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LIMIT THEOREMS FOR THE MAXIMAL EIGENVALUES OF THE MEAN-FIELD HAMILTONIAN WITH RANDOM POTENTIAL

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Abstract. Let $\bar{H}_V = \varkappa\bar{\Delta}_V + \xi_V(x)$, $x \in V \subset \mathbb{Z}^\nu$, be the mean-field Hamiltonian with $\varkappa > 0$ and random i.i.d. potential ξ_V. We prove limit theorems for the extreme eigenvalues of $\bar{H}_V$ as $|V| \to \infty$. The limiting distributions are the same as for the corresponding extremes of ξ_V only if either (i) ξ_V is unbounded and $\varkappa > 0$, or (ii) ξ_V is bounded with "sharp" peaks and $\varkappa \ll 1$. Localization properties for the corresponding eigenfunctions are also studied.

Key words: mean-field model, random potential, basic states, central limit theorem, extremal-type theorem.

1. INTRODUCTION

The mean-field (Curie–Weiss) model is given by the random (symmetric) operators $\bar{H}_V$, $V \subset \mathbb{Z}^\nu$, acting on the functions $\psi\colon V \to \mathbb{R}$ according to the formula

$$\bar{H}_V\psi(x) = \varkappa\bar{\Delta}_V\psi + \xi(x)\psi(x), \quad x \in V, \tag{1.1}$$

where $\bar{\Delta}_V\psi = N^{-1}\sum_{x\in V}\psi(x)$, N is the number of sites in V, $\varkappa \geqslant 0$, and the potential $\xi(x)$, $x \in \mathbb{Z}^\nu$, consists of independent identically distributed (i.i.d.) random variables with (continuous) distribution function $F(\cdot)$. The Hamiltonian (1.1) represents a simplified modification of the Anderson model

$$H_V\psi(x) = \varkappa\Delta_V\psi(x) + \xi(x)\psi(x), \quad x \in V, \tag{1.2}$$

with the Laplacian Δ (cf. [11]), and has been introduced by Bogachov and Molchanov [5] to investigate long-time intermittency phenomena for evolution problems with a Gaussian random potential.

Let

$$\lambda_{1,N} > \lambda_{2,N} > \cdots > \lambda_{N,N} \tag{1.3}$$

be the (random) eigenvalues of the Hamiltonian (1.1), and let $\psi(\cdot\,; \lambda_{k,N})$ be the corresponding (random) eigenfunctions normed by the condition $\sum_{x\in V}\psi^2\,(x; \lambda_{k,N}) = 1$, $1 \leqslant k \leqslant N$. Clearly, if $\varkappa = 0$, then (1.3) is simply the variational series

$$\xi(z_{1,N}) := \xi_{1,N} > \xi(z_{2,N}) := \xi_{2,N} > \cdots > \xi(z_{N,N}) := \xi_{N,N},$$

and the Kronecker symbols $\delta(x;\, z_{1,N}), \delta(x; z_{2,N}), \ldots, \delta(x;\, z_{N,N})$, $x \in V$, are the corresponding eigenfunctions. The purpose of this paper is to study the asymptotic properties (as $N \to \infty$) of eigenpairs $\lambda_{K,N}$, $\psi(\cdot\,; \lambda_{K,N})$, for fixed $K = 1, 2, \ldots$, for $\varkappa > 0$ and arbitrary $F(\cdot)$.

The asymptotic behavior of the maximal eigenvalues of $\bar{H}_V$ was earlier discussed in [5] for a Gaussian i.i.d. $\xi(\cdot)$ and in [7], [8] for an exponentially distributed i.i.d. $\xi(\cdot)$. This asymptotical analysis was shown to play a crucial role in the investigation of the long-time behavior of the evolution associated with $\bar{H}_V$. In [3], we discussed limit theorems for the maximal eigenvalue $\lambda_{1,N}$ of the Hamiltonian $\bar{H}_V$ for an arbitrary i.i.d. $\xi(\cdot)$,

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provided there exists a density $p(t) = F'(t)$ satisfying some condition on a continuity at a right endpoint of $\xi(0)$. The results of the present paper are proven under continuity conditions weaker than those in [3].

Clearly, asymptotic properties of the upper bound of the spectrum of the Hamiltonians (1.1), (1.2) in an increasing volume V depend strongly on both the diffusion constant $\varkappa$ and on the "tail behavior" $1 - F(t)$ as t goes to the right endpoint of $\xi(0)$.

For the Anderson model (1.2), we have shown in [4] that if $-\ln(1 - F(t)) = o(t^2)$ as $t \to \infty$ (the case of $\xi_V = \{\xi(x)\}_{x\in V}$ with extremely rare high peaks), then the asymptotic behavior of the maximal eigenvalues for $\varkappa > 0$ is similar to that for $\varkappa = 0$. Namely,

(L) a single-point localization takes place: $\lim_N \psi^2(z_{K,N}; \lambda_{K,N}) = 1$ in probability, and

(LT) if for some (normalizing) constants $a_N > 0$ and $b_N \in \mathbb{R}$ the weak (non-degenerate) limit $F^N(b_N + ta_N^{-1}) \xrightarrow{w} G(t)$ (as $N \to \infty$) exists, then

$$P\big(a_N(\lambda_{K,N} - b_N) < t\big) \xrightarrow{w} G_K(t)$$

as $N \to \infty$, with $G_K(t) := \frac{1}{(K-1)!}\int_{-\ln G(t)}^{\infty} v^{K-1}e^{-v}\,dv$.

Note that if $\underline{\lim}_{t\to\infty}(-t^{-2}\ln(1 - F(t))) > 0$, then the strong influence of the parameters of model (1.2) on both limit theorem for $\lambda_{K,N}$ and localization theorem for $\psi(\cdot\,; \lambda_{K,N})$ has been established (see also [2], [9]). The latter phenomenon is caused by the neighboring effects due to the local (strong) properties of the Laplacian Δ.

In contrast to Δ, the mean-field diffusion seems to be a long-range (weak) operator. This property, as well as the absence of neighboring effects in model (1.1), leads to the fact that every unbounded $\xi(\cdot)$ may be treated here as a potential with strongly pronounced peaks. The following two classes of i.i.d. $\xi(\cdot)$ lead to qualitatively different asymptotic behaviors of the maximal eigenvalues of $\bar{H}_V$ for any $\varkappa > 0$:

(1) for unbounded $\xi(\cdot)$ satisfying (2.9), properties (L), (LT) hold (cf. Theorem 5.1 and Corollary 2.1 below),

(2) for bounded $\xi(\cdot)$ such that

$$P\big(\xi(0) > t\big) = (1 - t)^{\alpha}, \quad 0 \leqslant t \leqslant 1, \quad \text{for } \alpha \leqslant 1 \tag{1.4}$$

(the case of ξ_V with weakly pronounced peaks), there exists a constant $\lambda^0 = \lambda^0(\alpha, \varkappa) > 1$ such that $\sqrt{N}(\lambda_{1,N} - \lambda^0)$ converges in distribution to the Gaussian variable. In addition, the eigenfunction $\psi(\cdot\,; \lambda_{1,N})$ is approximately "uniformly" distributed on V, i.e., a complete delocalization takes place (cf. Theorems 2.2° and 5.2° below).

We shall briefly illustrate a connection between asymptotics of the maximal eigenvalues of $\bar{H}_V$ (cf. (1) and (2)) and asymptotics (as $\tau \to \infty$ and $V \to \mathbb{Z}^\nu$) of the solution $u(\tau, x) \geqslant 0$ of the equation

$$\frac{\partial u(\tau, x)}{\partial \tau} = \frac{\varkappa}{N}\sum_{y\in V}\big(u(\tau, y) - u(\tau, x)\big) + \xi(x)u(\tau, x), \quad u(0, x) \equiv 1, \ \tau \geqslant 0, \quad x \in V.$$

In fact, the equation describes an evolution of a particle system of the branching type in a medium ξ_V, and $u(\tau, x)$ stands for the mean number of particles at site x at time τ. The notion of intermittency refers to the appearance (as $\tau \to \infty$) of extremely high isolated "peaks" of $u(\tau, \cdot)$, where most of the mass $\sum_x u(\tau, x)$ is concentrated (cf. [5], [7]–[9]). The solution $u(\cdot, \cdot)$ admits the spectral representation

$$u(\tau, x) = \sum_{k=1}^{N}\exp\{\tau\lambda_{k,N} - 2\tau\nu\varkappa\}\psi(x; \lambda_{k,N})\big(\psi(\cdot\,; \lambda_{k,N}), 1\big);$$

here $(\cdot, \cdot)$ stands for a scalar product in $L^2(V)$. Write

$$\bar{u}(\tau, x) = u(\tau, x)\Big/\sum_{y\in V}u(\tau, y)$$

for the "mass concentration function." Let $\tau \to \infty$ and $V \to \mathbb{Z}^\nu$ simultaneously, and $N = O(\tau^\beta)$ for some $\beta > 0$. Straightforward calculations based on Theorems 2.1 and 2.2° below show the validity of the following statements for any $\varkappa > 0$:

(i) for unbounded $\xi(\cdot)$ satisfying (2.9), a complete localization for $\bar{u}(\tau,\cdot)$ in the record point of ξ_V is observed, viz. $\bar{u}(\tau, z_{1,N}) \to 1$ in probability (intermittency effect);

(ii) for $\xi(\cdot)$ satisfying (1.4), $\bar{u}(\tau,\cdot)$ is approximately "uniformly" distributed on V, i.e., a complete delocalization takes place.

