aligned}
\mathbf{V}' * \mathbf{GM} * \mathbf{V} &= \sum_{k=0}^{\tilde{a}-1} \mathbf{V}' * \left(\pi^k \otimes \mathbf{D}^{(1,k+1)}\right) * \mathbf{V} \\
&= \sum_{k=0}^{\tilde{a}-1} \mathbf{F}'_{\tilde{a},\tilde{b},\tilde{N}} * \left(\pi^k \otimes \mathbf{D}^{(1,k+1)}\right) * \mathbf{F}_{\tilde{a},\tilde{b},\tilde{N}} \\
&= \sum_{k=0}^{\tilde{a}-1} \left(\mathbf{F}'_{\tilde{a}} \otimes \mathbf{F}'_{\tilde{b}}\right) * \left(\pi^k \otimes \mathbf{D}^{(1,k+1)}\right) * \left(\mathbf{F}_{\tilde{a}} \otimes \mathbf{F}_{\tilde{b}}\right)
\end{aligned}$$

$$= \sum_{k=0}^{\tilde{a}-1} \left(\mathbf{F}'_{\tilde{a}} * \pi^k * \mathbf{F}_{\tilde{a}}\right) \otimes \left(\mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,k+1)} * \mathbf{F}_{\tilde{b}}\right)$$

$$= \sum_{k=0}^{\tilde{a}-1} \overline{\Omega}^k \otimes \left(\mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,k+1)} * \mathbf{F}_{\tilde{b}}\right) ,$$

where $\overline{\Omega}$ is a diagonal matrix [6, p.72]:

$$\overline{\Omega} = \operatorname{diag}\left(1, w, \dots, w^{\tilde{a}-1}\right) \text{ with } w = \exp(-2\pi i/\tilde{a})$$

and

$$\overline{\Omega}^k = \operatorname{diag}\left(1, w^k, \dots, w^{(\tilde{a}-1)k}\right) .$$

The last equality follows from [6, Theorem 3.2.1].

Since $\mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,k+1)} * \mathbf{F}_{\tilde{b}}$ $(k = 0, 1, \dots, \tilde{a}-1)$ are *diagonal* matrices, $\mathbf{V}' * \mathbf{GM} * \mathbf{V}$ is a diagonal matrix.

Equation (3.4) follows from the above formula that $\mathbf{V}' * \mathbf{GM} * \mathbf{V} = \sum_{k=0}^{\tilde{a}-1} \overline{\Omega}^k \otimes \left(\mathbf{F}'_{\tilde{b}} * \mathbf{D}^{(1,k+1)} * \mathbf{F}_{\tilde{b}}\right)$.

The proof is complete. □

Corollary 4.

If $ab = N$, then $(\mathbf{g}, a, b)$ generates a Gabor frame if and only if all the entries of $\mathbf{d} = \mathcal{F}_{0,\tilde{b},\tilde{N}}\left(\mathcal{F}_{\tilde{a},0,\tilde{N}}\left(\mathbf{g} * GAB_{(\mathbf{g},a,b)}\right)\right)$ *are* strictly positive. *The Gabor frame upper and lower bounds are given by the maximal and minimal elements of* $\mathbf{d}$.

Since the nonzero eigenvalues of $\mathbf{GM}_{(\mathbf{g},a,b)} = [\mathrm{GAB}_{(\mathbf{g},a,b)}] * [\mathrm{GAB}_{(\mathbf{g},a,b)}]'$ are the same as those of $\mathbf{S} = [\mathrm{GAB}_{(\mathbf{g},a,b)}]' * [\mathrm{GAB}_{(\mathbf{g},a,b)}]$ from [10, p.53], Corollary 4 follows.

4. Computations of Gabor Coefficients and Dual Gabor Windows

In this section, we show that the UEGM matrix-vector multiplications can be performed efficiently via $\widetilde{\mathbf{B}}$. Then we are able to present CG-algorithms for computing the Gabor coefficients and the dual Gabor windows. For the cases that N divides ab, simple algorithms are developed.

4.1 UEGM Matrix-Vector Multiplication

Algorithm 1 (UEGM Matrix-Vector Multiplication) For a vector $\tilde{\mathbf{x}} \in \mathbb{C}^{\tilde{N}}$, $\tilde{\mathbf{y}} = \tilde{\mathbf{x}} * \widetilde{\mathbf{GM}}$ is determined as follows.

