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TESTING FOR PARAMETER CHANGES IN ARCH MODELS[1]

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Abstract. The paper develops the asymptotic theory for CUSUM-type tests for a change point in parameters of an ARCH(∞) model. Special attention is given to asymptotics under local alternatives.

Key words: ARCH model, change-point testing, local alternatives.

1. INTRODUCTION

This paper studies CUSUM-type tests for a change point in the parameters of an ARCH model defined by the equations

$$r_k = \sigma_k \varepsilon_k, \qquad \sigma_k^2 = b_0 + \sum_{j=1}^{\infty} b_j r_{k-j}^2, \tag{1.1}$$

where the random variables ε_j, $j = 0, \pm 1, \pm 2, \ldots$, are zero mean iid errors and the coefficients b_j are non-negative constants. We assume that the ε_j have finite moments up to order eight and that the following condition holds:

$$(E\varepsilon_0^8)^{1/4} \sum_{j=1}^{\infty} b_j < 1. \tag{1.2}$$

Condition (1.2) guarantees that Eqs. (1.1) have a strictly stationary solution $\{r_k\}$ such that the sequence $X_k = r_k^2$ is fourth-order stationary and satisfies the functional limit theorem (see Theorem 1.1 below and Giraitis *et al.* [5], [6] for more details).

The main feature of ARCH processes is that while the r_k's are uncorrelated, the sequence of the squares r_k^2 has a rich dependence structure. In modeling financial data, the squares of the *returns* r_k are used to estimate the so-called *volatility*, which is an important parameter in asset pricing models. An excellent account of the theory and applications of ARCH models is given by Gouriéroux [9].

For the reasons outlined above, we focus on the squares $X_k = r_k^2$. Denoting by $\mathbf{b} := (b_0, b_1, \ldots)$ the parameter sequence in (1.1), we write $\{X_k\} \in \mathcal{R}(\mathbf{b})$ if the X_k are obtained from Eqs. (1.1) with $\mathbf{b}$ satisfying condition (1.2).

We test the null hypothesis

$$H_0\colon\ X_1, \ldots, X_N \text{ is a sample from } \{X_k\} \in \mathcal{R}(\mathbf{b}) \text{ for some } \mathbf{b}$$

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against the change-point alternative

$$H_1\colon\ \exists\ \mathbf{b}^{(1)}, \mathbf{b}^{(2)},\ \text{satisfying } \mathbf{b}^{(1)} \neq \mathbf{b}^{(2)},$$

and such that the sample $X_1, \dots, X_N$ has the form

$$X_k = \begin{cases} X_k^{(1)}, & \text{if } 1 \leqslant k \leqslant k^*, \\ X_k^{(2)}, & \text{if } k^* < k \leqslant N, \end{cases} \tag{1.3}$$

where $\{X_k^{(1)}\} \in \mathcal{R}(\mathbf{b}^{(1)})$, $\{X_k^{(2)}\} \in \mathcal{R}(\mathbf{b}^{(2)})$, $k^* = [N\tau^*]$, $0 < \tau^* < 1$ is fixed. The sequences $\{X_k^{(1)}\}$ and $\{X_k^{(2)}\}$ are generated by the same noise sequence $\{\varepsilon_k\}$.

Testing for parameter constancy against some kind of unstability in the parameters of the conditional variance of the returns r_k is very important in financial data analysis. In particular, in order to avoid spurious inferences about a model, it is important to test the stability hypothesis against an alternative of structural breaks in parameters (see, e.g., Lamoureux and Lastrapes [12]; Hamilton and Susmel [10]).

Despite its importance, the problem of detecting parameter changes has not received as much attention in the context of ARCH models as in the setting of linear time series models. In the latter area, the literature is so extensive that we restrict ourselves only to citing the recent monograph [4]. As far as we are aware, the only tests available for ARCH sequences are Lagrange multiplier tests developed by Chu [3] and Lundbergh and Teräsvirta [14]. Change-point estimation in ARCH models was studied by us in [11].

Our aim is to consider a family of tests for the constancy of the parameter $\mathbf{b}$ and to investigate their asymptotic properties. We study CUSUM-type tests based on the process $\{U_N(t), t \in [0, 1]\}$, where

$$U_N(t) = N^{1/2}\,\frac{[Nt](N-[Nt])}{N^2}\left(\frac{1}{[Nt]}\sum_{j=1}^{[Nt]} X_j - \frac{1}{N-[Nt]}\sum_{j=[Nt]+1}^{N} X_j\right). \tag{1.4}$$

The partial sums in (1.4) are estimators of the variance of the r_k, so the tests are designed to detect a change in parameters, which leads to a change in variance.

We show that the process $\{U_N(t)\}$ satisfies invariance principles in $D[0, 1]$ under both the null hypothesis and local alternatives that converge to the null hypothesis at the rate $1/\sqrt{N}$. Thus, the asymptotic theory for the standard statistics $\sup_{0\leqslant t\leqslant 1} |U_N(t)|$, $\int_0^1 U_N^2(t)\,dt$ and their modifications follows automatically. The main attention in the paper is devoted to the asymptotic behavior of the process $\{U_N(t)\}$ under local alternatives. Our results show that the tests based on continuous functionals of the process $\{U_N(t)\}$ have positive asymptotic power against such alternatives. In the context of linear time series models, similar results were obtained by Giraitis and Leipus [7] and Leipus [13], who studied statistics based on the periodogram, and by Giraitis *et al.* [8], who used an approach based on the empirical distribution function. Bai [1] studied the asymptotics of a test based on the empirical distribution function under local alternatives of a change in regression parameters.

In the proofs, we frequently use the following Volterra expansion:

$$\begin{aligned} X_j &= b_0\varepsilon_j^2 + b_0 \sum_{l=1}^{\infty}\sum_{j_1,\dots,j_l=1}^{\infty} b_{j_1}\dots b_{j_l}\varepsilon_j^2\varepsilon_{j-j_1}^2\dots\varepsilon_{j-j_1-\dots-j_l}^2 \\ &= \sum_{l=0}^{\infty}\sum_{j_1,\dots,j_l=1}^{\infty}\left(\prod_{m=0}^{l} b_{j_m}\right)\left(\prod_{m=0}^{l}\varepsilon_{j-j_1-\dots-j_m}^2\right). \end{aligned} \tag{1.5}$$

In (1.5) and in the sequel, we assume that $j_0 = 0$. For the convenience of the reader, we state here the following result established by Giraitis *et al.* [5], which is extensively used in the proofs below. In the sequel, $\xrightarrow{D[0,1]}$ stands for the weak convergence in the space $D[0, 1]$ endowed with the Skorokhod topology.