The exact asymptotics of evolution associated with $\bar{H}_V$ for an arbitrary i.i.d. $\xi(\cdot)$ is studied in our forthcoming paper.

Our paper is organized as follows. In Sec. 2, we formulate the limit theorems for $\lambda_{K,N}$. Sections 3 and 4 are devoted to the proof of the results of Sec. 2. In fact, in Sec. 3 Theorems 2.1–2.3 are restated under conditions expressed in terms of $\xi_{K,N}$ and some functionals on $\xi_{K,N}$. The asymptotic behavior of such functionals is considered in Sec. 4. Finally, Sec. 5 is devoted to the localization theorems for the eigenfunction $\psi(\cdot\,;\lambda_{K,N})$.

2. LIMIT THEOREMS FOR $\lambda_{K,N}$

Let $\xi(x)$, $x \in \mathbb{Z}^\nu$, be a sequence of i.i.d. random variables on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a common (continuous) distribution function $F(\cdot)$. Given a realization $\xi_V = \{\xi(x)\}_{x\in V}$, we first consider the spectral problem

$$\bar{H}_V\psi(x) = \lambda\psi(x), \quad x \in V. \tag{2.1}$$

From (2.1) we conclude that

$$\psi(x) = \frac{\varkappa\bar{\Delta}_V\psi}{\lambda - \xi(x)}. \tag{2.2}$$

Averaging (2.2) with respect to $x \in V$ (note that $\bar{\Delta}_V\psi \neq 0$), we arrive at the dispersion equation

$$\frac{1}{\varkappa} = N^{-1}\sum_{x\in V}\big(\lambda - \xi(x)\big)^{-1}, \tag{2.3}$$

where $N := |V|$. Again by (2.2) the eigenfunction $\psi(x;\lambda)$, $x \in V$, corresponding to the eigenvalue λ and normed by the condition $\sum_{x\in V}\psi^2(x;\lambda) = 1$, can be written as

$$\psi(x;\lambda) = \big(\lambda - \xi(x)\big)^{-1}\Big(\sum_{y\in V}\big(\lambda - \xi(y)\big)^{-2}\Big)^{-1/2}. \tag{2.4}$$

Let

$$\xi_{1,N} > \xi_{2,N} > \cdots > \xi_{N,N} \tag{2.5}$$

be the variational series based on a sample $\xi(x)$, $x \in V$. The inequalities in (2.5) are strict with probability 1 because of the continuity of $F(\cdot)$. Thus, with probability 1 Eq. (2.3) has exactly N roots $\lambda_{1,N} > \lambda_{2,N} > \cdots > \lambda_{N,N}$ such that

$$\lambda_{1,N} > \xi_{1,N}, \qquad \xi_{K,N} < \lambda_{K,N} < \xi_{K-1,N} \quad (2 \leqslant K \leqslant N). \tag{2.6}$$

To formulate our results (here and in the sequel) we need some additional notation. Given $F(t)$, we write $\bar{F}(t) = 1 - F(t)$. Let ω_F stand for the right endpoint of $\xi := \xi(0)$:

$$\omega_F = \sup\{t\colon\ \bar{F}(t) > 0\}.$$

For $\omega_F \in (-\infty, \infty]$ and any $l \in (0,\infty)$, we write

$$e_{F,l} = \begin{cases} \langle(\omega_F - \xi)^{-l}\rangle & \text{for } \omega_F < \infty,\\ 0 & \text{for } \omega_F = \infty, \end{cases} \tag{2.7}$$

and $e_F := e_{F,1}$, where $\langle\cdot\rangle$ denotes the expectation with respect to P. Further, introduce the following functional:

$$I_{F,l}(t) = \begin{cases} \int \mathbb{I}\{\bar{F}^{1/l}(t) < v < 1\}v^{-l-1}\big(\bar{F}(t-v) - \bar{F}(t)\big)\,dv, & \text{if } \omega_F = \infty,\\ \int \mathbb{I}\{\bar{F}^{1/l}(t) < v < \omega_F - t\}v^{-l-1}\big(\bar{F}(t-v) - \bar{F}(t)\big)\,dv, & \text{if } \omega_F < \infty, \end{cases} \tag{2.8}$$

where $\int := \int_{\mathbb{R}}$, and $\mathbb{I}\{A\}$ denotes the indicator of a set A.

Finally, for a sequence of random variables $X_{N,M} \geqslant 0$, we write $X_{N,M} \asymp 1$ (as first $N \to \infty$ and afterwards $M \to \infty$) in probability if and only if

$$0 < P\text{-}\varliminf_{M} \varliminf_{N} X_{N,M} \leqslant P\text{-}\varlimsup_{M} \varlimsup_{N} X_{N,M} < \infty$$

or, equivalently, if and only if

$$\lim_{\varepsilon \downarrow 0} \varlimsup_{M} \varlimsup_{N} P(X_{N,M} < \varepsilon) = \lim_{\varepsilon \downarrow 0} \varlimsup_{M} \varlimsup_{N} P\Big(X_{N,M} > \frac{1}{\varepsilon}\Big) = 0.$$

These values of limits mean that the sequence of distributions $\{P(X_{N,M} \in \mathrm{d}t),\ N = 1, 2, \dots,\ M = 1, 2, \dots\}$ is weakly compact and any of its weak limits (as first $N \to \infty$ and afterwards $M \to \infty$) has no atom at zero.

In what follows, we consider the pair $(e_F, \varkappa)$ as a vector parameter of the model and distinguish between the following three zones of $(e_F, \varkappa)$:

(A) $1/\varkappa > e_F$ (the case of weak diffusion or strongly pronounced asymptotic structure of the "peaks" of ξ_V),

(B) $1/\varkappa < e_F$ (the case of strong diffusion or weakly pronounced asymptotic structure of the "peaks" of ξ_V), and

(C) $1/\varkappa = e_F$ ("critical" points).

Note that, by definition, if ξ is unbounded (viz., $\omega_F = \infty$), then, for any $\varkappa > 0$, the parameter $(e_F, \varkappa)$ belongs to (A), whereas, if ξ is bounded and $\langle(\omega_F - \xi)^{-1}\rangle = \infty$, then, for any $\varkappa > 0$, the parameter $(e_F, \varkappa)$ belongs to (B).

In case (A) we have

THEOREM 2.1. *Let $K \geqslant 1$ and $1/\varkappa > e_F$. If, moreover,*

$$I_{F,1}(t) \longrightarrow 0 \quad \text{as } t \uparrow \omega_F, \tag{2.9}$$

then

$$P\text{-}\lim_{N} N(\lambda_{K,N} - \xi_{K,N}) = 1\Big/\Big(\frac{1}{\varkappa} - e_F\Big)$$

and

$$P\text{-}\lim_{N} \frac{\lambda_{K,N} - \xi_{K,N}}{\xi_{K-1,N} - \lambda_{K,N}} = 0; \tag{2.10}$$

here $\xi_{0,N} := \omega_F$.

Theorem 2.1 and Lemma 4.1 below imply the following extremal-type limit theorem for eigenvalues (cf. also Remark 2.1).

