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INTEGRABLE GEODESIC FLOWS OF NONHOLONOMIC METRICS

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ABSTRACT. It is shown how to produce new examples of integrable Hamiltonian dynamical systems of differential geometric origin. These are normal geodesic flows of homogeneous Carnot–Carathéodory metrics. The relation to previous descriptions of such flows via non-Hamiltonian methods and to problems of analytic mechanics is discussed.

1. INTRODUCTION

In the present article we show how to produce new examples of integrable dynamical systems of differential geometric origin.

This is based on a construction of a canonical Hamiltonian structure for the geodesic flows of Carnot–Carathéodory metrics [7], [17] via the Pontryagin maximum principle. This Hamiltonian structure is achieved by introducing Lagrange multiplier bundles which are the phase spaces of these Hamiltonian flows. These bundles are diffeomorphic to cotangent bundles but have another meaning. A transfer to this phase space is given by a generalized Legendre transform.

We analyze the geodesic flow of the left-invariant Carnot–Carathéodory metric on the three-dimensional Heisenberg group as a super-integrable Hamiltonian system (Theorem 1). Moreover, its super-integrability explains the foliation of its phase space into one- and two-dimensional invariant submanifolds, as was pointed out in [17].

The geodesic flows of left-invariant Carnot–Carathéodory metrics on Lie groups are reduced to equations on Lie algebras in the same manner as the geodesic flows of left-invariant Riemannian metrics are reduced to the Euler equations on Lie algebras ([1], Theorem 2). These flows comprise many integrable systems. Moreover, this reduction to equations on Lie algebras gives a Hamiltonian explanation for the description of such flows

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on three-dimensional Lie groups given in [17] with the use of Euler–Lagrange equations (the desire to explain this in terms of integrability was the starting point of the present work).

In comparison with Riemannian geodesic flows, there is another class of invariant flows corresponding to left-invariant metrics and right-invariant distributions. In Sec. 6 we examine the simplest example of such flow on $\mathcal{H}^3$ and, in particular, show that this flow is integrable (Theorem 3).

In Sec. 7 we consider a Hamiltonian structure for the equations of motion of a heavy rigid body with a fixed point.

For completeness of explanation in Sec. 8 we discuss another approach to the definition "straight lines" in nonholonomic geometry which does not lead to Hamiltonian systems, but some "straight line" flows have an important mechanical meaning (for instance, the Chaplygin top [6]).

We also discuss some problems concerning dynamics and, in particular, integrability of these systems (see Sec. 9, Concluding Remarks).

2. The geodesic flows of Carnot–Carathéodory metrics

A. Carnot–Carathéodory metrics. Let M^n be a smooth manifold of dimension n.

A family $\mathcal{F}$ of k-dimensional subspaces of the tangent spaces to M^n is called a k-dimensional smooth distribution if $\mathcal{F}_x$ is a smooth section of the Grassmann bundle on M^n.

In what follows we suppose that distributions are smooth.

Let $V_{\mathcal{F}}$ be the linear space spanned by vector fields tangent to $\mathcal{F}$. Denote by $A_{\mathcal{F}}$ the algebra generated by fields from $V_{\mathcal{F}}$ via commutation. A distribution is called *nonholonomic* if $A_{\mathcal{F}}$ does not coincide with $V_{\mathcal{F}}$ as a linear space. Otherwise a distribution is called holonomic and, by the Frobenius theorem, locally looks like a family of spaces tangent to the leaves of a foliation. It is easily seen that near a generic point the distribution corresponding to $A_{\mathcal{F}}$ is holonomic. The distribution $\mathcal{F}$ is called *completely nonholonomic* if the algebra $A_{\mathcal{F}}$ coincides with the whole algebra of vector fields on M^n. In [13] such distributions are said to satisfy the *bracket generating hypothesis.*

In the sequel we assume that distributions are completely nonholonomic. This assumption is not very strong because otherwise we may restrict geodesic flows to the leaves of the foliation and consider the restricted distributions as completely nonholonomic. In fact, we use this to define a Carnot–Carathéodory metric as a correct intrinsic metric.

Thus, we assume now that M^n is endowed with a completely nonholonomic distribution $\mathcal{F}$. We also assume that M^n is a complete Riemannian manifold with a metric $\tilde{g}^{ij}$.

A piecewise smooth curve in M^n is called admissible if it is tangent to $\mathcal{F}$. A *Carnot–Carathéodory metric* $d_{CC}(x,y)$ is defined as follows. Denote by $\Omega_{x,y}$ the set of admissible curves with ends at the points x and y in M^n. Then

$$d_{CC}(x,y) = \inf_{\gamma \in \Omega_{x,y}} \text{length}(\gamma) \tag{1}$$

with the lengths of curves taken with respect to the metric $\tilde{g}^{ij}$.

By the Chow–Rashevskii theorem, any pair of points in a complete Riemannian manifold endowed with a completely nonholonomic distribution is connected by an admissible curve, hence equality (1) correctly defines an inner metric on M^n.

Carnot–Carathéodory metrics are the simplest examples of nonholonomic metrics which are defined by (1) for different choices of $\Omega_{x,y}$ corresponding to non-integrable constraints. For Carnot–Carathéodory metrics these constraints are linear in velocities.