THEOREM 1.1. *If condition* (1.2) *holds, then*

$$N^{-1/2}\sum_{j=1}^{[Nt]}(X_j - EX_j) \xrightarrow{D[0,1]} \sigma W(t),$$

where $W(t)$ *is a standard Wiener process and*

$$\sigma^2 = \sum_{j=-\infty}^{\infty} \operatorname{Cov}(X_j, X_0).$$

In the proofs, C stands for a positive constant whose value may change from line to line. We denote in the sequel $\lambda_k := E|\varepsilon_0|^k$.

In Section 2, we establish the asymptotics for the statistic (1.4) under the null hypothesis. In order to obtain a parameter-free asymptotic distribution, the statistic $U_N(\cdot)$ must be normalized by a variance estimator $\hat{s}_{N,q}$, which is also introduced in Section 2. In Sections 3 and 4, the local alternatives are investigated. In Section 3, we describe the local alternatives and give the corresponding asymptotics of the process $U_N(\cdot)$. In Section 4, we obtain the asymptotics for the rescaled process $U_N(\cdot)/\hat{s}_{N,q}$.

2. ASYMPTOTICS UNDER THE NULL HYPOTHESIS

THEOREM 2.1. *Under the null hypothesis* H_0,

$$U_N(t) \xrightarrow{D[0,1]} \sigma W^0(t),$$

where

$$\sigma^2 = \sum_{j=-\infty}^{\infty} \operatorname{Cov}(X_j, X_0)$$

and $W^0(t)$ *is a Brownian bridge.*

Proof. The proof follows from the representation

$$U_N(t) = N^{-1/2}\left(\sum_{j=1}^{[Nt]}(X_j - EX_j) - \frac{[Nt]}{N}\sum_{j=1}^{N}(X_j - EX_j)\right)$$

and Theorem 1.1.

In order to use Theorem 2.1 to construct asymptotic critical regions, we must estimate the parameter σ. Define

$$\hat{s}^2_{N,q} = \sum_{|j|\leqslant q} \omega_j(q)\hat{\gamma}_j, \tag{2.1}$$

where the $\hat{\gamma}_j$ are the sample covariances:

$$\hat{\gamma}_j = \frac{1}{N}\sum_{i=1}^{N-|j|}(X_i - \bar{X})(X_{i+|j|} - \bar{X}), \quad |j| < N, \tag{2.2}$$

$\bar{X}$ is the sample mean $N^{-1}\sum_{j=1}^{N} X_j$ and

$$\omega_j(q) = 1 - \frac{|j|}{q+1}, \quad |j| \leqslant q,$$

are the Bartlett weights. Then the following theorem holds.

THEOREM 2.2. *Assume that $q \to \infty$, $q/N \to 0$. Then under the hypothesis H_0*

$$\frac{U_N(t)}{\hat{s}_{N,q}} \xrightarrow{D[0,1]} W^0(t). \tag{2.3}$$

Proof. First of all, note that by Theorem 3.1 in [5]

$$\hat{s}^2_{N,q} \xrightarrow{P} \sigma^2 \tag{2.4}$$

for any fourth-order stationary process $\{X_j\}$ satisfying

$$\sum_{j=-\infty}^{\infty} \left|\mathrm{Cov}(X_j, X_0)\right| < \infty \tag{2.5}$$

and

$$\sum_{l,m=-\infty}^{\infty} \left|\mathrm{Cum}(X_0, X_k, X_l, X_m)\right| < \infty \tag{2.6}$$

uniformly in k. Here

$$\begin{aligned}\mathrm{Cum}(X_0, X_k, X_l, X_m) = {} & E\big[(X_0 - \mu)(X_k - \mu)(X_l - \mu)(X_m - \mu)\big] \\ & - \mathrm{Cov}(X_0, X_k)\mathrm{Cov}(X_l, X_m) - \mathrm{Cov}(X_0, X_l)\mathrm{Cov}(X_m, X_k) \\ & - \mathrm{Cov}(X_0, X_m)\mathrm{Cov}(X_k, X_l)\end{aligned}$$

is the fourth-order cumulant and $\mu = EX_0$. Giraitis *et al.* [5] showed that conditions (2.5) and (2.6) are satisfied by the process $\{X_k\} \in \mathcal{R}(\mathbf{b})$. Therefore, convergence (2.3) follows from (2.4) and Theorem 2.1.

Remark 2.1. Chu [3] considered, among other tests, the Lagrange multiplier test (LMU) for the constancy of unconditional variance against a single structural break. This test is based on the maximum of the statistic

$$L_N(t) = \frac{1}{\hat{s}_{N,q}} \left(\frac{1}{[Nt]} \sum_{j=1}^{[Nt]} (X_j - \bar{X}) + \frac{1}{N - [Nt]} \sum_{j=[Nt]+1}^{N} (X_j - \bar{X}) \right).$$

The theory was developed in the context of a finite-order GARCH model. Noting that

$$\frac{k(N-k)}{N^2} \left(\frac{1}{k} \sum_{j=1}^{k} (X_j - \bar{X}) + \frac{1}{N-k} \sum_{j=k+1}^{N} (X_j - \bar{X}) \right) = N^{-1} \left(1 - 2\,\frac{k}{N}\right) \sum_{j=1}^{k} (X_j - \bar{X}),$$

we obtain that, under the assumptions of Theorem 2.2,

$$N^{1/2}\, \frac{[Nt](N - [Nt])}{N^2}\, L_N(t) \xrightarrow{D[0,1]} (1 - 2t) W^0(t).$$

3. ASYMPTOTICS UNDER LOCAL ALTERNATIVES

In the investigation of statistical properties such as the efficiency of tests, it is important to know the behavior of the statistics under local alternatives. As indicated in the introduction, we focus our attention on the local alternatives that converge at the rate $1/\sqrt{N}$. We show in Theorem 3.1 that under such alternatives the statistic $U_N(\cdot)$ has an asymptotic distribution that differs from the one occurring in Theorem 2.1 by a deterministic function. In conjunction with Lemma 4.1, this allows us, in principle, to compute the asymptotic power of tests based on functionals of the rescaled process $U_N(\cdot)/\hat{s}_{N,q}$.