1. Calculate the first column vector of **GM** by the *short-time Fourier transform (STFT)*:

$$\mathbf{u} = \mathrm{GAB}_{(\mathbf{g}_1,a,b)} * \mathbf{g}'_2 := (u_k)_{k=0}^{\tilde{N}-1} . \tag{4.1}$$

2. Compute $\widetilde{\mathbf{B}}$:

$$\widetilde{\mathbf{B}} = \begin{pmatrix} \mathcal{F}_{\tilde{b}}^{-1}(\mathbf{b}_1) \\ \mathcal{F}_{\tilde{b}}^{-1}(\mathbf{b}_2) \\ \vdots \\ \mathcal{F}_{\tilde{b}}^{-1}(\mathbf{b}_{\tilde{a}}) \end{pmatrix} , \tag{4.2}$$

where $\mathbf{b}_j = \left(u_{(j-1)\tilde{b}+l}\right)_{l=0}^{\tilde{b}-1}$ $(j = 1, 2, \ldots, \tilde{a})$ are $\tilde{b}$-dimensional row vectors.

3. Write $\widetilde{\mathbf{B}} = \left(\tilde{B}_{k,l}\right)_{\tilde{a}\times\tilde{b}}$, then compute $\tilde{\mathbf{y}}$: for $k = p\tilde{b} + r$ with $p = 0, 1, \ldots, \tilde{a} - 1$ and $r = 1, 2, \ldots, \tilde{b}$,

$$\tilde{y}_k = \sum_{s=1}^{\tilde{a}} \tilde{x}_{1+\mathrm{mod}(r+(p+s-1)\tilde{b}-1,\tilde{N})} \tilde{B}_{s,1+\mathrm{mod}(r+pa-1,\tilde{b})} \,. \tag{4.3}$$

Proof. The first two steps are based on Theorem 2. The third step follows from Theorem 3. □

4.2 Iterative Algorithms for Gabor Coefficients and Dual Gabor Windows

The derivations of the following algorithms are based on the properties of pseudo-inverse [8, 10]: $pinv(\mathbf{A}) = \mathbf{A} * pinv(\mathbf{A} * \mathbf{A}')$ and $pinv(\mathbf{A}') = pinv(\mathbf{A} * \mathbf{A}') * \mathbf{A}$, where $pinv(\mathbf{A})$ denotes the *pseudo-inverse* of a matrix $\mathbf{A}$. The idea has been used by Feichtinger [8].

Algorithm 2 (Gabor Coefficients) For a signal $\mathbf{x} \in \mathbb{C}^N$, the Gabor coefficients $\mathbf{c} = \{c_{n,m}\}_{n,m}$ with respect to (g, a, b) are determined via the following:

1st step: Calculate $\mathbf{x}_1 = \mathbf{x} * [GAB_{(\mathbf{g},a,b)}]'$ by STFT.

2nd step: Calculate $\mathbf{x}_2 = \mathcal{F}_{0,\tilde{b},\tilde{N}}(\mathbf{x}_1)$ by TDFT.

3rd step: Solve the linear equation $\mathbf{x}_2 = \mathbf{x}_3 * \widetilde{\mathbf{GM}}$ by CG-algorithm, where $\widetilde{\mathbf{GM}}$ is the UEGM associated with $(\mathbf{g}, a, b)$. The matrix-vector multiplications in each iterative CG-step are performed by Algorithm 1.

4th step: Compute $\mathbf{c} = \{c_{n,m}\}_{n,m}$ by applying ITDFT. Set $\mathbf{c}_v = \mathcal{F}^{-1}_{0,\tilde{b},\tilde{N}}(\mathbf{x}_3) := \left(v_j\right)_{j=0}^{\tilde{N}-1}$, then $c_{n,m} = v_{n\tilde{b}+m}$ for $n = 0, 1, \ldots, \tilde{a} - 1$; $m = 0, 1, \ldots, \tilde{b} - 1$.

Proof. First, we need to show the CG-algorithm of the third step is convergent. In fact, it is easy to verify that $\mathbf{x}_2$ is in the column space of $\widetilde{\mathbf{GM}}$. By the same argument used in [20, Algorithm 3], we can show the CG-convergence.