B. The geodesic flow of a Carnot–Carathéodory metric. By definition, the lengths of admissible curves depend only on the restrictions of the metric $\tilde{g}^{ij}$ on $\mathcal{F}_x$. Denote these restricted forms by Q_x. This family of bilinear forms on $\mathcal{F}$ enables us to define the canonical mapping

$$g(x) : T^*M^n \to \mathcal{F}_x \subset TM^n \tag{2}$$

taking for $g(x)\xi \in \mathcal{F}_x$ a vector determined uniquely by the condition

$$Q_x(Y, g(x)\xi) = \langle Y, \xi \rangle \quad \text{for every } Y \in \mathcal{F}_x. \tag{3}$$

The symmetric tensor g^{ij} is called a *Carnot–Carathéodory metric tensor*. It generalizes a Riemannian metric tensor into which it degenerates when $\mathcal{F}_x = T_x M^n$.

A curve $\tilde{\gamma}$ in T^*M^n is called a cotangent lift of a curve γ in M^n if

$$g(\gamma(t))\xi(t) = \frac{d\gamma(t)}{dt}, \tag{4}$$

where $\tilde{\gamma}(t) = (\gamma(t), \xi(t))$, $\xi(t) \in T^*_{\gamma(t)}M^n$.

By (3), in terms of cotangent lifts the length of an admissible curve $\gamma(t)$ in M^n is expressed by

$$L(\gamma) = \int \sqrt{\langle g(\gamma(t))\xi(t), \xi(t) \rangle}\, dt, \tag{5}$$

and the energy of γ equals

$$E(\gamma) = \frac{1}{2} \int \langle g(\gamma(t))\xi(t), \xi(t) \rangle\, dt. \tag{6}$$

An admissible curve is called a *geodesic of the Carnot–Carathéodory metric* g^{ij} if locally it is an energy-minimizing curve.

The geodesics of the Carnot–Carathéodory metric g^{ij} are described by the Euler–Lagrange equations for a Lagrange function

$$L(x,\dot{x}) = \frac{1}{2}\tilde{g}_{pq}\dot{x}^p\dot{x}^q + \sum_{\alpha=1}^{n-k}\mu_\alpha\langle\dot{x},\omega^{(\alpha)}\rangle, \tag{7}$$

where $\omega^{(1)},\dots,\omega^{(n-k)}$ is a basis for $\mathcal{F}_x^\perp$. Here $\mathcal{F}_x^\perp$ is the annihilator of $\mathcal{F}_x$, i.e., the subset of $T_x^*M^n$ formed by covectors ξ such that $\langle\xi,v\rangle = 0$ for every $v\in\mathcal{F}_x$.

These equations are written as follows:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}^i} - \frac{\partial L}{\partial x^i} =$$

$$= \left\{\frac{\partial\tilde{g}_{iq}}{\partial x^p}\dot{x}^p\dot{x}^q + \tilde{g}_{iq}\ddot{x}^q + \sum_\alpha\left(\dot{\mu}_\alpha\omega_i^{(\alpha)} + \mu_\alpha\frac{\partial\omega^{(\alpha)}}{\partial x^p}\dot{x}^p\right)\right\} - \tag{8}$$

$$-\left\{\frac{1}{2}\frac{\partial\tilde{g}_{pq}}{\partial x^i}\dot{x}^p\dot{x}^q + \sum_\alpha\mu_\alpha\dot{x}^p\frac{\partial\omega_p^{(\alpha)}}{\partial x^i}\right\} = 0,$$

$$\frac{\partial L}{\partial\mu_\alpha} = \langle\dot{x},\omega^{(\alpha)}\rangle = 0. \tag{9}$$

In fact, although the Riemann metric tensor $\tilde{g}^{ij}$ enters these equations, the geodesic flow is determined by the restriction of this tensor on the distribution, the Carnot–Carathéodory metric tensor g^{ij}, only.

3. The Pontryagin maximum principle and a Hamiltonian structure for the geodesic flows of a Carnot–Carathéodory metric. A generalized Legendre transform

A. The Pontryagin maximum principle and the geodesics of Carnot–Carathéodory metrics. In [13] Strichartz showed that the geodesic flow of a Carnot-Carathéodory metric was described by equations derived from the Pontryagin maximum principle.

We explain his idea in brief. First, roughly quote the Pontryagin maximum principle in a weak form sufficient for our study referring to [5], Theorem 5.1, for an absolutely rigorous statement.

The Pontryagin maximum principle. *Consider the minimum problem for the functional*

$$I\big[x(t),u(t)\big]=\int_{t_1}^{t_2} f^0(x,u)\,dt \tag{10}$$

in the class of admissible functions $(x(t),u(t))$ *such that*

$$\dot{x}^k=f^k(x,u) \tag{11}$$

and some constraints $x\in A$, $u\in U$.

Introduce the functions

$$\tilde{H}(x,u,\tilde{\lambda})=\lambda_0 f^0(x,u)+\lambda_1 f^1(x,u)+\dots+\lambda_n f^n(x,u) \tag{12}$$

and

$$M(x,\tilde{\lambda})=\inf_{u\in U}\tilde{H}(x,u,\tilde{\lambda}), \tag{13}$$

where $\tilde{\lambda}=(\lambda_0,\lambda_1,\dots,\lambda_n)$.