Formally, we consider the local alternatives

$$H_1^{(\mathrm{loc})}\colon \ \exists\ \mathbf{b}^{(1,N)}, \mathbf{b}^{(2,N)}, \ \text{satisfying } \mathbf{b}^{(1,N)} \neq \mathbf{b}^{(2,N)}$$

and such that

$$X_k^{(N)} = \begin{cases} X_k^{(1,N)}, & \text{if } 1 \leqslant k \leqslant k^*, \\ X_k^{(2,N)}, & \text{if } k^* < k \leqslant N. \end{cases} \tag{3.1}$$

Here $X_k^{(i,N)} \in \mathcal{R}(\mathbf{b}^{(i,N)})$, $i = 1, 2$, with $\mathbf{b}^{(i,N)} \equiv (b_0^{(i,N)}, b_1^{(i,N)}, \ldots)$ satisfying the following assumption:

ASSUMPTION A. Assume that

$$b_j^{(i,N)} = b_j + \frac{\beta_j^{(i,N)}}{\sqrt{N}}, \quad j \geqslant 0, \ i = 1, 2, \tag{3.2}$$

with $b_j \geqslant 0$,

$$\beta_j^{(i,N)} \longrightarrow \beta_j^{(i)} \ \text{as } N \to \infty, \tag{3.3}$$

and

$$\lambda_8^{1/4} \sum_{j=1}^{\infty} (b_j + \beta_j^*) < 1, \tag{3.4}$$

where we denote

$$\beta_j^* := \max_{i=1,2} \sup_{N \geqslant 1} |\beta_j^{(i,N)}| < \infty. \tag{3.5}$$

Set $B^* := \sum_{j=1}^{\infty} (b_j + \beta_j^*)$, $B := \sum_{j=1}^{\infty} b_j$ and $B^{(i,N)} = \sum_{j=1}^{\infty} b_j^{(i,N)}$. We see that assumptions (3.2)–(3.4) imply, in particular,

$$\lim_{N\to\infty} B^{(i,N)} = B \quad \text{and} \quad \lim_{N\to\infty} \sqrt{N}(B^{(1,N)} - B^{(2,N)}) = \sum_{j=1}^{\infty} (\beta_j^{(1)} - \beta_j^{(2)}).$$

The last two relations are used in the proof of the following lemma:

LEMMA 3.1. *Under the hypothesis* $H_1^{(\mathrm{loc})}$,

$$EU_N(t) \longrightarrow (t \wedge \tau^* - t\tau^*)\Delta \ \ \text{as } N \to \infty, \tag{3.6}$$

where

$$\Delta = \frac{\lambda_2\big[(\beta_0^{(1)} - \beta_0^{(2)})(1 - \lambda_2 B) + \lambda_2 b_0 \sum_{j=1}^{\infty} (\beta_j^{(1)} - \beta_j^{(2)})\big]}{(1 - \lambda_2 B)^2}.$$

Proof. By a straightforward verification we obtain

$$EU_N(t) = \begin{cases} N^{1/2} \dfrac{[Nt]}{N} \left(1 - \dfrac{k^*}{N}\right) (EX_0^{(1,N)} - EX_0^{(2,N)}), & \text{if } 1 \leqslant [Nt] \leqslant k^*, \\[2ex] N^{1/2} \left(1 - \dfrac{[Nt]}{N}\right) \dfrac{k^*}{N} (EX_0^{(1,N)} - EX_0^{(2,N)}), & \text{if } k^* < [Nt] \leqslant N. \end{cases} \tag{3.7}$$

Since

$$X_0^{(i,N)} = \varepsilon_0^2 b_0^{(i,N)} \sum_{l=0}^{\infty} \sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{m=1}^{l} \left(b_{j_m}^{(i,N)} \varepsilon_{-j_1-\ldots-j_m}^2 \right),$$

we have

$$EX_0^{(i,N)} = \lambda_2 b_0^{(i,N)} \sum_{l=0}^{\infty} \sum_{j_1,\ldots,j_l=1}^{\infty} \prod_{m=1}^{l} (b_{j_m}^{(i,N)} \lambda_2) = \lambda_2 b_0^{(i,N)} \sum_{l=0}^{\infty} (\lambda_2 B^{(i,N)})^l = \frac{\lambda_2 b_0^{(i,N)}}{1 - \lambda_2 B^{(i,N)}}.$$

By (3.7) and Assumption A, the last relation implies (3.6).

Before establishing our main result, Theorem 3.1, which gives the asymptotics for the statistic $U_N(\cdot)$ under the local alternatives, we state a useful lemma that follows from representation (1.5) and elementary algebra.

LEMMA 3.2. *Suppose* $\{X_k^{(i)}\} \in \mathcal{R}(\mathbf{b}^{(i)})$, $i = 1, 2$. *For any integers* k, l,

(i) $\mathrm{Cov}(X_k^{(1)}, X_l^{(2)}) \geqslant 0$;

(ii) *if* $\max\{b_j^{(1)}, b_j^{(2)}\} \leqslant b_j^*$, $j \geqslant 0$, *where* $\mathbf{b}^* \equiv (b_0^*, b_1^*, \ldots)$ *satisfies assumption* (1.2), *then*

$$\mathrm{Cov}\big(X_k^{(i)}, X_l^{(i')}\big) \leqslant \mathrm{Cov}\left(X_k^*, X_l^*\right), \quad i, i' = 1, 2,$$

with $\{X_k^*\} \in \mathcal{R}(\mathbf{b}^*)$.