By Eq. (1.1), $\mathbf{x} = \mathbf{c}_v * \mathrm{GAB}_{(\mathbf{g},a,b)}$. Write $\mathbf{GM} = \mathbf{GM}_{(\mathbf{g},a,b)} = [\mathrm{GAB}_{(\mathbf{g},a,b)}] * [\mathrm{GAB}_{(\mathbf{g},a,b)}]'$, then

$$\mathbf{x}_1 = \mathbf{x} * \left[\mathrm{GAB}_{(\mathbf{g},a,b)}\right]' = \mathbf{c}_v * \mathbf{GM} \,. \tag{4.4}$$

By Theorem 2, $\widetilde{\mathbf{GM}} = \mathbf{U}' * \mathbf{GM} * \mathbf{U}$ with $\mathbf{U} = \mathbf{F}_{0,\tilde{b},\tilde{N}}$. Equation (4.4) implies

$$\mathbf{x}_2 = \mathbf{x}_1 * \mathbf{U} = \mathbf{c}_v * \mathbf{U} * \widetilde{\mathbf{GM}} = \mathbf{x}_3 * \widetilde{\mathbf{GM}} \,, \tag{4.5}$$

where $\mathbf{x}_3 = \mathbf{c}_v * U$. Hence, $\mathbf{c}_v = \mathbf{x}_3 * U' = \mathcal{F}_{0,\tilde{b},\tilde{N}}(\mathbf{x}_3)$. □

Algorithm 3 (Dual Gabor Window $\tilde{\mathbf{g}}$) Given $(\mathbf{g}, a, b)$, $\tilde{\mathbf{g}}$ is determined as follows.

1st step: Set $\mathbf{x}_1 = \frac{\sqrt{ab}}{N}(\underbrace{1, \ldots, 1}_{a}, \underbrace{0, \ldots, 0}_{ab-a})$.

2nd step: Solve the linear equation $\mathbf{x}_1 = \mathbf{x}_2 * \widetilde{\mathbf{GM}}$ by either (i) or (ii), where $\widetilde{\mathbf{GM}}$ is the UEGM associated with $(\mathbf{g}, \tilde{b}, \tilde{a})$.

(i) The sparse techniques: $\mathbf{x}_2 = \mathbf{x}_1/\mathrm{sparse}(\widetilde{\mathbf{GM}})$.

(ii) The standard CG-method if $(\mathbf{g}, a, b)$ generates a Gabor frame.

3rd step: Applying ITDFT, $\mathbf{x}_3 = \mathcal{F}^{-1}_{0,a,ab}(\mathbf{x}_2)$.

4th step: Compute $\tilde{\mathbf{g}} = \mathbf{x}_3 * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})}$ by STFT.

Proof. It is easy to check that $\mathbf{x}_1 = \mathcal{F}_{0,a,ab}(\frac{ab}{N}\mathbf{e}_{ab}) = \frac{ab}{N}\mathcal{F}_{0,a,ab}(\mathbf{e}_{ab})$. Let $\mathbf{GM} = \mathbf{GM}_{(\mathbf{g},\tilde{b},\tilde{a})}$. If $(\mathbf{g}, a, b)$ generates a Gabor frame, then $\mathbf{GM}$ is invertible. By Proposition 1,

$$
\begin{aligned}
\tilde{\mathbf{g}} &= \tilde{\mathbf{g}} * \left\{\mathrm{GAB}'_{(\mathbf{g},\tilde{b},\tilde{a})} * \mathbf{GM}^{-1} * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})}\right\} \\
&= \left(\tilde{\mathbf{g}} * \mathrm{GAB}'_{(\mathbf{g},\tilde{b},\tilde{a})}\right) * \left\{\mathbf{GM}^{-1} * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})}\right\} \\
&= \left(\frac{ab}{N}\mathbf{e}_{ab}\right) * \left\{\mathbf{GM}^{-1} * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})}\right\} \\
&= \left(\frac{ab}{N}\mathbf{e}_{ab} * \mathbf{GM}^{-1}\right) * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})} \\
&= \left(\mathbf{x}_1 * \widetilde{\mathbf{GM}}^{-1} * \mathbf{F}^{-1}_{0,a,ab}\right) * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})} \\
&= \left(\mathbf{x}_2 * \mathbf{F}^{-1}_{0,a,ab}\right) * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})} \\
&= \mathbf{x}_3 * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})},
\end{aligned}
$$

where $\mathbf{x}_2$ can be determined by (i) or (ii). If $(\mathbf{g}, a, b)$ does not generate a frame, we use (i) to solve for $\mathbf{x}_2$.