COROLLARY 2.1. *Let the conditions of Theorem 2.1 be fulfilled. Assume, in addition, that there exist constants $a_N > 0$, $b_N \in \mathbb{R}$, and a nondegenerate distribution function $D(\cdot)$ such that*

$$F^N\Big(b_N + \frac{t}{a_N}\Big) \xrightarrow{w} D(t) \quad \text{as } N \to \infty.$$

Then for any (fixed) $K = 1, 2, \dots,$

$$P\big(a_N(\lambda_{K,N} - b_N) < t\big) \xrightarrow{w} D_K(t) \quad \text{as } N \to \infty,$$

where

$$D_K(t) = \frac{1}{(K-1)!} \int_{-\ln D(t)}^{\infty} s^{K-1} \mathrm{e}^{-s}\, \mathrm{d}s.$$

The class of possible (extreme value) limit distributions $D(\cdot)$ is discussed, for example, in the monograph [10] of Leadbetter *et al.* (Chap. 1).

Cases (B) and (C) are more delicate. For (B) and $K = 1$, the following central limit theorem holds:

THEOREM 2.2°. *Let $K = 1$ and $1/\varkappa < e_F$, and suppose that $\lambda^0 > \omega_F$ is the solution of the equation*

$$\frac{1}{\varkappa} = \langle(\lambda^0 - \xi)^{-1}\rangle.$$

Then

$$\lim_N P\left(\sqrt{N}(\lambda_{1,N} - \lambda^0) < t\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left\{-\frac{v^2}{2\sigma^2}\right\} dv \tag{2.11}$$

for all $t \in \mathbb{R}$, where $\sigma^2 = 1 - \langle(\lambda^0 - \xi)^{-1}\rangle^2 \langle(\lambda^0 - \xi)^{-2}\rangle^{-1}$.

THEOREM 2.2. *Let $K \geqslant 2$ and $1/\varkappa < e_F$.*

(i) *If $e_F < \infty$ and $I_{F,1}(t) \to 0$ as $t \uparrow \omega_F$, then*

$$P\text{-}\lim_N N(\xi_{K-1,N} - \lambda_{K,N}) = 1\Big/\left(e_F - \frac{1}{\varkappa}\right)$$

and

$$P\text{-}\lim_N \frac{\xi_{K-1,N} - \lambda_{K,N}}{\lambda_{K,N} - \xi_{K,N}} = 0. \tag{2.12}$$

(ii) *If*

$$\bar{F}(t) = (\omega_F - t)^\alpha \left(1 + o(1)\right) \quad \text{as } t \uparrow \omega_F, \tag{2.13}$$

and $\alpha \in (0, 1)$, then

$$N^{1/\alpha}(\xi_{K-1,N} - \lambda_{K,N}) \asymp N^{1/\alpha}(\lambda_{K,N} - \xi_{K,N}) \asymp 1 \tag{2.14}$$

in probability as $N \to \infty$.

(iii) *If $\lim_{t\downarrow 0} \bar{F}(\omega_F - vt)/\bar{F}(\omega_F - t) = 1$ for all $v > 0$ (i.e., $\bar{F}(\omega_F - t)$, $t > 0$, is slowly varying at zero), then*

$$P\text{-}\lim_N \frac{\xi_{K-1,N} - \lambda_{K,N}}{\lambda_{K,N} - \xi_{K,N}} = K - 1. \tag{2.15}$$

Case (C) below differs slightly from (B):

THEOREM 2.3°. *Let $K = 1$ and $1/\varkappa = e_F$.*

(i) *If $e_{F,2} < \infty$, then the central limit theorem* (2.11) *holds with $\lambda^0 = \omega_F$.*

(ii) *If* (2.13) *is fulfilled with $\alpha \in (1, 2)$, then*

$$N^{1/\alpha}(\lambda_{1,N} - \xi_{1,N}) \asymp N^{1/\alpha}|\omega_F - \lambda_{1,N}| \asymp 1$$

in probability as $N \to \infty$.

THEOREM 2.3. *Let $K \geqslant 2$ and $1/\varkappa = e_F$.*

(i) *If $e_{F,2} < \infty$ and $I_{F,2}(t) \to 0$ as $t \uparrow \omega_F$, then both the limits*

$$P\text{-}\lim_N N(\omega_F - \xi_{K-1,N})(\xi_{K-1,N} - \lambda_{K,N}) = 1/e_{F,2}$$

and (2.12) *hold.*

(ii) *If* (2.13) *is fulfilled with $\alpha \in (1, 2)$, then* (2.14) *holds.*

Remark 2.1. For any $l \in (0, \infty)$, the condition $I_{F,l}(t) \to 0$ as $t \uparrow \omega_F$ implies

$$\frac{\widetilde{F}\big(t - \widetilde{F}^{1/l}(t)\big)}{\widetilde{F}(t)} \longrightarrow 1 \quad \text{as } t \uparrow \omega_F, \tag{2.16}$$

provided $e_{F,l} < \infty$. For $l = 0$, condition (2.16) (i.e., $\widetilde{F}(t-0)/\widetilde{F}(t) \to 1$) is well known in the extreme value theory for i.i.d. random variables as sufficient and necessary for a Poisson character of the occurrence of large values of ξ_V in limit as $N \to \infty$ (cf. [10], Sec. 1.7).

On the other hand, if $e_{F,1}$ is finite, assumption (2.9) of Theorem 2.1 is sufficient for the existence of the limit

$$\frac{1}{N} \sum_{m=K+1}^{N} \frac{1}{\xi_{K,N} - \xi_{m,N}} \longrightarrow e_F$$

in probability, which in turn implies the assertions of Theorem 2.1 (the necessity of (2.9) follows by a slight extension of the proof of Lemma 4.2 (ii)). The class of distribution functions satisfying (2.9) includes, for example, $F(\cdot)$ from (2.13) with $\alpha > 1$ (but not with $\alpha \leqslant 1$).

Proof of Theorems 2.1–2.3. Theorems 2.2° and 2.3° are proved in [3].

Theorem 2.1 for $K = 1$ follows from Lemma 4.2 (i), (ii) (with $l = 1$) by the same argument as in the proof of Theorem 1 in [3]. Theorem 2.1 for $K \geqslant 2$ follows from a combination of Lemma 4.2 (i), (ii) (with $l = 1$) and Lemma 3.1.

In Theorem 2.2, part (i) follows from a combination of Lemma 4.2 (i), (ii) (with $l = 1$) and Lemma 3.2 (i); part (ii) follows from Lemmas 4.4 (i) (with $l = 1$) and 3.3 (i), and part (iii) follows from Lemmas 4.4 (ii) and 3.3 (ii).

Finally, in Theorem 2.3, part (i) follows from Lemmas 4.2 (with $l = 2$) and 3.2 (ii); part (ii) follows from a combination of Lemma 4.4 (i) (with $l = 2$), its Corollary 4.1, and Lemma 3.3 (i).

3. ASYMPTOTIC RELATION BETWEEN $\lambda_{K,N}$ AND $\xi_{K,N}$ FOR $K \geqslant 2$

We assume throughout this section that $\xi_V(x)$, $x \in V$, are random (not necessarily i.i.d.) series; $N := |V| \geqslant 1$. Let $\xi_{1,N} > \xi_{2,N} > \cdots > \xi_{N,N}$ denote the variational series based on a sample ξ_V. (Assume that the inequalities are strict with probability 1.) Let $\lambda_{1,N} > \lambda_{2,N} > \cdots > \lambda_{N,N}$ be the solutions to the equation

$$\varkappa^{-1} = N^{-1} \sum_{k=1}^{N} (\lambda - \xi_{k,N})^{-1}. \tag{3.1}$$

Recall that, with probability 1, $\lambda_{1,N} \in (\xi_{1,N}, \infty)$ and $\lambda_{K,N} \in (\xi_{K,N}, \xi_{K-1,N})$ for all $2 \leqslant K \leqslant N$.

We write

$$r_{K,L,N} = \xi_{K,N} - \xi_{L,N}, \qquad r_{K,N} = r_{K,K+1,N}, \tag{3.2}$$

$$R^{(l)}_{K,L,N} = \sum_{k=L}^{N} r^{-l}_{K,k,N}, \qquad R^{(l)}_{K,N} = R^{(l)}_{K,K+1,N} \tag{3.3}$$

and $R_{K,N} = R^{(1)}_{K,N}$; here $L \geqslant 1$, $K \geqslant 1$, and $l \in [1, \infty)$.

The purpose of this section is to show in what manner the asymptotic properties of $r_{L,N}$, $R^{(l)}_{L,N}$ influence the asymptotic behavior of $\lambda_{K,N}$ in probability as $N \to \infty$, for each (fixed) $K \geqslant 2$. (Note that $r_{L,N}$ and $R_{L,N}$ depend only on the variational series $\xi_{1,N} > \xi_{2,N} > \cdots > \xi_{N,N}$.) Below, we formulate three lemmas in the situations where $r_{K-1,N}(\frac{N}{\varkappa} - R_{K,N}) \to \infty$, $\to -\infty$ and $= \mathrm{O}(1)$ in probability, respectively.