THEOREM 3.1. *Under the hypothesis* $H_1^{(\mathrm{loc})}$,

$$U_N(t) \xrightarrow{D[0,1]} \sigma W^0(t) + G(t), \tag{3.8}$$

where

$$G(t) := (t \wedge \tau^* - t\tau^*)\Delta,$$

with σ *defined by means of* $\{X_j\} \in \mathcal{R}(\mathbf{b})$ *given by* (1.5) *as in Theorem* 2.1 *and* Δ *as in Lemma* 3.1.

Proof. By Lemma 3.1, (3.8) follows if we prove that under hypothesis $H_1^{(\mathrm{loc})}$

$$U_N(t) - EU_N(t) \xrightarrow{D[0,1]} \sigma W^0(t). \tag{3.9}$$

To prove (3.9) we will show: (1) the convergence of the finite-dimensional distributions of $\{U_N(t) - EU_N(t), t \in [0, 1]\}$ under $H_1^{(\mathrm{loc})}$; (2) the tightness of the corresponding probability measures on $D[0, 1]$ under $H_1^{(\mathrm{loc})}$.

Since, by the definition of the local alternatives $H_1^{(\mathrm{loc})}$, the process $\{X_j\}$ defined by (1.5), with $\{b_j\}$ from Assumption A, is in the class $\mathcal{R}(\mathbf{b})$, we obtain from Theorem 2.1 that

$$N^{1/2} \frac{[Nt](N - [Nt])}{N^2} \left(\frac{1}{[Nt]} \sum_{j=1}^{[Nt]} (X_j - EX_j) - \frac{1}{N - [Nt]} \sum_{j=[Nt]+1}^{N} (X_j - EX_j) \right) \xrightarrow{D[0,1]} \sigma W^0(t).$$

Denote

$$
\begin{aligned}
Z_j^{(i,N)} &:= (X_j^{(i,N)} - EX_j^{(i,N)}) - (X_j - EX_j) \\
&= \sum_{l=0}^{\infty} \sum_{j_1,\dots,j_l=1}^{\infty} \prod_{m=0}^{l} b_{j_m}^{(i,N)} \left(\prod_{m=0}^{l} \varepsilon_{j-j_1-\dots-j_m}^2 - \lambda_2^{l+1} \right) - \sum_{l=0}^{\infty} \sum_{j_1,\dots,j_l=1}^{\infty} \prod_{m=0}^{l} b_{j_m} \left(\prod_{m=0}^{l} \varepsilon_{j-j_1-\dots-j_m}^2 - \lambda_2^{l+1} \right) \\
&= \sum_{l=0}^{\infty} \sum_{j_1,\dots,j_l=1}^{\infty} \left(\prod_{m=0}^{l} b_{j_m}^{(i,N)} - \prod_{m=0}^{l} b_{j_m} \right) \left(\prod_{m=0}^{l} \varepsilon_{j-j_1-\dots-j_m}^2 - \lambda_2^{l+1} \right), \quad i = 1, 2, \qquad (3.10)
\end{aligned}
$$

and

$$
\tilde{U}_N(t) := N^{1/2} \frac{[Nt](N-[Nt])}{N^2} \left(\frac{1}{[Nt]} \sum_{j=1}^{[Nt]} Z_j^{(N)} - \frac{1}{N-[Nt]} \sum_{j=[Nt]+1}^{N} Z_j^{(N)} \right),
$$

where

$$
Z_k^{(N)} = \begin{cases} Z_k^{(1,N)}, & \text{if } 1 \leqslant k \leqslant k^*, \\ Z_k^{(2,N)}, & \text{if } k^* < k \leqslant N. \end{cases} \qquad (3.11)
$$

The convergence of finite-dimensional distributions will follow if we prove that finite-dimensional distributions of the process $\{\tilde{U}_N(t),\ t \in [0,1]\}$ satisfy $\tilde{U}_N(t) \xrightarrow{d} 0$. Here and below $\xrightarrow{d}$ means the convergence in distribution. Define the partial sums $A_N^{(i)}(t) = N^{-1/2} \sum_{j=1}^{[Nt]} Z_j^{(i,N)}$. Then, by straightforward verification,

$$
\tilde{U}_N(t) = \begin{cases} A_N^{(1)}(t) - \frac{[Nt]}{N} A_N^{(1)}(\tau^*) - \frac{[Nt]}{N} A_N^{(2)}(1) + \frac{[Nt]}{N} A_N^{(2)}(\tau^*), & \text{if } [Nt] \leqslant [N\tau^*], \\ \left(1 - \frac{[Nt]}{N}\right) A_N^{(1)}(\tau^*) - \left(1 - \frac{[Nt]}{N}\right) A_N^{(2)}(\tau^*) + A_N^{(2)}(t) - \frac{[Nt]}{N} A_N^{(2)}(1), & \text{if } [Nt] > [N\tau^*]. \end{cases}
$$

Therefore, it is sufficient to prove that for any fixed $0 < t \leqslant 1$

$$
A_N^{(i)}(t) \xrightarrow{P} 0, \quad i = 1, 2. \qquad (3.12)
$$

For simplicity, assume that $i = 1$ and $t = 1$. Relation (3.12) will be proved if we show that

$$
E(A_N^{(1)}(1))^2 = N^{-1} \sum_{j,j'=1}^{N} E Z_j^{(1,N)} Z_{j'}^{(1,N)} \longrightarrow 0.
$$