The proof is complete. □

Remark. By Algorithm 1, the UEGM matrix-vector multiplication is determined by $\widetilde{\mathbf{B}}$. Since $\widetilde{\mathbf{B}}$ can be pre-calculated, the matrix-vector multiplications of the CG-iterations in Algorithms 2 and 3 are efficiently performed. The algorithms work fast.

For Algorithm 3, without the assumption that $(\mathbf{g}, a, b)$ generates a Gabor frame, $\mathbf{x}_1$ may not be in the column space of $\widetilde{\mathbf{GM}}$. The assumption is necessary for the CG-convergence. For Algorithm 2, however, we have shown the CG-convergence without assuming that $(\mathbf{g}, a, b)$ generates a frame. □

4.3 Special Cases: N Divides ab

In this section, based on the results developed in Section 3.2, we present simple algorithms for computing the Gabor coefficients and the dual Gabor windows for the cases that N divides ab. By making use of the results in [18, 21], we can derive similar methods for the cases that ab divides N.

Algorithm 4 (Gabor Coefficients: N divides ab) Given $\mathbf{x} \in \mathbb{C}^N$ and assume that N divides ab, the Gabor coefficients $\mathbf{c} = \{c_{nm}\}_{n,m}$ with respect to $(\mathbf{g}, a, b)$ are determined via the following steps:

1st step: Calculate $\mathbf{x}_1 = \mathbf{x} * [GAB_{(\mathbf{g},a,b)}]'$ and $\mathbf{u} = \mathbf{g} * [GAB_{(\mathbf{g},a,b)}]'$ by STFT.

2nd step: Calculate $\mathbf{d} = \mathcal{F}_{\tilde{a},\tilde{b},\tilde{N}}(\mathbf{u}) := (d_j)_{j=0}^{\tilde{N}-1}$, by TDFT.

3rd step: Calculate $\mathbf{x}_2 = \mathcal{F}_{\tilde{a},\tilde{b},\tilde{N}}(\mathbf{x}_1) := (t_j)_{j=0}^{\tilde{N}-1}$, by TDFT.

4th step: Compute $\mathbf{v} = \mathcal{F}^{-1}_{\tilde{a},\tilde{b},\tilde{N}}\left((s_j)_{j=0}^{\tilde{N}-1}\right) := (v_j)_{j=0}^{\tilde{N}-1}$, by TDFT, where s_j is given by $s_j = \begin{cases} t_j/d_j & \text{if } d_j \neq 0 \\ 0 & \text{if } d_j = 0 \end{cases}$ for $j = 0, 1, \ldots, \tilde{N}-1$. Then $c_{n,m} = v_{n\tilde{b}+m}$, for $n = 0, 1, \ldots, \tilde{a}-1$

and $m = 0, 1, \ldots, \tilde{b} - 1$. The Gabor coefficients $\mathbf{c} = \{c_{n,m}\}_{n,m}$ are determined.

Algorithm 5 (Dual Gabor Window: ab divides N) If ab divides N, then the *dual Gabor window* $\tilde{\mathbf{g}}$ is computed as follows.

1st step: Calculate $\mathbf{u} = \mathbf{g} * [\mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})}]'$ by STFT.

2nd step: Calculate $\mathbf{d} = \mathcal{F}_{b,a,ab}(\mathbf{u}) := (d_j)_{j=0}^{ab-1}$ by TDFT and set $\mathbf{v} = (v_j)_{j=0}^{ab-1}$ with v_j given by $v_j = \begin{cases} \sqrt{ab}/(Nd_j) & \text{if } d_j \neq 0 \\ 0 & \text{if } d_j = 0 \end{cases}$ for $j = 0, 1, \ldots, ab - 1$.