LEMMA 3.1. *Suppose that*

$$P\text{-}\lim_N r_{K-1,N}\Big(\frac{N}{\varkappa} - R_{K,N}\Big) = \infty. \tag{3.4}$$

Then, for each $N \geqslant 1$,

$$\Big(\frac{N}{\varkappa} - R_{K-1,N}\Big)^{-1}(1 + \rho_N) \leqslant \lambda_{K,N} - \xi_{K,N} \leqslant \Big(\frac{N}{\varkappa} - R_{K,N}\Big)^{-1},$$

where

$$P\text{-}\lim_N \rho_N = 0,$$

and, consequently, (2.10) *holds.*

Proof. We have from Eq. (3.1) that

$$(\lambda_{K,N} - \xi_{K,N})^{-1} \equiv \frac{N}{\varkappa} - \sum_{k \neq K}(\lambda_{K,N} - \xi_{k,N})^{-1} \geqslant \frac{N}{\varkappa} - R_{K,N}. \tag{3.5}$$

Note that $P\text{-}\lim_N(\xi_{K-1,N} - \lambda_{K,N})/r_{K-1,N} = 1$ by (3.5) and (3.4). Thus, again by (3.1), the left-hand side of (3.5) does not exceed

$$\frac{N}{\varkappa} + \frac{K-1}{\xi_{K-1,N} - \lambda_{K,N}} + \frac{1}{r_{K-1,N}} - R_{K-1,N} = \frac{N}{\varkappa} - R_{K-1,N} + \frac{K + o(1)}{r_{K-1,N}} = \Big(\frac{N}{\varkappa} - R_{K-1,N}\Big)(1 + o(1))$$

in probability by (3.4). Lemma 3.1 is proved. □

LEMMA 3.2. *Suppose that*

$$P\text{-}\lim_N r_{K-2,N}\Big(R_{K-1,N} - \frac{N}{\varkappa}\Big) = \infty \quad \textit{for } K \geqslant 3. \tag{3.6}$$

(i) *If, moreover,*

$$P\text{-}\lim_N r_{K-1,N}\Big(R_{K-1,N} - \frac{N}{\varkappa}\Big) = \infty, \tag{3.7}$$

then for each $N \geqslant 1$,

$$\Big(R_{K,N} - \frac{N}{\varkappa}\Big)^{-1}(1 + \rho_{1,N}) \leqslant \xi_{K-1,N} - \lambda_{K,N} \leqslant \Big(R_{K-1,N} - \frac{N}{\varkappa}\Big)^{-1}(1 + \rho_{2,N}), \tag{3.8}$$

where

$$P\text{-}\lim_N \rho_{i,N} = 0 \quad (i = 1, 2),$$

and, consequently, (2.12) *holds.*

(ii) *If, in addition to* (3.6),

$$P\text{-}\lim_N (R^{(2)}_{K-1,N})^{-1/2}\Big(R_{K-1,N} - \frac{N}{\varkappa}\Big) = \infty, \tag{3.9}$$

then both (3.8) *and* (2.12) *hold with the difference that in (3.8)* $R_{K,N}$ *is replaced by* $R_{K-1,N}$.

Proof. (i) By (3.1),

$$\frac{1}{\xi_{K-1,N}-\lambda_{K,N}} \geqslant -\frac{K-2}{r_{K-2,N}} + R_{K-1,N} - \frac{N}{\varkappa} = \Big(R_{K-1,N} - \frac{N}{\varkappa}\Big)(1+o(1)) \tag{3.10}$$

in probability, because of (3.6) for $K \geqslant 3$. On the other hand, by (3.1), the left-hand side of (3.10) does not exceed

$$\frac{1}{\lambda_{K,N}-\xi_{K,N}} + R_{K,N} - \frac{N}{\varkappa} = \Big(R_{K,N} - \frac{N}{\varkappa}\Big)(1+o(1)),$$

since $\lambda_{K,N} - \xi_{K,N} = r_{K-1,N}(1+o(1))$ in probability by (3.10) and (3.7). Consequently, (3.8) and (2.12) follow.

(ii) Note that assumptions (3.6) and (3.9) imply (3.10). Then we have from (3.1) that

$$\begin{aligned}\frac{1}{\xi_{K-1,N}-\lambda_{K,N}} &\leqslant \sum_{k=K}^{N}(\lambda_{K,N}-\xi_{k,N})^{-1} - \frac{N}{\varkappa} = R_{K-1,N} - \frac{N}{\varkappa} \\ &\quad + (\xi_{K-1,N}-\lambda_{K,N}) \sum_{k=K}^{N} \Big(r_{K-1,k,N}\big(r_{K-1,k,N} - (\xi_{K-1,N}-\lambda_{K,N})\big)\Big)^{-1} \\ &= \Big(R_{K-1,N} - \frac{N}{\varkappa}\Big)(1+o(1))\end{aligned}$$

in probability by virtue of (3.10) and (3.9). Lemma 3.2 is proved. □

The case

$$P\text{-}\overline{\lim_N}\, r_{K-1,N}\Big|R_{K,N} - \frac{N}{\varkappa}\Big| < \infty \tag{3.11}$$

is more delicate.

LEMMA 3.3. (i) *If* (3.11) *holds and if* $P\text{-}\overline{\lim}_N\, r_{K-1,N}|R_{K-1,N} - \frac{N}{\varkappa}| < \infty$, *then*

$$\xi_{K-1,N} - \lambda_{K,N} \asymp \lambda_{K,N} - \xi_{K,N} \asymp r_{K-1,N} \tag{3.12}$$

in probability as $N \to \infty$.

(ii) *If the conditions*

$$P\text{-}\lim_N N^{-1} R_{K,N} = \infty \tag{3.13}$$

and

$$P\text{-}\lim_N r_{k-1,N} R_{k,N} = 0 \quad \text{for any } 2 \leqslant k \leqslant K \tag{3.14}$$

are fulfilled, then (2.15) *holds.*

Proof. (i) Fix $K \geqslant 2$, and introduce the events

$$\Omega^{+}_{\delta,N} = \{\lambda_{K,N} - \xi_{K,N} < \delta r_{K-1,N}\}, \tag{3.15}$$

$$\Omega^{-}_{\varepsilon,N} = \{\xi_{K-1,N} - \lambda_{K,N} < \varepsilon r_{K-1,N}\}, \tag{3.16}$$

where $\varepsilon \in (0,1)$ and $\delta \in (0,1)$. We get from (3.1) that, under the event (3.15),

$$\frac{1}{\delta r_{K-1,N}} \leqslant \frac{1}{\lambda_{K,N}-\xi_{K,N}} \equiv \sum_{k=1}^{K-1}(\xi_{k,N}-\lambda_{K,N})^{-1} - \sum_{k=K+1}^{N}(\lambda_{K,N}-\xi_{k,N})^{-1} + \frac{N}{\varkappa}$$
$$\leqslant \frac{K}{(1-\delta)r_{K-1,N}} - R_{K-1,N} + \frac{N}{\varkappa}. \tag{3.17}$$

Hence,

$$\overline{\lim_N} P(\Omega^{+}_{\delta,N}) \leqslant \overline{\lim_N} P\Big(r_{K-1,N}\Big|R_{K-1,N}-\frac{N}{\varkappa}\Big| > \text{const}\,\delta^{-1}\Big) \longrightarrow 0 \quad \text{as } \delta \downarrow 0 \tag{3.18}$$

by the assumptions of (i). Similarly, again by (3.1), the event (3.16) implies

$$\frac{1}{\varepsilon r_{K-1,N}} \leqslant \frac{1}{\xi_{K-1,N}-\lambda_{K,N}} \equiv -\sum_{k=1}^{K-2}(\xi_{k,N}-\lambda_{K,N})^{-1} + \sum_{k=K}^{N}(\lambda_{K,N}-\xi_{k,N})^{-1} - \frac{N}{\varkappa}$$
$$\leqslant \frac{1}{(1-\varepsilon)r_{K-1,N}} + R_{K,N} - \frac{N}{\varkappa},$$

so that

$$\overline{\lim_N} P(\Omega^{-}_{\varepsilon,N}) \leqslant \overline{\lim_N} P\Big(r_{K-1,N}\Big|R_{K,N}-\frac{N}{\varkappa}\Big| > \frac{\text{const}}{\varepsilon}\Big) \longrightarrow 0 \quad \text{as } \varepsilon \downarrow 0. \tag{3.19}$$

Now (3.12) follows from (3.18) and (3.19).