Write

$$
N^{-1} \sum_{j,j'=1}^{N} E Z_j^{(1,N)} Z_{j'}^{(1,N)} = N^{-1} \sum_{\substack{j,j'=1 \\ |j-j'| \leqslant K}}^{N} E Z_j^{(1,N)} Z_{j'}^{(1,N)} + N^{-1} \sum_{\substack{j,j'=1 \\ |j-j'| > K}}^{N} E Z_j^{(1,N)} Z_{j'}^{(1,N)} =: I_1^{(N)} + I_2^{(N)},
$$

where $0 < K = K_N < N$ and $K/N \to 0$, $K \to \infty$ as $N \to \infty$. To show that

$$
I_1^{(N)} \longrightarrow 0 \qquad (3.13)
$$

write

$$|EZ_j^{(1,N)}Z_{j'}^{(1,N)}| = \Bigg| \sum_{l,l'=0}^{\infty} \sum_{j_1,\dots,j_l=1}^{\infty} \sum_{j'_1,\dots,j'_{l'}=1}^{\infty} \Big(\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m}\Big)\Big(\prod_{m'=0}^{l'} b_{j'_{m'}}^{(1,N)} - \prod_{m'=0}^{l'} b_{j'_{m'}}\Big)$$

$$\times E\Big(\prod_{m=0}^{l} \varepsilon^2_{j-j_1-\dots-j_m} - \lambda_2^{l+1}\Big)\Big(\prod_{m'=0}^{l'} \varepsilon^2_{j'-j'_1-\dots-j'_{m'}} - \lambda_2^{l'+1}\Big)\Bigg|$$

$$\leqslant \sum_{l,l'=0}^{\infty} \sum_{j_1,\dots,j_l=1}^{\infty} \sum_{j'_1,\dots,j'_{l'}=1}^{\infty} \Big|\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m}\Big| \Big|\prod_{m'=0}^{l'} b_{j'_{m'}}^{(1,N)} - \prod_{m'=0}^{l'} b_{j'_{m'}}\Big|$$

$$\times \Big(E\prod_{m=0}^{l} \varepsilon^4_{j-j_1-\dots-j_m}\Big)^{1/2}\Big(E\prod_{m'=0}^{l'} \varepsilon^4_{j'-j'_1-\dots-j'_{m'}}\Big)^{1/2}.$$

Note that

$$\Big(E\prod_{m=0}^{l} \varepsilon^4_{j-j_1-\dots-j_m}\Big)^{1/2}\Big(E\prod_{m'=0}^{l'} \varepsilon^4_{j'-j'_1-\dots-j'_{m'}}\Big)^{1/2} = \lambda_4\lambda_4^{(l+l')/2}.$$

Thus, since $\#\{(j,j') |\ 1 \leqslant j, j' \leqslant N,\ |j-j'| \leqslant K\} = 2KN + N - K(K+1)$, we have

$$|I_1^{(N)}| \leqslant N^{-1} \sum_{\substack{j,j'=1 \\ |j-j'|\leqslant K}}^{N} \lambda_4 \Big(\sum_{l=0}^{\infty} \lambda_4^{l/2} \sum_{j_1,\dots,j_l=1}^{\infty} \Big|\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m}\Big|\Big)^2 \tag{3.14}$$

$$\leqslant \Big(2K + 1 - \frac{K(K+1)}{N}\Big)\lambda_4 \Big(\sum_{l=0}^{\infty} \lambda_4^{l/2} \sum_{j_1,\dots,j_l=1}^{\infty} \Big|\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m}\Big|\Big)^2.$$

We will show that

$$r_N := \sum_{l=0}^{\infty} \lambda_4^{l/2} \sum_{j_1,\dots,j_l=1}^{\infty} \Big|\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m}\Big| \leqslant C\ N^{-1/2}. \tag{3.15}$$

Since, by (3.2),

$$|b_j^{(1,N)} - b_j| \leqslant N^{-1/2}|\beta_j^{(1,N)}|$$

and

$$\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m} = \sum_{m=0}^{l} b_{j_1}\cdots b_{j_{m-1}}(b_{j_m}^{(1,N)} - b_{j_m})b_{j_{m+1}}^{(1,N)}\cdots b_{j_l}^{(1,N)},$$

we obtain

$$\Big|\prod_{m=0}^{l} b_{j_m}^{(1,N)} - \prod_{m=0}^{l} b_{j_m}\Big| \leqslant N^{-1/2}\sum_{k=0}^{l} b_{j_1}\cdots b_{j_{k-1}}|\beta_{j_k}^{(1,N)}|b_{j_{k+1}}^{(1,N)}\cdots b_{j_l}^{(1,N)}.$$

Therefore, setting

$$\hat{b}_j^{(i,N)} := \max\{|\beta_j^{(i,N)}|, b_j,\ b_j^{(i,N)}\} \quad \text{and} \quad \hat{B}^{(i,N)} := \sum_{j=1}^{\infty} \hat{b}_j^{(i,N)}, \quad i = 1, 2, \tag{3.16}$$

we obtain

$$
\begin{aligned}
r_N &\leqslant N^{-1/2}\sum_{l=0}^{\infty}\lambda_4^{l/2}\sum_{j_1,\dots,j_l=1}^{\infty}\sum_{k=0}^{l}\left|\beta_{j_k}^{(1,N)}\right|\prod_{\substack{m=0\\ m\neq k}}^{l}\max\left\{b_{j_m}, b_{j_m}^{(1,N)}\right\}\\
&\leqslant N^{-1/2}\sum_{l=0}^{\infty}\hat{b}_0^{(1,N)}(l+1)\lambda_4^{l/2}\sum_{j_1,\dots,j_l=1}^{\infty}\prod_{m=1}^{l}\hat{b}_{j_m}^{(1,N)}\\
&\leqslant N^{-1/2}\sum_{l=0}^{\infty}\hat{b}_0^{(1,N)}(l+1)\left(\lambda_4^{1/2}\hat{B}^{(1,N)}\right)^l.
\end{aligned}
\tag{3.17}
$$

Since

$$\hat{b}_j^{(i,N)} \leqslant b_j + \left|\beta_j^{(i,N)}\right| \leqslant b_j + \beta_j^*,$$

and, by (3.4), $\lambda_4^{1/2}B^* < 1$, we obtain from (3.17) that

$$r_N \leqslant N^{-1/2}(b_0+\beta_0^*)\sum_{l=0}^{\infty}(l+1)\left(\lambda_4^{1/2}B^*\right)^l \leqslant CN^{-1/2}.$$

Thus (3.15) follows and hence

$$\left|EZ_j^{(1,N)}Z_{j'}^{(1,N)}\right| \leqslant \frac{C}{N}. \tag{3.18}$$

This, together with (3.14) and the assumption that $K/N \to 0$, implies (3.13).