3rd step: Calculate $\mathbf{w} = \mathcal{F}_{b,a,ab}^{-1}(\mathbf{v})$ by ITDFT.

4th step: Compute $\tilde{\mathbf{g}} = \mathbf{w} * \mathrm{GAB}_{(\mathbf{g},\tilde{b},\tilde{a})}$ by STFT.

Proof. Since ab divides the signal length N, N divides $\tilde{a}\tilde{b}$. By Theorem 4, $\mathbf{GM}_{(\mathbf{g},\tilde{b},\tilde{a})}$ is a block-circulant matrix and is unitarily diagonalized by $V = \mathbf{F}_{b,a,ab}$. Noticing that $\mathcal{F}_{b,a,ab}(\mathbf{e}_{ab}) = \frac{1}{\sqrt{ab}}(\underbrace{1, 1, \ldots, 1}_{ab})$, the algorithm is deduced. □

Remark. For undersampling cases in which N divides ab, we consider Gabor triple $(\mathbf{g}, \tilde{b}, \tilde{a})$. By Algorithm 5, we are able to determine the dual Gabor window $\tilde{\mathbf{g}}_o$ associated with $(\mathbf{g}, \tilde{b}, \tilde{a})$. Applying the results in [18, 21], $\tilde{\mathbf{g}} = \frac{ab}{N}\tilde{\mathbf{g}}_o$ is the dual Gabor window corresponding to $(\mathbf{g}, a, b)$.

In the 2D case, we can similarly define 2D-TDFT and show that 2D-TDFT can be performed by 2D-FFT. Considering the 2D separable discrete Gabor transform [18], we are able to formulate algorithms for computing the 2D Gabor coefficients and the dual Gabor windows. □

4.4 Connections with Previous Results

Discrete Gabor transforms have been studied by many authors via discrete Zak transforms [1, 16, 24]. However, the Zak transform methods are restricted to the cases that N divides ab, as we studied in Sections 3.2 and 4.3. In [16, p.1788] and [24, p.944], the authors gave analytic formulations of the Gabor coefficients $\{c_{n,m}\}$, by using the Zak transforms and employing the bi-orthogonal functions. The determination of the Gabor coefficients may be difficult because of the occurrence of zeros in the Zak transform. Advantages of our methods are as follows.

For the cases that N divides ab and ab divides N, simple results are presented in Sections 3.2 and 4.3 based on standard linear algebra. Algorithms 4 and 5 can be easily implemented.

Algorithms 2 and 3 are general approaches. The algorithms can be used without any restrictions on the lattice constants (a, b). We do not need to assume that $\{\mathbf{g}_{n,m}\}$ be a Gabor frame. Algorithm 2 is for computing the Gabor coefficients without precalculating the dual Gabor window.

Algorithms 2 and 3 are CG-iterative algorithms. For each CG-iteration, only a UEGM matrix-vector multiplication is required and can be done easily by Algorithm 1. Moreover, we only need to run a few CG-cycles to meet the required accuracy. This leads to efficient computations.

In [18, 19], different approaches have been derived for computing the dual Gabor window. The algorithms are based on the sparse structures of the Gabor matrix $\mathbf{S}$ [18, pp.2860–2862] and the block-diagonal matrix $\mathbf{D}$ [19, p.2874]. The total number of nonzero elements of $\mathbf{S}$ and $\mathbf{D}$ are Nb and Na, respectively. For Algorithm 3, $\widetilde{\mathbf{GM}}$ is associated with $(\mathbf{g}, \tilde{b}, \tilde{a})$. There are only ab^2 possible nonzero elements in $\widetilde{\mathbf{GM}}$. If $ab < N$ (oversampling), then the advantage of Algorithm 3 is clear.

5. Numerical Simulations

In this section, we illustrate some numerical results.

Figure 1 shows the explicit sparsity of the UEGM matrices. The structural properties of $\widetilde{GM}$ lead to efficient computations of Gabor coefficients. Cases (i) and (ii) are oversampling with $(a, b) = (20, 8), (24, 8)$. Cases (iii) and (iv) with $(a, b) = (30, 20), (48, 20)$ are undersampling.