Let $(x(t),u(t))$ *be a solution to this minimum problem. Then there exists an absolutely continuous vector function* $\tilde{\lambda}(t)$ *such that*

(i) $\lambda_0=\text{const}, \lambda_0\geq 0$, *and*

$$\frac{d\lambda_i}{dt}=-\frac{\partial\tilde{H}(x(t),u(t),\tilde{\lambda}(t))}{\partial x^i}; \tag{14}$$

(ii) $\tilde{H}(x(t),u(t),\tilde{\lambda}(t))=M(x(t),\tilde{\lambda})(t)$ *for every* $t\in[t_1,t_2]$;

(iii) $M(x(t),\tilde{\lambda}(t))=\text{const}$.

Now, it suffices only to note that in the case of geodesic flows of Carnot–Carathéodory metrics we have the minimum problem on the set of the cotangent lifts of admissible curves on M^n, where we consider the variables ξ_i as the control functions u_i. In this case

$$f^0(x,u)=\frac{1}{2}g^{ij}u_iu_j, \tag{15}$$

$$f^i(x,u)=g^{ij}u_j, \tag{16}$$

and

$$\tilde{H}(x,u,\tilde{\lambda})=\frac{\lambda_0}{2}g^{ij}u_iu_j+g^{ij}\lambda_iu_j. \tag{17}$$

Thus, we have

$$M(x,\tilde{\lambda}) = \begin{cases} -\frac{1}{2\lambda_0} g^{ij}\lambda_i\lambda_j & \text{for } \lambda_0 \neq 0 \\ 0 & \text{for } \lambda_0 = 0 \text{ and } g^{ij}\lambda_j \equiv 0 \\ -\infty & \text{otherwise.} \end{cases} \tag{18}$$

B. A Hamiltonian structure for geodesic flows of Carnot–Carathéodory metrics. Consider the bundle $\Lambda M^n \to M^n$ diffeomorphic to the cotangent bundle T^*M^n via the diffeomorphism $(x,p) \leftrightarrow (x,\lambda)$ but having another sense. We call it the *Lagrange multiplier bundle* on M^n. As well as the cotangent bundle, this bundle is endowed with the natural symplectic structure generated by the form

$$\Omega = \sum_{i=1}^{n} d\lambda_i \wedge dx^i. \tag{19}$$

Consider the Hamiltonian flow with a Hamiltonian function

$$H(x,\lambda) = -\frac{1}{2} g^{ij}\lambda_i\lambda_j \tag{20}$$

on the symplectic manifold.

Definition. A geodesic of a Carnot–Carathéodory metric is called *normal* if $\lambda_0 \neq 0$.

Otherwise, if $g^{ij}\lambda_j = 0$, then $M(x,\lambda) = 0$, and the Pontryagin maximum principle gives nothing.

Theorem HS. (On a Hamiltonian structure for normal geodesic flow.) *The projections of trajectories of the Hamiltonian flow on ΛM^n with the Hamiltonian function* (20) *are exactly the naturally-parametrized normal geodesics of the Carnot–Carathéodory metric g^{ij}. Moreover, $|\dot{x}|^2 = -2H(x,\lambda)$.*

Proof of Theorem HS. From (18) and from the homogeneity of $\tilde{H}(x,u,\tilde{\lambda})$, we infer that after change of the parameter on an extremal to a multiple one, if necessary, we obtain an extremal $\tilde{\gamma}$ with $\lambda_0 = 1$.

By (13), we derive

$$\frac{\partial \tilde{H}(x,u,\tilde{\lambda})}{\partial u_i} = 0,$$

which is equivalent to

$$g^{ij}u_j = -g^{ij}\lambda_j. \tag{21}$$

It follows from (21) that

$$g^{ij}u_iu_j = g^{ij}\lambda_i\lambda_j. \tag{22}$$

Recall that, by (4), we have

$$g^{ij}u_j = \dot{x}^i, \tag{23}$$

and together with (3) this implies

$$|\dot{x}|^2 = \tilde{g}_{ij}\dot{x}^i\dot{x}^j = g^{ij}u_iu_j = |u|^2. \tag{24}$$

From (22) and from statement (iii) of the Pontryagin maximum principle it follows that the extremal $\tilde{\gamma}$ is naturally-parametrized.

According to (4) and (21), we have that

$$\dot{x}^i = \frac{\partial H(x,\lambda)}{\partial \lambda_i}, \tag{25}$$

and we can regard (14) as

$$\dot{\lambda}_i = -\frac{\partial H(x,\lambda)}{\partial x^i}. \tag{26}$$

It is easily seen that equations (25)–(26) form a system of Hamilton equations on ΛM^n for the Hamiltonian function (20).

Thus we prove that the normal geodesics are the projections of trajectories of this Hamiltonian flow.

For the converse, we refer to [8] where a comprehensive examination of analytical properties of the energy functional for a Carnot–Carathéodory metric is given.