Next we show that

$$I_2^{(N)} \equiv N^{-1}\sum_{\substack{j,j'=1\\ |j-j'|>K}}^{N} EZ_j^{(1,N)}Z_{j'}^{(1,N)} \longrightarrow 0 \quad \text{as } N\to\infty. \tag{3.19}$$

Note that

$$EZ_j^{(1,N)}Z_{j'}^{(1,N)} = \operatorname{Cov}\left(X_j^{(1,N)}, X_{j'}^{(1,N)}\right) - \operatorname{Cov}\left(X_j^{(1,N)}, X_{j'}\right) - \operatorname{Cov}\left(X_j, X_{j'}^{(1,N)}\right) + \operatorname{Cov}(X_j, X_{j'}).$$

By Lemma 3.2 (i), for (3.19) it is sufficient to verify that

$$N^{-1}\sum_{\substack{j,j'=1\\ |j-j'|>K}}^{N}\left(\operatorname{Cov}\left(X_j^{(1,N)}, X_{j'}^{(1,N)}\right) + \operatorname{Cov}\left(X_j, X_{j'}\right) + \operatorname{Cov}\left(X_j^{(1,N)}, X_{j'}\right)\right) \longrightarrow 0. \tag{3.20}$$

Lemma 3.2 (ii) implies that

$$\operatorname{Cov}\left(X_j^{(1,N)}, X_{j'}^{(1,N)}\right) + \operatorname{Cov}(X_j, X_{j'}) + \operatorname{Cov}\left(X_j^{(1,N)}, X_{j'}\right) \leqslant 3\operatorname{Cov}\left(\hat{X}_j^{(1,N)}, \hat{X}_{j'}^{(1,N)}\right),$$

where

$$\hat{X}_j^{(i,N)} := \sum_{l=0}^{\infty}\sum_{j_1,\dots,j_l=1}^{\infty}\prod_{m=0}^{l}\hat{b}_{j_m}^{(i,N)}\prod_{m=0}^{l}\varepsilon_{j-j_1-\dots-j_m}^2, \quad i=1,2,$$

and the $\hat{b}_j^{(i,N)}$ are given by (3.16). Thus (3.20) will be proved if we show that

$$N^{-1}\sum_{\substack{j,j'=1\\ |j-j'|>K}}^{N}\operatorname{Cov}\left(\hat{X}_j^{(1,N)}, \hat{X}_{j'}^{(1,N)}\right) \longrightarrow 0. \tag{3.21}$$

Write

$$N^{-1} \sum_{\substack{j,j'=1 \\ |j-j'|>K}}^{N} \mathrm{Cov}\big(\hat{X}_j^{(1,N)}, \hat{X}_{j'}^{(1,N)}\big) = \sum_{K<|j|<N} \left(1 - \frac{|j|}{N}\right) \mathrm{Cov}\big(\hat{X}_j^{(1,N)}, \hat{X}_0^{(1,N)}\big)$$
$$\leqslant \sum_{K<|j|<N} \mathrm{Cov}\big(\hat{X}_j^{(1,N)}, \hat{X}_0^{(1,N)}\big).$$

Note that $\hat{b}_j^{(1,N)} \leqslant b_j + \beta_j^*$. Hence, relation (3.21) follows from (3.4), Lemma 3.2, and the assumption that $K \to \infty$ in conjunction with Proposition 3.1 of Giraitis *et al.* [6], which implies that the autocovariance function of any $\{X_k\} \in \mathcal{R}(\mathbf{b})$ is absolutely summable.

It remains to verify that under $H_1^{(\mathrm{loc})}$ the sequence $\{U_N(\cdot) - EU_N(\cdot)\}$ is tight. By Lemma 3.1, the functions $EU_N(\cdot)$ converge to a constant in $D[0,1]$, so it suffices to verify the tightness of the sequence $\{U_N(\cdot)\}$. Set

$$S_N(t) = N^{-1/2} \sum_{j=1}^{[Nt]} \big(X_j^{(N)} - EX_j^{(N)}\big) \tag{3.22}$$

and observe that $U_N(t) = S_N(t) - \frac{[Nt]}{N} S_N(1)$. Thus, the problem is reduced to the verification of the tightness of the sequence $\{S_N(\cdot)\}$. For this, it suffices to show that

$$E\big(S_N(t)\big)^4 \leqslant C\big([Nt]/N\big)^2. \tag{3.23}$$

Relation (3.23) implies that a tightness criterion for $S_N(t)$ is satisfied (see [2], Theorem 15.6). Indeed, for any $t_1 < t < t_2$,

$$E\big(S_N(t) - S_N(t_1)\big)^2 \big(S_N(t_2) - S_N(t)\big)^2 \leqslant \Big(E\big(S_N(t) - S_N(t_1)\big)^4 E\big(S_N(t_2) - S_N(t)\big)^4\Big)^{1/2}$$
$$\leqslant C\, \frac{[Nt] - [Nt_1]}{N} \frac{[Nt_2] - [Nt]}{N} \leqslant \frac{C}{2} \left(\frac{[Nt_2] - [Nt_1]}{N}\right)^2.$$

To verify (3.23), set $\tilde{B}^{(N)} := \sum_{j=1}^{\infty} \max\{b_j^{(1,N)}, b_j^{(2,N)}\}$ and $D_N := (E\varepsilon_0^8)^{1/4} \tilde{B}^{(N)}$. By (3.4), $D_N < \lambda_8^{1/4} B^* < 1$, and so (3.23) follows from the inequality

$$E\big(S_N(t)\big)^4 \leqslant C \left(\frac{[Nt]}{N}\right)^2 \left(\sum_{l=0}^{\infty} (l+2)^2 D_N^l\right)^4. \tag{3.24}$$

Inequality (3.24) was established in [5] with $B := \sum_{j=1}^{\infty} b_j$ instead of $\tilde{B}^{(N)}$; its proof carries over with b_j replaced by $\max\{b_j^{(1,N)}, b_j^{(2,N)}\}$.