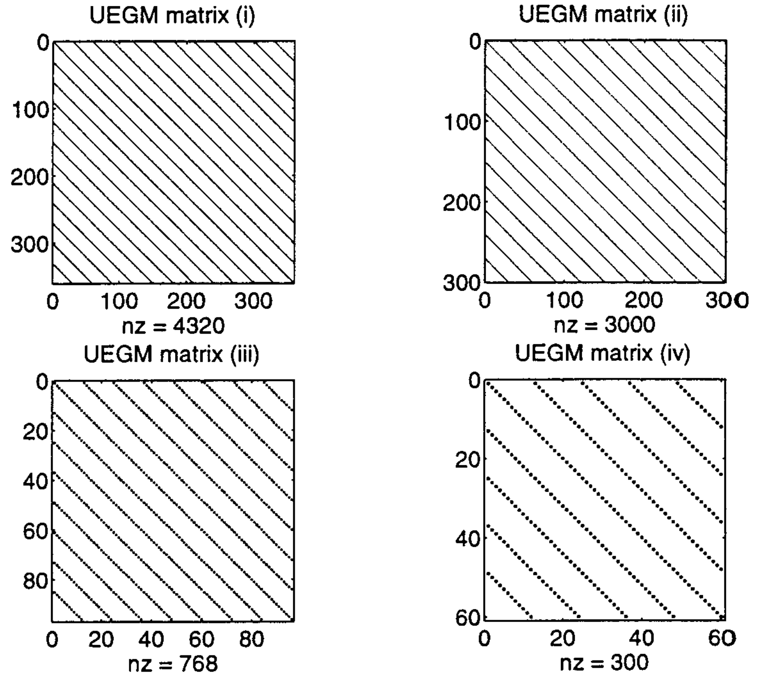

FIGURE 1. The sparse structures of the unitary equivalent Gabor-Gram matrices. The lattice constants (a, b) corresponding to (i), (ii), (iii), and (iv) are (20, 8), (24, 8), (30, 20), and (48, 20), respectively. The signal length N is 128.

Figure 2 illustrates a Gabor window $\mathbf{g}$ and a Chirp signal $\mathbf{x}$. The signal length $N = 240$. Figure 3 is the plot of reconstruction error $\frac{\|\mathbf{x}-\mathbf{x}_{\text{app}}^{(k)}\|}{\|\mathbf{x}\|}$ vs. iteration number k.

The iteration number refers to the CG-iteration number of Algorithm 2. Fix a maximal iteration number k, the Gabor coefficients $\{c_{n,m}\}$ are calculated by Algorithm 2. Then $\mathbf{x}_{\text{app}}^{(k)}$ is computed with $\{c_{n,m}\}$. For $(a, b) = (20, 10), (20, 8), (16, 10), (16, 8)$, the reconstruction (relative) errors reach the order of 10^{-6} with 10, 6, 6, and 5 CG-iterations, respectively. The examples show that the CG-algorithm converges fast. If the lattice constants (a, b) are smaller, the convergence is faster.

6. Conclusions

In this article, we introduced the tensor discrete Fourier transforms (TDFT). We showed that TDFT can be performed by FFT. We have studied a class of Gabor-Gram matrices. We have shown that these matrices are unitarily equivalent to sparse matrices by TDFT. Based on the structural