(ii) Clearly, the conditions of (ii) imply (3.12). Consequently, $\lambda_{K,N}-\xi_{K,N} = \mathrm{o}(r_{K,N})$ in probability by virtue of (3.14) with $k=K$. Combining this with (3.13), (3.14), we have

$$\Big|\frac{1}{\lambda_{K,N}-\xi_{K,N}} - \sum_{k=1}^{K-1}(\xi_{k,N}-\lambda_{K,N})^{-1}\Big|$$
$$\leqslant \sum_{k=K+1}^{N}(\lambda_{K,N}-\xi_{k,N})^{-1} + \frac{N}{\varkappa} = \mathrm{o}\Big(\frac{1}{r_{K-1,N}}\Big) = \mathrm{o}\Big(\frac{1}{\xi_{K-1,N}-\lambda_{K,N}}\Big) \tag{3.20}$$

in probability as $N \to \infty$. In particular, this implies (2.15) with $K=2$. On the other hand, (3.14) with $2 \leqslant k \leqslant K-1$ imply $P\text{-}\lim_N r_{l,N}/r_{K-1,N} = 0$ for any $1 \leqslant l \leqslant K-2$ and any $K \geqslant 3$. Thus, again by (3.12), we have, for any $1 \leqslant k \leqslant K-1$,

$$\xi_{k,N}-\lambda_{K,N} = \sum_{l=k}^{K-2} r_{l,N} + \xi_{K-1,N}-\lambda_{K,N} = (\xi_{K-1,N}-\lambda_{K,N})(1+\mathrm{o}(1)) \tag{3.21}$$

in probability. Substituting (3.21) into (3.20), we arrive at (2.15) for $K \geqslant 3$. Lemma 3.3 is proved. □

4. ON THE ASYMPTOTIC BEHAVIOR OF $\xi_{K,N}$

We consider here the asymptotic behavior (as $N \to \infty$) of the larger values $\xi_{k,N}$ (2.5) of an i.i.d. sample $\xi(x)$, $x \in V$, and the functions $r_{K,N}$, $R^{(l)}_{K,M,N}$, and $R^{(l)}_{K,N}$ introduced in Sec. 3 (Lemmas 4.2 and 4.4). Throughout, we use the notation introduced at the beginning of Sec. 2 and assume that $F(\cdot)$ is continuous and $\omega_F \in (-\infty, \infty]$.

LEMMA 4.1. *Let $F(\cdot)$ satisfy condition (2.16) with $l = 1$, and assume that there exist constants $a_N > 0$ and $b_N \in \mathbb{R}$ and a nondegenerate distribution function $D(\cdot)$ such that $F^N(b_N + ta_N^{-1}) \overset{\omega}{\to} D(t)$ as $N \to \infty$. Then*

$$\lim_N N^{-1} a_N = 0.$$

Proof. The assertion is an immediate consequence of the limit $a_N(\xi_{1,N} - \xi_{2,N}) \asymp 1$ in probability as $N \to \infty$ (which in turn follows from Theorem 2.3.2 in [10]) and assertion (i) of Lemma 4.2 (with $l = 1$) below. □

With $e_{F,l}$ and $I_{F,l}$ as in (2.7) and (2.8), and $K = 1, 2, \ldots$ fixed, we have:

LEMMA 4.2. (i) *Fix $l \in [1, \infty)$, and let $F(\cdot)$ satisfy* (2.16). *Then*

$$P\text{-}\lim_N N^{1/l} r_{K,N} = \infty. \tag{4.1}$$

(ii) *Fix $l \in [1, \infty)$, and let $F(\cdot)$ satisfy the following conditions*:

$$e_{F,l} < \infty, \quad \lim_{t \uparrow \omega_F} I_{F,l}(t) = 0. \tag{4.2}$$

Then

$$P\text{-}\lim_N N^{-1} R_{K,N}^{(l)} = e_{F,l}. \tag{4.3}$$

(iii) *If $\omega_F < \infty$ and if* (4.2) *is fulfilled with $l = 2$, then*

$$P\text{-}\lim_N (\omega_F - \xi_{K,N})^{-1}(N^{-1} R_{K,N}^{(1)} - e_{F,1}) = e_{F,2}.$$

Remark 4.1. Let us show that (2.16) implies

$$\lim_{t \uparrow \omega_F} \frac{\bar{F}(t - c\bar{F}^{1/l}(t))}{\bar{F}(t)} = 1 \tag{4.4}$$

for any $c > 0$. Indeed, fix $n = 1, 2, \ldots$ and assume that (4.4) is satisfied for any $c = 1, 2, \ldots, n$. Then, having written $t_n = t - n\bar{F}^{1/l}(t)$, we obtain

$$1 \leqslant \frac{\bar{F}(t - (n+1)\bar{F}^{1/l}(t))}{\bar{F}(t)} \leqslant \frac{\bar{F}(t_n - \bar{F}^{1/l}(t))}{\bar{F}(t_n)} \cdot \frac{\bar{F}(t - n\bar{F}^{1/l}(t))}{\bar{F}(t)} \longrightarrow 1$$

as $t \uparrow \omega_F$. Thus, (4.4) follows by induction.

Proof of Lemma 4.2. For any $N \geqslant 1$ and any $M \geqslant 1$, we write

$$t_{M,N} = \inf\{s\colon\ \bar{F}(s) = M/N\}, \qquad T_{M,N} = \sup\{s\colon\ \bar{F}(s) = 1/(MN)\}. \tag{4.5}$$

Write $\Omega_{K,M,N} = \{t_{M,N} \leqslant \xi_{K,N} \leqslant T_{M,N}\} \subset \Omega$ and $F_{M,N}(\mathrm{d}t) = P(\{\xi_{1,N} \in \mathrm{d}t\} \cap \Omega_{1,M,N})$. We need the following technical results.

LEMMA 4.3. *For some $C > 0$ and any $N \geqslant 1$, let $g\colon \mathbb{R}^{N-K+1} \to [0, C]$ be functions such that $g(s_K, s_{K+1}, \ldots, s_N) \equiv 0$ if $s_i < s_j$ for $i < j$, and assume, in addition, that the set of discontinuity points of g, written as $\Lambda \subset \mathbb{R}^{N-K+1}$, satisfies the condition $P((\xi_{K,N}, \xi_{K+1,N}, \ldots, \xi_{N,N}) \in \Lambda) = 0$. Then we have, for any $M \geqslant 100K^2$ and any $N \geqslant 2MK$,*

$$\langle g(\xi_{K,N}, \xi_{K+1,N}, \ldots, \xi_{N,N}) \mathbb{1}\{\Omega \backslash \Omega_{K,M,N}\} \rangle \leqslant 3C/M =: \delta_M \tag{4.6}$$

and

$$C_{K,M}\left(\left\langle\int g(t,\xi_{K+1,N},\ldots,\xi_{N,N})F_{M,N}(\mathrm{d}t)\right\rangle-\delta_M\right)\leqslant\left\langle g(\xi_{K,N},\xi_{K+1,N},\ldots,\xi_{N,N})\mathbb{I}\{\Omega_{K,M,N}\}\right\rangle \leqslant C'_{K,M}\left\langle\int g(t,\xi_{K+1,N},\ldots,\xi_{N,N})F_{M,N}(\mathrm{d}t)\right\rangle, \tag{4.7}$$

where the (positive) constants $C_{K,M}$ and $C'_{K,M}$ do not depend on N.

Proof. By the definition of $t_{M,N}$ and $T_{M,N}$,

$$P(\xi_{K,N}<t_{M,N})=\sum_{k=0}^{K-1}\binom{k}{N}F^{N-k}(t_{M,N})\big(1-F(t_{M,N})\big)^k\leqslant M^K\exp\left\{1-\frac{M}{2}\right\},\quad\text{provided } N\geqslant 2MK,$$

and

$$P(\xi_{K,N}>T_{M,N})\leqslant 1-F^N(T_{M,N})\leqslant 2/M,\quad\text{provided } M\geqslant 3.$$

Thus, (4.6) is proved.