4. ASYMPTOTIC BEHAVIOR OF THE VARIANCE ESTIMATOR $\hat{s}_{N,q}^2$ UNDER LOCAL ALTERNATIVES

As pointed out in Section 2, the tests should be based on the rescaled statistic $U_N(\cdot)/\hat{s}_{N,q}$. In order to prove the consistency of such tests against the local alternatives introduced in Section 3, we must show that under such alternatives the statistic $U_N(\cdot)/\hat{s}_{N,q}$ tends to a nondegenerated random process different from a Brownian bridge. This is the content of Corollary 4.1, which follows from Theorem 3.1 and the following lemma:

LEMMA 4.1. *Under hypothesis* $H_1^{(\mathrm{loc})}$, *if* $q \to \infty$ *and* $q/\sqrt{N} \to 0$, *then* $\hat{s}_{N,q}^2 \xrightarrow{P} \sigma^2$, *where* $\hat{s}_{N,q}^2$ *and* σ^2 *are defined in* (2.1) *and in Theorem* 2.1, *respectively.*

Proof. For the process $\{X_j\}$ defined by (1.5), with the $\{b_j\}$ appearing in Assumption A, we have (see (2.4))

$$\sum_{|j|\leqslant q} \omega_j(q)\hat{\gamma}_j \longrightarrow \sigma^2,$$

where $\hat{\gamma}_j$ is defined in (2.2). Therefore it is sufficient to verify that

$$\sum_{|j|\leqslant q} \omega_j(q)(\hat{\gamma}_j^{(N)} - \hat{\gamma}_j) \xrightarrow{P} 0. \tag{4.1}$$

Recall that the covariance estimator $\hat{\gamma}_j^{(N)}$ is based on the observations $X_1^{(N)}, \ldots, X_N^{(N)}$ given by (3.1). Let $\tilde{\gamma}_j^{(N)}$ and $\tilde{\gamma}_j$ be the estimators of the covariance function with $\bar{X}^{(N)}$, $\bar{X}$ replaced by corresponding theoretical expectations:

$$\tilde{\gamma}_j^{(N)} := \frac{1}{N} \sum_{i=1}^{N-|j|} (X_i^{(N)} - EX_i^{(N)})(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)})$$

and

$$\tilde{\gamma}_j := \frac{1}{N} \sum_{i=1}^{N-|j|} (X_i - EX_i)(X_{i+|j|} - EX_{i+|j|}), \quad |j| < N.$$

To establish (4.1) we show that

$$P_1^{(N)} := \sum_{|j|\leqslant q} \omega_j(q)(\hat{\gamma}_j^{(N)} - \tilde{\gamma}_j^{(N)}) \xrightarrow{P} 0 \tag{4.2}$$

and

$$P_2^{(N)} := \sum_{|j|\leqslant q} \omega_j(q)(\tilde{\gamma}_j^{(N)} - \tilde{\gamma}_j) \xrightarrow{P} 0. \tag{4.3}$$

We prove that $E|P_1^{(N)}| \leqslant Cq/N$; therefore (4.2) holds under the weaker assumption $q/N \to 0$. Rewrite

$$\begin{aligned} P_1^{(N)} &= \frac{1}{N} \sum_{|j|\leqslant q} \omega_j(q) \sum_{i=1}^{N-|j|} \Big[(\bar{X}^{(N)} - EX_i^{(N)})(\bar{X}^{(N)} - EX_{i+|j|}^{(N)}) \\ &\quad - (\bar{X}^{(N)} - EX_i^{(N)})(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)}) - (\bar{X}^{(N)} - EX_{i+|j|}^{(N)})(X_i^{(N)} - EX_i^{(N)})\Big] \\ &=: P_{1,1}^{(N)} - P_{1,2}^{(N)} - P_{1,3}^{(N)}. \end{aligned}$$

We first prove that

$$E|P_{1,1}^{(N)}| \leqslant Cq/N \longrightarrow 0. \tag{4.4}$$

For this, we estimate

$$\begin{aligned} E|P_{1,1}^{(N)}| &\leqslant \frac{1}{N} \sum_{|j|\leqslant q} \omega_j(q) \sum_{i=1}^{N-|j|} E|\bar{X}^{(N)} - EX_i^{(N)}||\bar{X}^{(N)} - EX_{i+|j|}^{(N)}| \\ &\leqslant \frac{1}{N} \sum_{|j|\leqslant q} \sum_{i=1}^{N-|j|} \Big(E(\bar{X}^{(N)} - EX_i^{(N)})^2\Big)^{1/2} \Big(E(\bar{X}^{(N)} - EX_{i+|j|}^{(N)})^2\Big)^{1/2}. \end{aligned} \tag{4.5}$$

Now we show that

$$E(\bar{X}^{(N)} - EX_i^{(N)})^2 \leqslant C/N. \tag{4.6}$$

Write

$$E(\bar{X}^{(N)} - EX_i^{(N)})^2 = N^{-2}\sum_{s=1}^{N}\sum_{s'=1}^{N} E(X_s^{(N)} - EX_i^{(N)})(X_{s'}^{(N)} - EX_i^{(N)})$$

$$= N^{-2}\sum_{s=1}^{N}\sum_{s'=1}^{N}\operatorname{Cov}(X_s^{(N)}, X_{s'}^{(N)}) + N^{-2}\left(\sum_{s=1}^{N}(EX_s^{(N)} - EX_i^{(N)})\right)^2.$$

Using Lemma 3.2, we have

$$\operatorname{Cov}(X_s^{(N)}, X_{s'}^{(N)}) \leqslant \operatorname{Cov}(\tilde{X}_s^{(N)}, \tilde{X}_{s'}^{(N)}), \tag{4.7}$$

where

$$\tilde{X}_s^{(N)} := \sum_{l=0}^{\infty}\sum_{j_1,\dots,j_l=1}^{\infty}\prod_{m=0}^{l}(\tilde{b}_{j_m}^{(N)}\varepsilon^2_{s-j_1-\dots-j_m}), \qquad \tilde{b}_j^{(N)} = \max\{b_j^{(1,N)}, b_j^{(2,N)}\}.$$

By Lemma 3.2, Assumption A, and the absolute summability of covariances under condition (1.2)

$$N^{-2}\sum_{s=1}^{N}\sum_{s'=1}^{N}\operatorname{Cov}(\tilde{X}_s^{(N)}, \tilde{X}_{s'}^{(N)}) \leqslant C/N. \tag{4.8}$$