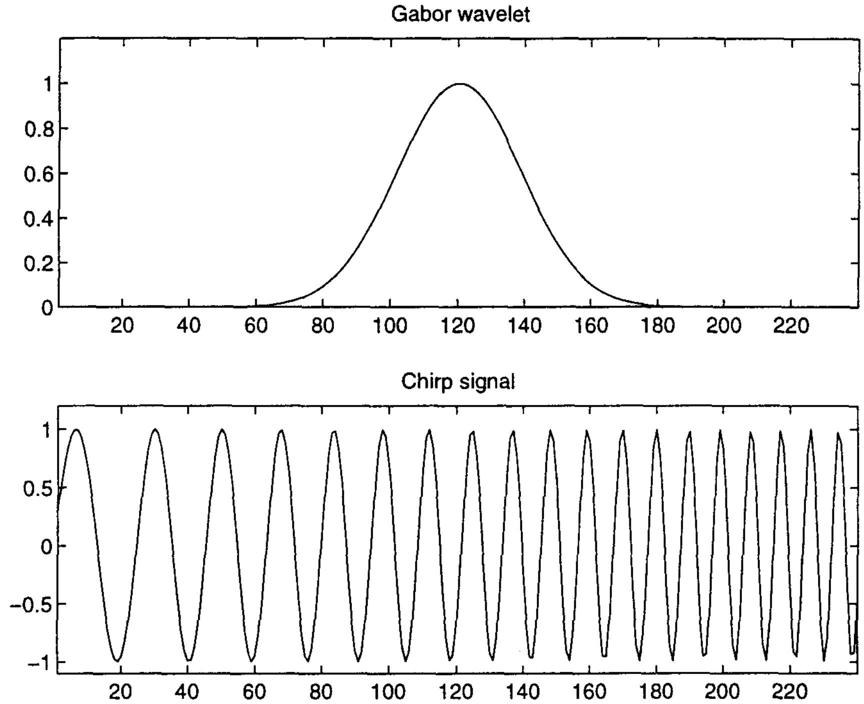

FIGURE 2. The Gabor window and the Chirp signal. The signal length $N = 240$.

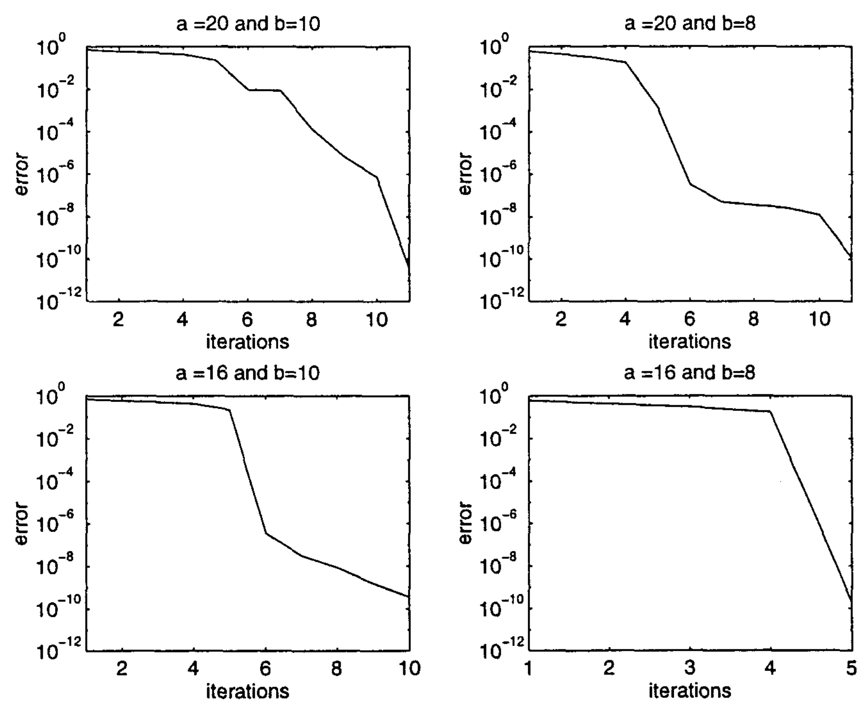

FIGURE 3. Reconstruction error $\frac{\|\mathbf{x}-\mathbf{x}^{(k)}\text{app}\|}{\|\mathbf{x}\|}$ vs. iteration number k. The Gabor window and the original Chirp signal are illustrated in Figure 2.

properties of the Gabor-Gram matrices, we were able to present CG-methods for computing the Gabor coefficients and the dual Gabor windows. The algorithms can be applied to the critical sampling, oversampling, and undersampling cases. We do not need to assume that the family $\{g_{n,m}\}_{n,m}$ be a Gabor frame. In particular, if ab divides N (or N divides ab), computations of the Gabor coefficients and the dual Gabor windows are simple.

Acknowledgments

The author acknowledges fruitful discussions on the subject with Prof. Hans G. Feichtinger. The author thanks his referee for the insightful suggestions.

References

[1] Auslander, L., Gertner, I.C., and Tolimieri, R. (1991). The Discrete Zak Transform Application to Time-Frequency Analysis and Synthesis of Nonstationary Signals, *IEEE Trans. Signal Processing,* **39**(4), 825–835.