Clearly, it is sufficient to prove (4.7) for $g(s_K,s_{K+1},\ldots,s_N)=g_1(s_K)g_2(s_{K+1},\ldots,s_N)$ for $s_K>s_{K+1}$. (For this we learn from the assumptions of Lemma 4.3 that the expectations in (4.7) can be regarded as Riemann's integrals with respect to the corresponding measures on $\mathbb{R}^{N-K+1}$ and, consequently, g in (4.7) can be approximated by linear combinitions of indicator functions of the $(N-K+1)$-dimensional rectangles with sides parallel to the $\mathbb{R}^{N-K+1}$ axes.) Noting that $\xi_{N-k,N}$, $k=0,1,\ldots,N-1$, is a Markov chain with transition probability $P(\xi_{K,N}>t|\xi_{K+1,N}=v)=(1-F(t))^K(1-F(v))^{-K}$, $t\geqslant v$, for such g one obtains that

$$\begin{aligned}&\left\langle g_1(\xi_{K,N})g_2(\xi_{K+1,N},\ldots,\xi_{N,N})\mathbb{I}\{\Omega_{K,M,N}\}\right\rangle\\&=\left\langle\left\langle g_1(\xi_{K,N})\mathbb{I}\{\Omega_{K,M,N}\}|\xi_{K+1,N}\right\rangle g_2(\xi_{K+1,N},\ldots,\xi_{N,N})\right\rangle\\&\leqslant KF^{1-N}(t_{M,N})\big(N\bar F(T_{M,N})\big)^{-1}\left\langle\int g(t,\xi_{K+1,N},\ldots,\xi_{N,N})F_{M,N}(\mathrm{d}t)\right\rangle.\end{aligned}$$

Next, applying (4.5), we arrive at the right-hand inequality in (4.7). The proof of the left-hand inequality in (4.7) is similar. □

We now turn to the proof of Lemma 4.2.

(i) To show (4.1), write $g(s_K,s_{K+1},\ldots,s_N)=\mathbb{I}\{s_K-s_{K+1}<N^{-1/l}\varepsilon^{-1}\}$ in (4.6), (4.7). Recall that $\lim_N P(\Omega\backslash\Omega_{K+1,M,N})\leqslant\delta_M$ (cf. (4.6)), and note that the event $\Omega_{K+1,M,N}$ implies $\{(M\bar F(\xi_{K+1,N}))^{-1}\leqslant N\leqslant M/\bar F(\xi_{K+1,N})\}$. Thus,

$$\begin{aligned}P\left(r_{K,N}<N^{-1/l}\varepsilon^{-1}\right)&\leqslant c\left\langle\mathbb{I}\{\Omega_{K+1,M,N}\}\left(F^N(\xi_{K+1,N}+N^{-1/l}\varepsilon^{-1})-F^N(\xi_{K+1,N})\right)\right\rangle+\delta_M\\&\leqslant c\left\langle\mathbb{I}\{\Omega_{K+1,M,N}\}N\big(\bar F(\xi_{K+1,N})-\bar F(\xi_{K+1,N}+N^{-1/l}\varepsilon^{-1})\big)\right\rangle+\delta_M\\&\leqslant c'\left\langle\mathbb{I}\{\Omega_{K+1,M,N}\}\left(1-\frac{\bar F\big(\xi_{K+1,N}+c''\bar F^{-1/l}(\xi_{K+1,N})\big)}{\bar F(\xi_{K+1,N})}\right)\right\rangle+\delta_M\longrightarrow\delta_M\end{aligned}$$

as $N\to\infty$, by (4.4). Since $\delta_M\to 0$, we get (4.1).

(ii) To prove (4.3) for $\omega_F < \infty$ and $l \in [1, \infty)$, we write $s_{K,N} = \omega_F - \xi_{K,N}$. Then, applying the law of large numbers to the random variables $(\omega_F - \xi(x))^{-l}$, $x \in V$, and using (4.1) (cf. Remark 2.1), we rewrite (4.3) as

$$P\text{-}\lim_N N^{-1} \sum_{k=K+1}^{N} \left(r_{K,k,N}^{-l} - s_{k,N}^{-l}\right) = 0. \tag{4.8}$$

To prove (4.8), we note that the kth summand in (4.8) does not exceed $\frac{1}{N}\sum_{i=1}^{2} h_i(\xi_{K,N}, \xi_{k,N})$, where the functions h_i are given by

$$h_1(t, v) = l(\omega_F - t)(\omega_F - v)^{-1}(t - v)^{-l}\mathbf{I}\{t - v > \omega_F - t\},$$
$$h_2(t, v) = (t - v)^{-l}\mathbf{I}\{0 < t - v \leqslant \omega_F - t\}.$$

Thus, it suffices to show that

$$\sum\nolimits_{(i)} := N^{-1}\sum_{k=1}^{N} h_i(\xi_{K,N}, \xi_{k,N}) \longrightarrow 0 \tag{4.9}$$

in probability as $N \to \infty$ for each $i = 1, 2$.

For each $i = 1, 2$, we use Lemma 4.3 with

$$g_i(t, t_{K+1}, \ldots, t_N) := \mathbf{I}\left\{\frac{1}{N}\sum_{k=K+1}^{N} h_i(t, t_k) > \varepsilon\right\}$$

to get

$$P\left(\sum\nolimits_{(i)} > \varepsilon\right) \leqslant c_{K,M}\left\langle\int g_i(t, \xi_{K+1,N}, \ldots, \xi_{N,N})F_{M,N}(\mathrm{d}t)\right\rangle + \delta_M$$
$$\leqslant c_{K,M}\left\langle\int \mathbf{I}\left\{N^{-1}\sum_{x\in V} h_i\big(t, \xi(x)\big) > \varepsilon\right\} F_{M,N}(\mathrm{d}t)\right\rangle + \delta_M,$$

where $\delta_M < \text{const}/M$ for $M \geqslant 100K^2$. Thus, by the definition of $F_{M,N}$, it suffices to prove that for any (large) M and any (non-random) sequence τ_N, $N \geqslant 1$, such that $\tau_N \in (t_{M,N}, T_{M,N})$,

$$\frac{1}{N}\sum_{x\in V} h_i\big(\tau_N, \xi(x)\big) \longrightarrow 0 \tag{4.10}$$

in probability as $N \to \infty$ for each $i = 1, 2$.

For $i = 1$, we have (via integration by parts)

$$\big\langle h_1(\tau_N, \xi)\big\rangle \leqslant \text{const}(\omega_F - \tau_N)\left(1 + \int_{-\infty}^{2\tau_N - \omega_F} \frac{1 - F(t)}{(\omega_F - t)(\tau_N - t)^{l+1}}\mathrm{d}t\right) \longrightarrow 0 \quad \text{as } N \to \infty,$$

since $(\omega_F - t)^{-l}(1 - F(t)) \to 0$ as $t \uparrow \omega_F$, due to $e_{F,l} < \infty$. This implies (4.10) with $i = 1$.

For $i = 2$, (4.10) is satisfied if and only if

$$NP\big(h_2(\tau_N, \xi) > N\varepsilon\big) + \big\langle h_2(\tau_N, \xi)\mathbf{I}\{h_2(\tau_N, \xi) < N\varepsilon\}\big\rangle \longrightarrow 0 \quad (\text{as } N \to \infty) \tag{4.11}$$

for any $\varepsilon > 0$ (see [6], Sec. IX.9). Elementary calculation shows that the left-hand side of (4.11) is equal (with an accuracy of $o(1)$) to

$$\mathrm{const}N\left(F(\tau_N) - F\left(\tau_N - (\varepsilon N)^{-1/l}\right)\right) + l\int \mathrm{d}t\,\mathbf{I}\left\{(\varepsilon N)^{-1/l} < t < \omega_F - \tau_N\right\} t^{-l-1}\left(F(\tau_N) - F(\tau_N - t)\right). \quad (4.12)$$

From the definition of the numbers $t_{M,N} \leqslant \tau_N \leqslant T_{M,N}$, it follows that $N \in [1/(M\widetilde{F}(\tau_N)), M/\widetilde{F}(\tau_N)]$. Substituting this into (4.12) and using both (4.2) and Remarks 2.1 and 4.1, we note that (4.12) tends to zero as $\tau_N \uparrow \omega_F$. This implies (4.10) for $i = 2$. The proof of (4.3) for $\omega_F < \infty$ is completed.

To show (4.3) in the case $\omega_F = \infty$, we write

$$h(t, v) = (t - v)^{-l}\mathbf{I}\{0 < t - v < M\} \quad \text{for } M > 1.$$

Since (by definition) $e_{F,l} = 0$, we only need to check

$$P\text{-}\lim_N \sum_k \frac{1}{N} h(\xi_{K,N}, \xi_{k,N}) = 0.$$

The proof of the last relation is similar to that of (4.10) (with $i = 2$ and $\omega_F < \infty$) and can be omitted.