This completes the proof of the theorem. □

The variables $\lambda_1, \dots, \lambda_n$ have no physical meaning in comparison with the momenta $u_1, \dots, u_n$. The correspondence between them and velocities is given by (25) and is one-to-one in the case where the form g^{ij} is nondegenerate, i.e., for Riemannian metrics only. In this case the existence of a Hamiltonian formalism for the geodesic flows of Riemannian metrics follows from both the Legendre transform and the Pontryagin maximum principle. Thus the transfer to the new variables (x, λ) is to be regarded as a *generalized Legendre transform.*

Theorem HS is contained implicitly in [13]. However, this fact did not attract the attention of specialists on integrable systems, because the problem of regularity of geodesics and that of exponential maps were studied in [8], [16]. From the analytical point of view this also coincides with the introduction of a Hamiltonian in the "vakonomic mechanics" [2] where the momenta are introduced by the implicit theorem procedure and the Hamiltonian system is also regarded as a system on a cotangent bundle. However, the difference between T^*M^n and ΛM^n is essential for applications to mechanics and physics because, at least, the variables $\lambda_1, \dots, \lambda_n$ are not observable.

Although it has been assumed for a long time that all geodesics are normal, recently Montgomery has shown that abnormal geodesics exist [10].

The geodesics found by him do not admit end-point $\mathcal{F}$-tangent perturbations and, thus, are solutions to any variational problem on the space of admissible curves.

But if the space $\mathcal{V}_{\mathcal{F}} + [\mathcal{V}_{\mathcal{F}}, \mathcal{V}_{\mathcal{F}}]$ coincides with the whole algebra of vector fields on M^n, then every geodesic is normal [13].

In the sequel, speaking about geodesic flows, we shall suppose that they are normal.

4. Integrability of the geodesic flow of the left-invariant Carnot–Carathéodory metric on the three-dimensional Heisenberg group

We mean by a left-invariant Carnot–Carathéodory metric a left-invariant metric restricted to a left-invariant distribution. It is known that such metric on the three-dimensional Heisenberg group is unique up to isomorphism [17].

This flow is described in [4], [8], [17], [19], in some of them with certain generalizations. However, it is nowhere regarded as a completely integrable Hamiltonian system.

The three-dimensional Heisenberg group $\mathcal{H}^3$ is the group of matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

with respect to multiplication, where $x, y, z \in \mathbb{R}$. Its Lie algebra $\mathcal{L}$ is spanned by the following elements:

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

We denote by $\mathcal{L}_0$ the linear subspace spanned by e_1 and e_2.

The group $\mathcal{H}^3$ acts on itself by the left and right translations:

$$L_g : \mathcal{H}^3 \to \mathcal{H}^3 : \quad L_g(h) = gh,$$

$$R_g : \mathcal{H}^3 \to \mathcal{H}^3 : \quad R_g(h) = hg.$$

The left-invariant distribution generated by $\mathcal{L}_0$ consists of the 2-planes $\mathcal{F}_x = L_{g*}\mathcal{L}_0$.

Since the following commutation relations hold:

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = [e_2, e_3] = 0,$$

this distribution is completely nonholonomic.

We consider the left-invariant metric on $\mathcal{H}^3$ which at the unit of the group takes the form

$$(e_i, e_j) = \delta_{ij}. \tag{29}$$

Identify $\mathcal{H}^3$ with $\mathbb{R}^3$ by the diffeomorphism which assigns the point in $\mathbb{R}^3$ with the coordinates (x, y, z) to the matrix (27). Thus we identify the tangent space at every point of $\mathcal{H}^3$ with the vector space generated by matrices (28). In this case the left translations act on $T\mathcal{H}^3$ as follows:

$$L_{g*}(e_1) = e_1, \quad L_{g*}(e_2) = e_2 + xe_3, \quad L_{g*}(e_3) = e_3, \tag{30}$$

where g is the element of $\mathcal{H}^3$ given by the matrix (27). It follows from (30) that in these coordinates the mapping $L_{g*} : T_e\mathcal{H}^3 \to T_g\mathcal{H}^3$ is written as follows:

$$L_{g*} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & x & 1 \end{pmatrix}, \tag{31}$$

where e is the unit of $\mathcal{H}^3$. Thus we have

$$[g^{ij}(x, y, z)] = (L_{g*})^{**} \cdot [g^{ij}(0, 0, 0)] \cdot (L_{g*})^*, \tag{32}$$

and since

$$g^{ij}(0, 0, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{33}$$

we derive

$$[g^{ij}(x, y, z)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & x & x^2 \end{pmatrix}. \tag{34}$$

The left-invariant Riemannian metric on $\mathcal{H}^3$ in these coordinates takes the form

$$[\tilde{g}_{ij}(x, y, z)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1 + x^2) & -x \\ 0 & -x & 1 \end{pmatrix}. \tag{35}$$

Now, it follows from Theorem HS that the geodesic flow of the left-invariant Carnot–Carathéodory metric corresponding to the Riemannian metric (35) and the distribution $L_{g*}\mathcal{L}_0$ is Hamiltonian on $\Lambda\mathcal{H}^3$ with the following Hamiltonian function:

$$H(q, \lambda) = \frac{1}{2}(\lambda_1^2 + \lambda_2^2 + x^2\lambda_3^2 + 2x\lambda_2\lambda_3), \tag{36}$$

where $q = (x, y, z)$.