[2] Bastiaans, M.J. (1980). Gabor's signal expansion of a signal into Gaussian elementary signals, *Proc. IEEE,* **68**(4), 538–539.

[3] Benedetto, J.J. and Walnut, D.F. (1993). Gabor frames for L^2 and related spaces, *WAVELETS Mathematics and Applications,* J. Benedetto and M. Frazier, Eds., CRC Press, Boca Raton, 97–162, 1994.

[4] Daubechies, I. (1990). The wavelet transform, time-frequency localization, and signal analysis, *IEEE Trans. Inform. Theory,* **36**(5), 961–1005.

[5] Daugman, J. (1988). Complete Discrete 2-D Gabor Transforms by Neural Networks for Image Analysis and Compression, *IEEE Trans. Acoust. Speech Signal Proc.,* **36**(7), 1169–1179.

[6] Davis, P.J. (1979). Circulant Matrices, Wiley-Interscience, New York.

[7] Ebrahimi, T. and Kunt, M. (1991). Image Compression by Gabor Expansion, *Optical Eng.,* **30**(7), 873–880.

[8] Feichtinger, H.G. (1991). Pseudo-inverse matrix methods for signal reconstruction from partial data, *SPIE-Proc.* **1606**, 766–772.

[9] Gabor, D. (1946). Theory of Communication, *J. Electric. Eng.,* **93**, 429–441.

[10] Horn, R.A. and Johson, C.A. (1985). Matrix Analysis, Cambridge University Press, Cambridge.

[11] Horn, R.A. (1989). The Hadamard Product, *Matrix Theory and Applications,* C. R. Johnson, Ed., 87–169.

[12] Janssen, A.J.E.M. (1981). Gabor representation of generalized functions, *J. Math. Anal. Appl.,* **83**, 377–394.

[13] Lancaster, P. and Tismenetsky, M. (1985). The Theory of Matrices, Academic Press, Orlando, San Diego, New York, London, et al.", second edition, 1985.

[14] Orr, R.S. (1993). The Order of Computation for Finite Discrete Gabor Transform, *IEEE Trans. Signal Processing,* **41**(2), 122–130.

[15] Polyak, N., Pearlman, W.A., and Zeevi, Y.Y. (1995). Orthogonalization of Circular Stationary Vector Sequences and Its Application to the Gabor Decomposition, *IEEE Trans. Signal Processing,* **43**(8), 1778–1789.

[16] Porat, M. and Zeevi, Y.Y. (1990). Gram-Gabor Approach to Optimal Image Representation, *Proc. SPIE,* **1360**, 1474–1477.

[17] Qian, S. and Chen, D. (1993). Discrete Gabor Transform, *IEEE Trans. Signal Processing,* **41**(7), 2429–2438.

[18] Qiu, S. and Feichtinger, H.G. (1995). Discrete Gabor Structure and Optimal Representation, *IEEE Trans. Signal Processing,* **43**(10), 1146–1157.

[19] Qiu, S. (1995). Block-Circulant Gabor-Matrix Structure and Gabor Transforms, *Optical Eng.,* **34**(10), 2872–2878.

[20] Qiu, S. (1996). Generalized Dual Gabor Atoms and Best Approximations by Gabor Family, *Signal Processing,* **49**(3), 167–186.

[21] Qiu, S. (1996). Gabor Transform with Undersampling, *Proc. TFTS* (Paris), 317–320.

[22] Stewart, D., Potter, L., and Ahalt, S.C. (1995). Computationally Attractive Real Gabor Transform, *IEEE Trans. Signal Processing,* **43**(1), 77–84.

[23] Wexler, J. and Raz, S. (1990). Discrete Gabor Expansion, *Signal Processing,* **21**, 207–220.

[24] Zibulski, M. and Zeevi, Y.Y. (1994). Frame Analysis of the Discrete Gabor Scheme, *IEEE Trans. on Signal*

Processing, **42**(4), 942–945.

[25] Zibulski, M. and Zeevi, Y.Y. (1994). Oversampling in the Gabor Scheme, *IEEE Trans. Signal Processing,* **41**(8), 2679–2687.

Received April 30, 1995

Revision received January 23, 1997

Cirrus Logic, NCDC, Raleigh, NC 27615

e-mail: sigang@ncdc.cirruc.com