(iii) First, applying the CLT and (4.1) with $l = 2$ (cf. also Remark 2.1), we have

$$P\text{-}\overline{\lim_N} \frac{1}{\sqrt{N}} \sum_{k=K+1}^{N} \left(s_{k,N}^{-1} - e_F\right) < \infty.$$

Further on,

$$\sum_{k=K+1}^{N} \left(\frac{1}{r_{K,k,N}} - \frac{1}{s_{k,N}}\right) = s_{K,N} N \left(e_{F,2} + o(1)\right)$$

as $N \to \infty$, because of the law of large numbers for the random variables $s_{k,N}^{-2}$ $(k = 1, 2, \ldots, N)$ and Lemma 4.2 (i), (ii) with $l = 2$. This proves (iii). □

The remainder of this section concerns generalization of Lemma 4.2 when $e_{F,l} = \infty$ for $l \geqslant 1$. Let $r_{K,N}$ and $R_{K,N}^{(l)}$ be given by (3.2) and (3.3), respectively.

LEMMA 4.4. *The following assertions hold in probability for any (fixed) $K = 1, 2, \ldots$ and $l \in [1, \infty)$:*

(i) *If $\widetilde{F}(t)(\omega_F - t)^{-\alpha} \to 1$ (as $t \uparrow \omega_F$) and $\alpha < l$, then*

$$N^{-1/\alpha} r_{K,N}^{-1} \asymp N^{-1/\alpha}\left(R_{K,N}^{(l)}\right)^{1/l} \asymp 1 \quad \text{as } N \to \infty \quad (4.13)$$

and

$$\widetilde{R}_{M,N} := M^{l/\alpha - 1} N^{-l/\alpha} R_{K,M,N}^{(l)} \asymp 1 \quad (4.14)$$

as $N \to \infty$ and $M \to \infty$.

(ii) *If $\widetilde{F}(t - \omega_F)$, $t \geqslant 0$, is slowly varying at zero, then*

$$\lim_N r_{L,N}\left(R_{L+1,N}^{(l)}\right)^{1/l} = 0.$$

Lemma 4.4 (i) (with $l = 1, 2$) implies the following:

COROLLARY 4.1. *Under the condition of Lemma 4.4 (i) with $\alpha < 2$,*

$$P\text{-}\overline{\lim_N} |R_{K,N} - N e_F| N^{-1/\alpha} < \infty.$$

Proof of Lemma 4.4. (i) We exploit the fact that the random variables $\eta(x) := -\ln(1 - F(\xi(x)))$ are independent exponentially distributed with mean 1, so that the variables $\eta_{k,N}$ and $\xi_{k,N}$ are related by

$$\omega_F - \xi_{k,N} = \tau_{k,N} \exp\{-\alpha^{-1}\eta_{k,N}\} =: s_{k,N} \quad \text{for any } k = 1, 2, \ldots, N, \tag{4.15}$$

where $\tau_{k,N} := \left(\widetilde{F}(\xi_{k,N})(\omega_F - \xi_{k,N})^{-\alpha}\right)^{-1/\alpha}$. We need some properties of $\eta_{k,N}$ (cf. [6], Sec. I.6):

LEMMA 4.5. *For any* $N \geqslant 1$,

$$(\eta_{k,N})_{1 \leqslant k \leqslant N} \stackrel{\mathrm{d}}{=} \left(T_k k^{-1} + T_{k+1}(k+1)^{-1} + \cdots + T_N N^{-1}\right)_{1 \leqslant k \leqslant N}; \tag{4.16}$$

here T_k, $k \geqslant 1$, *are independent exponentially distributed random variables with mean 1, and* $\stackrel{\mathrm{d}}{=}$ *means that the random vectors have the same distribution. Moreover, almost surely (a.s.)*

$$\left|\sum_{k=K}^{N} T_k k^{-1} - \ln N\right| \asymp 1 \quad \text{as } N \to \infty \tag{4.17}$$

for fixed $K \geqslant 1$, *and*

$$\lim_N \sum_{k=[N\delta]}^{N} T_k k^{-1} = -\ln \delta \tag{4.18}$$

for all $0 < \delta < 1$, *where* $[t]$ *denotes the integer part of* t.

Now we are in a position to prove (4.14) (the proof of (4.13) is similar and can be omitted). Applying (4.16)–(4.18) to (4.15) with $k = [N\delta]$ and $0 < \delta < 1$, we get

$$0 < c(\delta) < P\text{-}\varliminf_N s_{[N\delta],N} \leqslant P\text{-}\varlimsup_N s_{[N\delta],N} \leqslant C(\delta) < \infty, \tag{4.19}$$

where (non-random) constants $c(\delta)$ and $C(\delta)$ tend to zero as $\delta \downarrow 0$. Thus,

$$P\text{-}\lim_N N^{-l/\alpha} \sum_{k=[N\delta]}^{N} r_{K,k,N}^{-l} = 0, \tag{4.20}$$

since $\alpha < l$.

Further, by (4.15) and (4.16), the remaining kth summands in $R_{K,M,N}^{(l)}$ can be rewritten as

$$\left(r_{K,k,N}^{-l}\right)_{M \leqslant k \leqslant [N\delta]} \stackrel{\mathrm{d}}{=} \left(s_{K,N}^{-l}\left(\frac{\tau_{k,N}}{\tau_{K,N}} \exp\left\{\frac{1}{\alpha}\sum_{n=K}^{k-1}\frac{T_n}{n}\right\} - 1\right)^{-l}\right)_{M \leqslant k \leqslant [N\delta]} := \left(s_{K,N}^{-l}\widetilde{r}_{K,k,N}\right)_{M \leqslant k \leqslant [N\delta]}.$$

Consequently, combining (4.20) and the fact that $s_{K,N} \asymp N^{-1/\alpha}$ in probability as $N \to \infty$, we see that the left-hand side in (4.14) can be replaced by

$$\widetilde{R}_{M,N} := M^{\frac{l}{\alpha}-1} \sum_{k=M}^{[N\delta]} \widetilde{r}_{K,k,N}. \tag{4.21}$$

Let us estimate (4.21). Again by (4.19) we choose (non-random) $\rho(\delta) > 0$ such that $\rho(\delta) \to 0$ as $\delta \downarrow 0$ and the probability of the event $\Omega_{\delta,N} := \{1 - \rho(\delta) \leqslant \inf_{1 \leqslant k \leqslant N\delta} \frac{\tau_{k,N}}{\tau_{K,N}} \leqslant \sup_{1 \leqslant k \leqslant N\delta} \frac{\tau_{k,N}}{\tau_{K,N}} \leqslant 1 + \rho(\delta)\} \subset \Omega$ tends to 1, as $N \to \infty$, for any (small) $\delta > 0$. Write also $\Omega_\delta = \{\exp\{\alpha^{-1}\sum_{m=K}^{M-1} T_m m^{-1}\} > \frac{2}{1-\rho(\delta)}\} \subset \Omega$. With the above notation,

$$\lim_N P\left(\widetilde{R}_{M,N} > 1/\varepsilon\right) \leqslant \overline{\lim_N} P\left(\{\widetilde{R}_{M,N} > 1/\varepsilon\} \cap \Omega_{\delta,N} \cap \Omega_\delta\right) + P(\Omega\backslash\Omega_\delta)$$
$$\leqslant P\left(\left(\frac{2}{1-\rho(\delta)}\right)^l M^{\frac{l}{\alpha}-1} \sum_{k\geqslant M} \exp\left\{-\frac{l}{\alpha}\sum_{m=K}^{k-1}\frac{T_m}{m}\right\} > \frac{1}{\varepsilon}\right) + P(\Omega\backslash\Omega_\delta) \tag{4.22}$$

for any $\varepsilon > 0$, and, similarly,

$$\overline{\lim_N} P\left(\widetilde{R}_{M,N} < \varepsilon\right) \leqslant P\left(\left(\frac{1}{1+\rho(\delta)}\right)^l M^{\frac{l}{\alpha}-1} \sum_{k\geqslant M} \exp\left\{-\frac{l}{\alpha}\sum_{m=K}^{k-1}\frac{T_m}{m}\right\} < \varepsilon\right) + P(\Omega\backslash\Omega_\delta). \tag{4.23}$$

By applying (4.17) to the kth summands in (4.22), (4.23) when, first, $M \to \infty$ and then $\varepsilon \downarrow 0$, we arrive at (4.14). Part (i) is proved.