Observe also that

$$\left|\sum_{s=1}^{N}(EX_s^{(N)} - EX_i^{(N)})\right| = \left|k^*\frac{\lambda_2 b_0^{(1,N)}}{1-\lambda_2 B^{(1,N)}} + (N-k^*)\frac{\lambda_2 b_0^{(2,N)}}{1-\lambda_2 B^{(2,N)}} - NEX_i^{(N)}\right|$$

$$\leqslant k^*\left|\frac{\lambda_2 b_0^{(1,N)}}{1-\lambda_2 B^{(1,N)}} - \frac{\lambda_2 b_0^{(2,N)}}{1-\lambda_2 B^{(2,N)}}\right| + N\left|\frac{\lambda_2 b_0^{(2,N)}}{1-\lambda_2 B^{(2,N)}} - EX_i^{(N)}\right| \tag{4.9}$$

$$\leqslant CN^{1/2}.$$

Thus, (4.8) and (4.9) imply (4.6), and therefore (4.4) follows.

Next we prove that $E|P_{1,2}^{(N)}| \leqslant Cq/N$. Write

$$E|P_{1,2}^{(N)}| = \frac{1}{N}E\left|\sum_{|j|\leqslant q}\omega_j(q)\sum_{i=1}^{N-|j|}(\bar{X}^{(N)} - EX_i^{(N)})(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)})\right|$$

$$\leqslant \frac{1}{N}E\left|\sum_{|j|\leqslant q}\omega_j(q)\sum_{i=1}^{N-|j|}(\bar{X}^{(N)} - EX_0^{(1,N)})(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)})\mathbf{1}_{\{i\leqslant k^*\}}\right|$$

$$+ \frac{1}{N}E\left|\sum_{|j|\leqslant q}\omega_j(q)\sum_{i=1}^{N-|j|}(\bar{X}^{(N)} - EX_0^{(2,N)})(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)})\mathbf{1}_{\{i> k^*\}}\right|$$

$$=: t_1^{(N)} + t_2^{(N)}.$$

We now show that $t_1^{(N)} \leqslant Cq/N$. The same estimate holds for $t_2^{(N)}$. Estimate $t_1^{(N)}$ as follows:

$$t_1^{(N)} \leqslant \frac{1}{N}\sum_{|j|\leqslant q}E|\bar{X}^{(N)} - EX_0^{(1,N)}|\left|\sum_{i=1}^{N-|j|}(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)})\mathbf{1}_{\{i\leqslant k^*\}}\right|$$

$$\leqslant \frac{1}{N}\sum_{|j|\leqslant q}\left(E(\bar{X}^{(N)} - EX_0^{(1,N)})^2\right)^{1/2}\left(\sum_{i,i'=1}^{N-|j|}\operatorname{Cov}(X_{i+|j|}^{(N)}, X_{i'+|j|}^{(N)})\right)^{1/2}.$$

By inequalities (4.7) and (4.8) we have

$$\sum_{i,i'=1}^{N-|j|} \mathrm{Cov}\big(X_{i+|j|}^{(N)}, X_{i'+|j|}^{(N)}\big) \leqslant CN.$$

Thus, by (4.6) we have

$$t_1^{(N)} \leqslant C\,\frac{1}{N}\sum_{|j|\leqslant q} N^{-1/2}N^{1/2} = C\,\frac{2q+1}{N} \longrightarrow 0.$$

An analogous argument shows that

$$E\big|P_{1,3}^{(N)}\big| \leqslant Cq/N \longrightarrow 0.$$

It remains to show that $P_2^{(N)} \xrightarrow{P} 0$. Write

$$\begin{aligned}&\big(X_i^{(N)} - EX_i^{(N)}\big)\big(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)}\big) - (X_i - EX_i)(X_{i+|j|} - EX_{i+|j|})\\ &\quad = Z_i^{(N)}\big(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)}\big) + Z_{i+|j|}^{(N)}(X_i - EX_i),\end{aligned}$$

where $Z_i^{(N)}$ is defined in (3.11), (3.10). Since by (3.18)

$$E\big(Z_i^{(N)}\big)^2 = E\big(Z_i^{(1,N)}\mathbf{1}_{\{i\leqslant k^*\}} + EZ_i^{(2,N)}\mathbf{1}_{\{i>k^*\}}\big)^2 \leqslant E\big(Z_i^{(1,N)}\big)^2 + E\big(Z_i^{(2,N)}\big)^2 \leqslant C/N,$$

we obtain

$$\begin{aligned}E\big|P_2^{(N)}\big| &= \frac{1}{N}\,E\left|\sum_{|j|\leqslant q}\omega_j(q)\sum_{i=1}^{N-|j|}\Big(Z_i^{(N)}\big(X_{i+|j|}^{(N)} - EX_{i+|j|}^{(N)}\big) + Z_{i+|j|}^{(N)}(X_i - EX_i)\Big)\right|\\ &\leqslant \frac{1}{N}\sum_{|j|\leqslant q}\sum_{i=1}^{N-|j|}\left[\Big(E\big(Z_i^{(N)}\big)^2\Big)^{1/2}\Big(\mathrm{Var}X_{i+|j|}^{(N)}\Big)^{1/2} + \Big(E\big(Z_{i+|j|}^{(N)}\big)^2\Big)^{1/2}(\mathrm{Var}X_i)^{1/2}\right]\\ &\leqslant \frac{1}{N}\sum_{|j|\leqslant q}\sum_{i=1}^{N} CN^{-1/2} = C\,\frac{2q+1}{\sqrt{N}} \longrightarrow 0.\end{aligned}$$

Thus, (4.1) follows. This completes the proof of the lemma.

COROLLARY 4.1. *Let $q \to \infty$ and $q/\sqrt{N} \to 0$. Under the hypothesis $H_1^{(\mathrm{loc})}$,*

$$\frac{U_N(t)}{\hat{s}_{N,q}} \xrightarrow{D[0,1]} W^0(t) + \sigma^{-1}G(t).$$

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