Hamilton equations for (36) have the simple form

$$\dot{x} = \frac{\partial H}{\partial \lambda_1} = \lambda_1, \quad \dot{y} = \frac{\partial H}{\partial \lambda_2} = \lambda_2 + x\lambda_3,$$

$$\dot{z} = \frac{\partial H}{\partial \lambda_3} = x\lambda_2 + x^2\lambda_3, \quad \dot{\lambda}_1 = -\frac{\partial H}{\partial x} = -x\lambda_3^2 - \lambda_2\lambda_3, \tag{37}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial y} = 0, \quad \dot{\lambda}_3 = -\frac{\partial H}{\partial z} = 0,$$

and we immediately infer from (37) that this Hamiltonian system is completely integrable, because it has three first integrals

$$I_1 = H, \quad I_2 = \lambda_2, \quad I_3 = \lambda_3, \tag{38}$$

which are in involution and functionally independent almost everywhere. In particular, these integrals are functionally independent in the domain $H \neq 0$.

Moreover, by (37),

$$\lambda_3 = \text{const}, \quad \dot{z} = x\dot{y} \tag{39}$$

along trajectories of the flow; we can restrict this flow to the level set $\{\lambda_3 = C = \text{const}\}$ and project this restriction of the flow to the plane (x, y). We denote this system by $\mathcal{P}_C$ and note that it is defined on the 4-dimensional symplectic manifold $\mathcal{M}_C$ diffeomorphic to the cotangent bundle of the 2-plane with the coordinates (x, y) but with another Poisson structure.

Introduce the new variables on $\mathcal{M}_C$,

$$u = \lambda_1, \quad v = \lambda_2 + x\lambda_3. \tag{40}$$

Then by (19) and (40) the Poisson structure on $\mathcal{M}_C$ induced from $\Lambda\mathcal{H}^3$ is written as

$$\{x, u\} = \{y, v\} = 1, \{u, v\} = -C \ (= -\lambda_3), \tag{41}$$

$$\{x, v\} = \{y, u\} = \{x, y\} = 0,$$

in the coordinates (x, y, u, v). The flow $\mathcal{P}_C$ is also a Hamiltonian system with the following Hamiltonian function:

$$H(x, y, u, v) = \frac{u^2 + v^2}{2}. \tag{42}$$

It is easily seen that the flow (41)–(42) describes nothing else but the motion of a charged particle on the Euclidean plane (x, y) in the constant magnetic field $F = -\lambda_3\, dx \wedge dy$ [11]. This system has three first integrals, which are functionally independent almost everywhere

$$\tilde{I}_1 = H, \quad \tilde{I}_2 = \lambda_3 x - v, \quad \tilde{I}_3 = \lambda_3 y + u, \tag{43}$$

and thus we conclude that these systems are super-integrable, i.e., have more first integrals than $\dim \mathcal{M}_C/2$. This is also true for the main flow.

Hence, we conclude

Theorem 1. (1) *The geodesic flow of the left-invariant Carnot–Carathéodory metric on $\mathcal{H}^3$, corresponding to the left-invariant Riemannian metric* (35) *and the left-invariant distribution $L_{g*}\mathcal{L}_0$, is a Hamiltonian system on $\Lambda\mathcal{H}^3$, integrable in the Liouville sense via the first integrals* (38)*, which are in involution and functionally independent almost everywhere.*

Moreover, since this flow possesses the fourth first integral $I_4 = \lambda_3 y + \lambda_1$ functionally independent on (38)*, it is super-integrable, and the subset $\{\lambda_3 \neq 0\}$ of its phase space is foliated into the 2-dimensional invariant Liouville tori $S^1 \times \mathbb{R}$.*

(2) *Let us restrict this geodesic flow to the level set $\{\lambda_3 = C\}$ and project this restriction of the flow to the plane (x, y). The flow $\mathcal{P}_C$ constructed by this procedure is equivalent to the Hamiltonian system describing the motion of a charged particle on the Euclidean 2-plane (x, y) in the constant magnetic field $F = -\lambda_3 dx \wedge dy$. This flow is super-integrable, and its phase space $\mathcal{M}_C$ is foliated into closed trajectories for $\lambda_3 \neq 0$.*

5. Left-invariant Carnot–Carathéodory metrics on Lie groups and their geodesic flows

Let $\mathcal{G}$ be a Lie group, let G be its Lie algebra, and let G_0 be a subspace of G generating G. Take a scalar product $\mathcal{J}$ in G and decompose G into a direct sum of G_0 and its orthogonal complement:

$$G = G_0 \oplus G_0^{\perp}.$$

We take another bilinear form $\mathcal{J}_0$ on G uniquely defined by the following conditions:

$$\mathcal{J}_0(x, y) = 0 \ \text{ for every } \ x \in G_0^{\perp}, y \in G,$$

$$\mathcal{J}_0(x, y) = \mathcal{J}(x, y) \ \text{ for every } \ x, y \in G_0.$$

Now, take the left-invariant distribution $L_{g*}G_0$ on $\mathcal{G}$ and the left-invariant Riemannian metric generated by $\mathcal{J}$. To this pair there is a uniquely assigned left-invariant Carnot–Carathéodory metric.