(ii) It suffices to prove the assertion for $l = 1$. Set $S_{L,N} = \sum_{k=L}^N s_{k,N}^{-1}$. Under the condition of Lemma 4.4 (ii), $P\text{-}\lim_N s_{L,N} S_{L+1,N} = 0$ for any (fixed) $L = 1, 2, \ldots$, by Theorem 6 of [1]. Moreover, for any $L + 1 \leqslant k \leqslant N$, $r_{L,k,N} = s_{k,N}(1 - s_{L,N}/s_{k,N}) \geqslant s_{k,N}(1 - s_{L,N}S_{L+1,N})$. Thus,

$$r_{K,N}R_{K+1,N} \leqslant s_{K+1,N}\left(1 - s_{K+1,N}S_{K+2,N}\right)^{-1} S_{K+2,N} \to 0$$

in probability as $N \to \infty$. Lemma 4.4 is proved. □

5. ON LOCALIZATION OF THE EIGENFUNCTIONS

In this section, we investigate the asymptotic structure (as $N \to \infty$) of the support of the normalized eigenfunctions $\psi_K(x) := \psi(x; \lambda_{K,N})$, $x \in V$ (2.4), for any (fixed) $K \geqslant 1$. For any $N \geqslant 1$ and any $1 \leqslant k \leqslant N$, let $z_{1,N}, z_{2,N}, \ldots, z_{N,N} \in V$ and $z_{1,N}^{(k)}, z_{2,N}^{(k)}, \ldots, z_{N,N}^{(k)} \in V$ denote (random) coordinates of ξ_V-peaks and ψ_k^2-peaks, respectively, i.e.,

$$\xi(z_{1,N}) := \xi_{1,N} > \xi(z_{2,N}) := \xi_{2,N} > \cdots > \xi(z_{N,N}) := \xi_{N,N}$$

and

$$\psi_k^2(z_{1,N}^{(k)}) > \psi_k^2(z_{2,N}^{(k)}) > \cdots > \psi_k^2(z_{N,N}^{(k)}).$$

Note that $z_{l,N}^{(1)} = z_{l,N}$ for any $1 \leqslant l \leqslant N$.

Definition 5.1. Given $K = 1, 2, \ldots$, we will say that $\psi_K(\cdot)$ has the property of
(a) *M-point localization* if

$$\min\left\{L:\ P\text{-}\lim_N \sum_{m=1}^{L} \psi_K^2\left(z_{m,N}^{(K)}\right) = 1\right\} = M < \infty; \tag{5.1}$$

(b) *partial localization* if

$$P\text{-}\lim_M \lim_N \sum_{m=1}^{M} \psi_K^2(z_{m,N}^{(K)}) = 1 \tag{5.2}$$

and if the (finite) minimum in (5.1) does not exist;
(c) *complete delocalization* if

$$P\text{-}\lim_{\varepsilon\downarrow 0} \overline{\lim_N} \sum_{0<m\leqslant\varepsilon N} \psi_K^2(z_{m,N}^{(K)}) = 0.$$

In view of Theorems 2.1–2.3 and Definition 5.1, we obtain the following results.

THEOREM 5.1. *Let* $1/\varkappa > e_F$ *and* $K = 1, 2, \ldots$. *Under the conditions of Theorem* 2.1, $\psi_K(\cdot)$ *satisfies the single-point localization, viz.*

$$P\text{-}\lim_N \psi_K^2(z_{K,N}) = 1.$$

THEOREM 5.2°. *Let* $1/\varkappa \leqslant e_F$ *and* $K = 1$.

(j) *If either the conditions of Theorem* 2.2° *or the conditions of Theorem* 2.3° (i) *are fulfilled, then* $\psi_1(\cdot)$ *satisfies complete delocalization.*

(jj) *Under the conditions of Theorem* 2.3° (ii), $\psi_1(\cdot)$ *satisfies partial localization.*

THEOREM 5.2. *Let* $1/\varkappa \leqslant e_F$ *and* $K = 2, 3, \ldots$.

(j) *If either the conditions of Theorem* 2.2 (i) *or the conditions of Theorem* 2.3 (i) *are fulfilled, then single-point localization holds, viz.*

$$P\text{-}\lim_N \psi_K^2(z_{K-1,N}) = 1.$$

(jj) *If either the conditions of Theorem* 2.2 (ii) *or the conditions of Theorem* 2.3 (ii) *hold, then* $\psi_K(\cdot)$ *satisfies partial localization.*

(jjj) *Under the conditions of Theorem* 2.2 (iii), *K-point localization holds, viz. for any* $1 \leqslant k \leqslant K-1$,

$$P\text{-}\lim_N \psi_K^2(z_{k,N}) = 1/\big(K(K-1)\big) \quad \text{and} \quad P\text{-}\lim_N \psi_K^2(z_{K,N}) = (K-1)/K.$$

Theorem 5.2° is proved in [3]. To prove the remaining statements, we use the assertions of Theorems 2.1–2.3 and Lemmas 4.2 and 4.4.

Proof of Theorem 5.1. Using (2.4), (3.2), and (3.3),

$$\psi_K^2(z_{K,N}) \geqslant \frac{(\lambda_{K,N} - \xi_{K,N})^{-2}}{(\lambda_{K,N} - \xi_{K,N})^{-2} + (K-1)(\lambda_{K,N} - \xi_{K-1,N})^{-2} + R_{K,N}^{(2)}} = 1 + \mathrm{o}(1) \tag{5.3}$$

in probability, due to Theorem 2.1 and Lemma 4.2 (i), (ii) with $l = 1$. □

Proof of Theorem 5.2. (j) follows by the same argument as in (5.3) with $z_{K,N}$ replaced by $z_{K-1,N}$.

(jj) Write

$$\bar{\psi}_{K,M,N} = \sum_{k=K+M}^{N} \psi_K^2(z_{k,N}) \quad \text{for } M \geqslant 1.$$

Simple estimates show that

$$\bar{\psi}_{K,M,N} \geqslant \frac{(\xi_{K-1,N} - \xi_{K+M,N})^{-2}}{(k-2)(\xi_{K-2,N} - \xi_{K-1,N})^{-2} + (\lambda_{K,N} - \xi_{K-1,N})^{-2} + (\lambda_{K,N} - \xi_{K,N})^{-2} + R_{K,N}^{(2)}}$$

and thus, by applying Theorems 2.2 (ii), 2.3 (ii), and assertion (4.13), we arrive at

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_N} P(\overline{\psi}_{K,M,N} < \varepsilon) = 0 \quad \text{for each } M \geqslant 3. \tag{5.4}$$

Similarly,

$$\overline{\psi}_{K,M,N} \leqslant \sum_{k=K+M}^{N} (\lambda_{K,N} - \xi_{k,N})^{-2} \left(\sum_{k=K}^{N} (\lambda_{K,N} - \xi_{k,N})^{-2} \right)^{-1} \leqslant \frac{R_{K,K+M,N}^{(2)}}{R_{K-1,N}^{(2)}}.$$

This and Lemma 4.4 (i) for $l = 2$ imply

$$\lim_M \overline{\lim_N} P(\overline{\psi}_{K,M,N} > \varepsilon) = 0 \quad \text{for any } \varepsilon > 0. \tag{5.5}$$

The claimed assertion follows from (5.4) and (5.5).

(jjj) For any $1 \leqslant l \leqslant K$, write

$$\psi_K^2(z_{l,N}) = (\lambda_{K,N} - \xi_{l,N})^{-2}\left(\sum_{k=1}^{K} + \sum_{k=K+1}^{N} (\lambda_{K,N} - \xi_{k,N})^{-2}\right)^{-1}. \tag{5.6}$$

First,

$$r_{K-1,N}^2 \sum_{k=K+1}^{N} \leqslant r_{K-1,N}^2 R_{K,N}^{(2)} = o(1) \tag{5.7}$$

in probability by virtue of Lemma 4.4 (ii). Further on, by (2.15),

$$P\text{-}\lim_N r_{K-1,N}^2 (\lambda_{K,N} - \xi_{K,N})^{-2} = K^2, \tag{5.8}$$

and, again by Lemma 4.4 (ii) for $1 \leqslant k \leqslant K-1$,

$$P\text{-}\lim_N r_{K-1,N}^2 (\xi_{k,N} - \lambda_{K,N})^{-2} = P\text{-}\lim_N r_{K-1,N}^2 (\xi_{K-1,N} - \lambda_{K,N})^{-2} = K^2/(K-1)^2. \tag{5.9}$$

Applying (5.7)–(5.9) to (5.6), we get the desired results. Theorem 5.2 is proved. □

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