We do not consider the details, but only mention that for the same arguments as for the geodesic flow of a left-invariant Riemannian metric on a Lie group, the geodesic flow of this Carnot–Carathéodory metric is reduced to equations on its Lie co-algebra, the Euler equations, cf. [1], and the following theorem holds.

Theorem 2. *The geodesic flow of a left-invariant Carnot–Carathéodory metric is reduced to the following equations on G^*:*

$$\dot{M} = ad^*_\omega M, \tag{44}$$

where $\omega = L_{g^{-1}*}\dot{g} \in G$ and $M = \mathcal{J}_0\omega$. Here $\mathcal{J}_0$ is regarded as an operator $\mathcal{J}_0 : G \to G^*$ acting as $\mathcal{J}_0(x, y) = \langle \mathcal{J}_0 x, y\rangle$.

If there exists an invariant nondegenerate bilinear form on G, we can identify G and G^*, and equations (44) take the form

$$\dot{M} = [\omega, M], \quad M = \mathcal{J}_0\omega. \tag{45}$$

Since these flows possess commutation representations (45), they give a lot of new examples of integrable Hamiltonian systems, and the well-known methods of integrating Euler equations on Lie algebras are immediately generalized for them.

Consider the simplest example. Let $\mathcal{G} = SO(3)$, and let e_1, e_2, e_3 be generators of this group satisfying the commutation relations

$$[e_i, e_j] = \varepsilon_{ijk}e_k;$$

assume that $\mathcal{J} = \text{diag}(1, 1, 1)$, and let G_0 be spanned by e_1 and e_2. In this case Eqs. (45) have two first integrals $\langle \mathcal{J}x, x\rangle$ and $\langle \mathcal{J}_0 x, x\rangle$ and, thus, are completely integrable.

We remark that passing from the Lagrange equations (8)–(9) to the Hamiltonian equations (44) simplifies the study of left-invariant Carnot–Carathéodory geodesic flows and explains the integrable behavior of such flows on three-dimensional Lie algebras given in [17].

6. The geodesic flow of the Carnot-Carathéodory metric on $\mathcal{H}^3$ corresponding to a left-invariant metric and a right-invariant distribution

Since a Carnot–Carathédory metric is defined by a pair of objects (a metric and a distribution), there are other classes of invariant Carnot–Carathédory metrics corresponding to metrics and distributions invariant with respect to different actions of a Lie group $\mathcal{G}$.

Examine, for example, the geodesic flow of the Carnot–Carathéodory metric on $\mathcal{H}^3$ which corresponds to the left-invariant metric (35) and the right-invariant distribution $R_{g*}\mathcal{L}_0$ and show that it is integrable in the Liouville sense.

By (3) and (35) we have

$$g^{11} = (1+x^2)V, \quad g^{12} = g^{21} = xyV,$$
$$g^{22} = (1+y^2)V, \quad g^{13} = g^{31} = y(1+x^2)V, \tag{46}$$
$$g^{23} = g^{32} = xy^2V, \quad g^{33} = y^2(1+x^2)V, \quad V = \frac{1}{1+x^2+y^2}.$$

This system has two obvious first integrals: $I_1 = H$ and $I_2 = \lambda_3$. As in Sec. 4, restrict this flow to the level set $\{\lambda_3 = C = \text{const}\}$ and successively project this restriction on the plane (x, y). Thus we obtain a Hamiltonian system $\mathcal{R}_C$ on the 4-dimensional symplectic manifold $\mathcal{M}_C$. Introduce the new variables

$$u = \lambda_1 + y\lambda_3, \quad v = \lambda_2.$$

Then the Poisson structure on $\mathcal{M}_C$ is given by

$$\{x, u\} = \{y, v\} = 1, \quad \{u, v\} = C \ (= \lambda_3), \tag{47}$$
$$\{x, v\} = \{y, u\} = \{x, y\} = 0.$$

The Hamiltonian functions are written as

$$H(x, y, u, v) = \frac{1}{2(1+x^2+y^2)}((1+x^2)u^2 + 2xyuv + (1+y^2)v^2). \tag{48}$$

By the same reasoning as in the proof of Theorem 2, we conclude that this flow is equivalent to a Hamiltonian system describing the motion of a charged particle on the 2-plane with the Riemannian metric

$$(1+y^2)\,dx^2 - 2xy\,dx\,dy + (1+x^2)\,dy^2 \tag{49}$$

in the constant magnetic field $F = \lambda_3\,dx \wedge dy$.

In the polar coordinates (r, φ), where $x = r\cos\varphi$ and $y = r\sin\varphi$, the metric (49) is written as

$$dr^2 + (r^2 + r^4)\,d\varphi^2, \tag{50}$$

and we infer that the flow $\mathcal{R}_C$ is Hamiltonian,

$$\frac{df}{dt} = \{f, H\},$$

with the Hamiltonian function

$$H(r, \varphi, p_r, p_\varphi) \doteq \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2+r^4}\right), \tag{51}$$

and the following Poisson structure on $\mathcal{M}_C$:

$$\{r, p_r\} = \{\varphi, p_\varphi\} = 1, \quad \{p_r, p_\varphi\} = C, \tag{52}$$

$$\{r, p_\varphi\} = \{\varphi, p_r\} = \{r, \varphi\} = 0.$$

This flow is defined on the four-dimensional symplectic manifold $\mathcal{M}_C$ and has two functionally independent first integrals $\tilde{I}_1 = H$ and $\tilde{I}_2 = p_\varphi + Cr^2/2$. It is clear that these functions are also first integrals of the main geodesic flow. In the initial coordinates the integral $\tilde{I}_2$ takes the form

$$\tilde{I}_2 = C\frac{x^2+y^2}{2} + xv - yu.$$

We conclude

Theorem 3. (1) *The geodesic flow, the Carnot–Carathéodory metric corresponding to the left-invariant Riemannian metric* (35), *and the right-invariant distribution* $R_{g*}\mathcal{L}_0$ *constitute a Hamiltonian system on* $\Lambda\mathcal{H}^3$ *with the following Hamiltonian function:*

$$H(q,\lambda) = \frac{1}{2(1+x^2+y^2)}((1+x^2)\lambda_1^2 + (1+y^2)\lambda_2^2 + \tag{53}$$

$$+y^2(1+x^2)\lambda_3^2 + 2xy\lambda_1\lambda_2 + 2y(1+x^2)\lambda_1\lambda_3 + 2xy^2\lambda_2\lambda_3),$$

where $q = (x, y, z)$;

(2) *this flow possesses the first integrals*

$$I_1 = H, \quad I_2 = \lambda_3, \quad I_3 = \lambda_3\frac{x^2-y^2}{2} + x\lambda_2 - y\lambda_1$$

which are involutive and functionally independent almost everywhere. Hence, this flow is integrable in the Liouville sense;

(3) *restrict this flow to the level set* $\{\lambda_3 = const\}$ *and project the restriction of the flow to the plane* (x, y). *The flow* $\mathcal{R}_C$ *constructed by this procedure is equivalent to the Hamiltonian system describing the motion of a charged particle on the* 2*-plane with the Riemannian metric* (49) *in the constant magnetic field* $\lambda_3\, dx \wedge dy$.

7. The equations for the motion of a heavy rigid body with a fixed point

First, recall that the Lie algebra $e(3)$ of the group of motions of the three-dimensional Euclidean space $E(3)$ is spanned by the elements e_14, e_2, e_3, f_1, f_2, and f_3 meeting the following commutation relations:

$$[e_i, e_j] = \varepsilon_{ijk}e_k, \quad [e_i, f_j] = \varepsilon_{ijk}f_k, \quad [f_i, f_j] = 0. \tag{54}$$

Denote by m_i and γ_j the adjoint basis in the co-algebra $e^*(3)$. The relations (54) determine the Lie–Poisson structure on the space of functions on $e(3)$ as follows:

$$\{m_i, m_j\} = \varepsilon_{ijk}m_k, \quad \{m_i, \gamma_j\} = \varepsilon_{ijk}\gamma_k, \quad \{\gamma_i, \gamma_j\} = 0. \tag{55}$$

As in the case of the geodesic flows of left-invariant metrics on Lie groups, any Lagrangian system corresponding to a left-invariant metric and a "left-invariant" potential field on the group $E(3)$ is reduced to a Hamiltonian system

$$\frac{df}{dt} = \{f, H\}$$

on the algebra $e(3)$ with a Hamiltonian

$$H(m,\gamma) = \frac{1}{2}a^{ij}m_i m_j + \frac{1}{2}b^{ij}(m_i\gamma_j + m^j\gamma_i) + \frac{1}{2}c^{ij}\gamma_i\gamma_j + V(m,\gamma), \quad (56)$$

where the matrices a^{ij}, b^{ij}, and c^{ij} are symmetric and U is a linear function in m and γ. Here we call a potential field $U(q)$ "left-invariant" if its gradient is left-invariant. Thus, denoting the local coordinates corresponding to e_i and f_j by x^i and y^j we conclude that

$$V(m,\gamma) = \sum_{i=1}^{3}\left(\frac{\partial U(0)}{\partial x^i}m_i + \frac{\partial U(0)}{\partial y^j}\gamma_j\right),$$

and a "left-invariant" potential field is determined uniquely by its gradient at the unit of the group.

The Kirchhoff equations for the free motion of a rigid body in a liquid correspond to $V \equiv 0$ [11]. In this case the configuration space is the whole group $E(3)$.

By Theorem HS, in the Hamiltonian formalism constraints enter Hamiltonian equations via a Hamiltonian function. Moreover, Theorem HS also holds for holonomic constraints.

Hence, starting with the problem of the free motion of a heavy body in a potential field with the configuration space $E(3)$ we pose holonomic constraints by fixing a point of the body. In this case the configuration space is homeomorphic to SO(3), but the Euler equations (see Sec. 5) are still written for the algebra $e(3)$ and correspond to the Hamiltonian function

$$H(m,\gamma) = \frac{(Im,m)}{2} + (r,\gamma). \quad (57)$$

Thus, we obtain the well-known Hamiltonian formalism for this problem.

First, it was derived in physical terms. Here I is the inertia tensor, m is the angular momentum, γ is the vector in the direction of gravity, and r is the center of mass. All coordinates are taken with respect to the orthogonal frame attached to the body with the fixed point as the center of coordinates.

8. Another definition of "straight lines" and problems of mechanics

There is another way to define "straight lines" in nonholonomic geometry. We discuss only the case of constraints linear in velocities.

Roughly speaking, geodesics of Carnot–Carathéodory metric are solutions of the following variational problem. Let $L(x,\dot{x}) = \tilde{g}_{ij}\dot{x}^i\dot{x}^j dt$ be an energy functional on a suitable space of curves (periodic or with fixed endpoints) in a manifold M^n and let $\tilde{g}_{ij}$ be a Riemannian metric tensor. A geodesic $\gamma(t)$ of the Carnot–Carathéodory metric corresponding to $\tilde{g}_{ij}$ and a distribution $\mathcal{F}$ are local extremals of this functional with respect to the set of variations of the form $\gamma(t,u)$, where $\gamma(t,0) = \gamma(t)$ and

$$\frac{\partial\gamma}{\partial t}(t,u) \in \mathcal{F}_{\gamma(t,u)}. \tag{58}$$

Of course, all variations belong to the space of curves under consideration.

But condition (58) can be replaced by another one:

$$\frac{\partial\gamma}{\partial u}(t,0) \in \mathcal{F}_{\gamma(t,0)}. \tag{59}$$

In this case the equations for "straight lines" are written as

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} - \frac{\partial L}{\partial x^i} = \sum_\alpha \mu_\alpha \omega_i^{(\alpha)}, \tag{60}$$

where

$$\langle \omega^{(\alpha)}, \dot{x}\rangle = \omega_i^{(\alpha)}\dot{x}^i = 0 \tag{61}$$

are the constraints, and the coefficients μ_α are derived from the condition that relations (61) hold. One can easily see the difference of these equations from Eqs. (8)–(9).

Notice also that these "straight lines" are determined by the Lagrange function defined on the whole tangent space TM^n but not only on $\mathcal{F}$.

The most famous example of such system is the Chaplygin top, the dynamically-asymmetric ball rolling on the horizontal plane and with the center of mass coinciding with its geometric center [6].

This system was integrated by Chaplygin without the use of the Liouville integrability theorem. Its algebraic origin was clarified by Veselov and Veselova who regarded it as a flow of "straight lines" on $E(3)$, the Lie group of motions of the three-dimensional Euclidean space, endowed with a left-invariant metric and a right-invariant nonholonomic distribution [18]. They also generalized this system to arbitrary Lie groups and succeeded in generalizing the Chaplygin integration method for three-dimensional groups.

9. Concluding remarks

(1) The procedure of constructing a Hamiltonian structure is generalized for nonholonomic systems with Lagrange functions of the form

$$L(x,\dot{x}) = \frac{|\dot{x}|^2}{2} + U(x)$$

in the usual manner. We already mentioned this in Sec. 7.

(2) We have not found essentially new examples of manifolds which admit the integrable geodesic flow of a Carnot–Carathéodory metric and have no Riemannian metrics with integrable geodesic flows. Nevertheless, we would like to mention that all methods of finding topological obstructions to integrability of the geodesic flows of Riemannian metrics [12], [14] (see also [15]) fail in the case of Carnot–Carathéodory metrics. The reason for this is clear. These methods use the compactness of the level set of a Hamiltonian which cannot be compact for Carnot–Carathéodory metrics. Thus one can expect that the class of manifolds admitting integrable Carnot–Carathéodory geodesic flows is wider than the class of manifolds admitting integrable Riemannian geodesic flows.

The lack of compactness of the level sets of a Hamiltonian also deters us from defining the entropy characteristics of such flows in the usual manner.

However, in Sec. 5 we give an example of an integrable flow on SO(3) with compact level sets of the first integral $(\mathcal{J}x, x)$ which is not a Hamiltonian function. This situation is typical for the geodesic flows of left-invariant metrics on compact Lie groups but it is not generic, as one can see from the example given in Sec. 6.

(3) Consider the simplest example of degeneration of integrable Riemannian geodesic flows into an integrable Carnot–Carathéodory geodesic flow.

Take the Lie group SO(3) and denote by e_1, e_2, and e_3 the generators of its Lie algebra so(3) with the following commutation relations:

$$[e_i, e_j] = \varepsilon_{ijk} e_k.$$

Denote by $\tilde{\mathcal{L}}_0$ the subspace of so(3) spanned by e_1 and e_2. Consider the family of left-invariant metrics generated by the metrics on so(3) of the form

$$G_D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & D \end{pmatrix}.$$

The geodesic flows of these metrics are integrable. Tending D to infinity, $D \to \infty$, we arrive at the geodesic flow of the Carnot–Caratheodory metric on SO(3) corresponding to the Riemannian metric G_1 and the left-invariant distribution $L_{g*}\tilde{\mathcal{L}}_0$, [7], [19]. It is shown in Sec. 5 that this flow is integrable.

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