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# LINEAR QUADRATIC NONZERO-SUM DIFFERENTIAL GAMES WITH RANDOM JUMPS *

WU Zhen (吴 臻), YU Zhi-yong (于志勇)

(School of Mathematics and System Science, Shandong University, Jinan 250100, P. R. China)

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**Abstract**: The existence and uniqueness of the solutions for one kind of forward-backward stochastic differential equations with Brownian motion and Poisson process as the noise source were given under the monotone conditions. Then these results were applied to nonzero-sum differential games with random jumps to get the explicit form of the open-loop Nash equilibrium point by the solution of the forward-backward stochastic differential equations.

**Key words**: stochastic differential equation; Poisson process; stochastic differential game

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## Introduction

Fully coupled forward-backward stochastic differential equations with Brownian motion can be encountered in the optimization problem when we apply stochastic maximum principle and in mathematics finance when we consider large investor in security market. Using partial differential equation method, Ma, Protter and Yong[1] successfully obtained the existence and uniqueness result for an arbitrarily prescribed time duration. Using probability method, Hu, Peng and Wu[2,3] obtained the existence and uniqueness result under some monotone assumptions. Yong[4] made the above method systematic and called it "continuation method".

The backward stochastic differential equation with Poisson process (BSDEP) was first discussed by Tang and Li[5]. Then Situ[6] obtained the existence and uniqueness result with non-Lipschitz coefficient for this kind of equation. Under some monotone assumptions,

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Corresponding author WU Zhen, Professor, Doctor, E-mail: wuzhen@sdu.edu.cn

Wu[7] obtained the existence and uniqueness of forward-backward stochastic differential equations with Brownian motion and Poisson process (FBSDEP in short). The stochastic process in the equations is discontinuous with random jump. In next section, we give another existence and uniqueness results of FBSDEP under some monotone assumptions suitable for the nonzero-sum differential games problems.

The differential games problems have been considered by Friedman[8], Bensoussan[9] and Eisele[10]. For stochastic case without random jump, Hamadene[11] gave the existence result of Nash equilibrium point under some assumptions. In Section 2, we study the linear quadratic nonzero-sum stochastic differential games problem with random jump. Using the preliminary result of FBSDEP in Section 1, we give the explicit form of Nash equilibrium point for this kind of stochastic differential games problems.

## 1 Preliminary Results of FBSDEP

Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},P)$ be a stochastic basis such that $\mathcal{F}_0$ contains all $P$-null elements of $\mathcal{F}$ and $\mathcal{F}_{t+}=\bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}=\mathcal{F}_t, t\geqslant 0$. We suppose that the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ is generated by the following two mutually independent processes: a $d$-dimensional standard Brownian motion $\{B_t\}_{t\geqslant 0}$ and a Poisson random measure $N$ on $\mathbb{R}_+\times Z$, where $Z\subset \boldsymbol{R}^l$ is a nonempty open set equipped with its Borel field $\mathcal{B}(Z)$, with compensator $\hat{N}(\mathrm{d}z,\mathrm{d}t)=\lambda(\mathrm{d}z)\mathrm{d}t$, such that $\tilde{N}(A\times[0,t])=(N-\hat{N})(A\times[0,t])_{t\geqslant 0}$ is a martingale for $\forall A\in\mathcal{B}(Z)$ satisfying $\lambda(A)<\infty$. $\lambda$ is assumed to be a $\sigma$-finite measure on $(Z,\mathcal{B}(Z))$ and called the characteristic measure.

In this section, we study the following FBSDEP:

$$\begin{cases}\mathrm{d}\boldsymbol{x}_t=\boldsymbol{b}(t,\boldsymbol{x}_t,\boldsymbol{p}_t)\mathrm{d}t+\boldsymbol{\sigma}(t,\boldsymbol{x}_t,\boldsymbol{p}_t)\mathrm{d}B_t+\int_Z \boldsymbol{g}(t_-,\boldsymbol{x}_{t_-},\boldsymbol{p}_{t_-})\tilde{N}(\mathrm{d}z\mathrm{d}t),\\ -\mathrm{d}\boldsymbol{p}_t=\boldsymbol{f}(t,\boldsymbol{x}_t,\boldsymbol{p}_t,\boldsymbol{q}_t,\boldsymbol{k}_t)\mathrm{d}t-\boldsymbol{q}_t\mathrm{d}B_t-\int_Z \boldsymbol{k}_{t_-}(z)\tilde{N}(\mathrm{d}z\mathrm{d}t),\\ \boldsymbol{x}_0=\boldsymbol{a},\quad \boldsymbol{p}_T=\boldsymbol{\Phi}(\boldsymbol{x}_T).\end{cases}\tag{1}$$

For notational simplification, we assume $d=1$, where $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{q},\boldsymbol{k})$ takes value in $\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n\times\boldsymbol{R}^n$, and $\boldsymbol{b},\boldsymbol{\sigma},\boldsymbol{f},\boldsymbol{g},\boldsymbol{\Phi}$ are the mappings with appropriate dimensions, for each fixed $(\boldsymbol{x},\boldsymbol{p},\boldsymbol{q},\boldsymbol{k})$, $\mathcal{F}_t$- progressively measurable, and $T>0$ is an arbitrarily fixed number and is called the time duration. We introduce the notations

$$\boldsymbol{u}=(\boldsymbol{x},\boldsymbol{p},\boldsymbol{q},\boldsymbol{k})^{\mathrm{T}},\quad A(t,\boldsymbol{u})=(-\boldsymbol{f},\boldsymbol{b},\boldsymbol{\sigma},\boldsymbol{g})^{\mathrm{T}}(t,\boldsymbol{u}),$$

and use the usual inner product, Euclidean norm in $\boldsymbol{R}^n$ and the following space:

$M^2=\Big\{v_t,\ 0\leqslant t\leqslant T,$ is a progressively measurable process s. t. $\boldsymbol{E}\Big[\int_0^T|v_t|^2\mathrm{d}t\Big]<\infty\Big\}$;

$L^2=\{\xi,\ \xi$ is a $\mathcal{F}_T$ measurable random variable s. t. $\boldsymbol{E}|\xi|^2<\infty\}$;

$F_N^2=\Big\{k_t(\cdot),\ 0\leqslant t\leqslant T,$ is a $\mathcal{F}_t$ predictable process s. t. $\boldsymbol{E}\Big[\int_0^T\int_Z|k_t(z)|^2 n(\mathrm{d}z)\mathrm{d}t\Big]<\infty\Big\}$.

We assume that

$$\begin{cases}(\text{i})\ A(t,\boldsymbol{u}) \text{ and } \boldsymbol{\Phi}(\boldsymbol{x}) \text{ are uniformly Lipschitz continuous with respect to their variables;}\\ (\text{ii})\ \text{for each } \boldsymbol{x}, \boldsymbol{\Phi}(\boldsymbol{x}) \text{ is in } L^2(\Omega,\mathcal{F}_T,P);\\ (\text{iii})\ \text{for } (\omega,t)\in\Omega\times[0,T], \text{ there are } \boldsymbol{l}(\omega,t,\boldsymbol{0},\boldsymbol{0})\in M^2(0,T),\ \boldsymbol{l}=\boldsymbol{b},\boldsymbol{\sigma}, \text{ respectively,}\\ \qquad \boldsymbol{g}(\omega,t,\boldsymbol{0},\boldsymbol{0})\in F_N^2(0,T) \text{ and } \boldsymbol{f}(\omega,t,\boldsymbol{0},\boldsymbol{0},\boldsymbol{0},\boldsymbol{0})\in M^2(0,T)\end{cases}\tag{2}$$

and also the following monotone conditions:

$$\begin{cases} \langle A(t,u) - A(t,\bar{u}), u - \bar{u}\rangle \leqslant -\beta_1 |\hat{x}|^2 - \beta_2 |\hat{p}|^2, \\ \langle \Phi(x) - \Phi(\bar{x}), x - \bar{x}\rangle \geqslant \mu_1 |\hat{x}|^2, \text{ a.s.}, \\ \forall \hat{u} = (u - \bar{u}) = (\hat{x},\hat{p},\hat{q},\hat{k}) = (x - \bar{x}, p - \bar{p}, q - \bar{q}, k - \bar{k}), \end{cases} \tag{3}$$

where $\beta_1$, $\beta_2$ and $\mu_1$ are given nonnegative constants with $\beta_1 + \beta_2 > 0$, $\mu_1 + \beta_2 > 0$. Then we have the following Theorem 1.

**Theorem 1** *We assume that the assumptions* (2) *and* (3) *hold, then there exists a unique quartet* $(x(\cdot), p(\cdot), q(\cdot), k(\cdot)) \in M^2(0,T;\mathbf{R}^{n+n+n}) \times F_N^2(0,T;\mathbf{R}^n)$ *satisfying FBSDEP*(1).

The proof method is similar to Theorem 3.1 in Ref. [7], we omit it.

## 2 Linear Quadratic Nonzero-Sum Stochastic Differential Games with Random Jump

In this section, we study linear quadratic nonzero-sum stochastic differential games problem with random jump. For notational simplification, we only consider two players, which is similar for $n$ players. The control system is

$$\begin{cases} \mathrm{d}x_t^v = (Ax_t^v + B^1 v_t^1 + B^2 v_t^2 + \alpha_t)\mathrm{d}t + (Cx_t^v + \beta_t)\mathrm{d}\beta_t \\ \qquad + \int_Z (Ex_{t_-}^v + \gamma_{t_-}(z))\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ x_0^v = a, \end{cases} \tag{4}$$

where $A$, $C$ and $E$ are $n \times n$ bounded matrices, and $v_t^1$ and $v_t^2$, $t \in [0,T]$, are two admissible control processes, *i.e.*, $\mathcal{F}_t$-adapted square-integrable processes taking values in $\mathbf{R}^k$. $B^1$ and $B^2$ are $n \times k$ bounded matrices. $\alpha_t$ and $\beta_t$ are two adapted square-integrable processes. $\gamma_t(\cdot)$ is the process in $F_N^2(0,T;\mathbf{R}^n)$. We denote $J^1(v(\cdot))$ and $J^2(v(\cdot))$, $v(\cdot) = (v^1(\cdot), v^2(\cdot))$, which are the cost functions corresponding to the two players 1 and 2:

$$\begin{cases} J^1(v(\cdot)) = \frac{1}{2}E\Big[\int_0^T (\langle R^1 x_t^v, x_t^v\rangle + \langle N^1 v_t^1, v_t^1\rangle)\mathrm{d}t + \langle Q^1 x_T^v, x_T^v\rangle\Big], \\ J^2(v(\cdot)) = \frac{1}{2}E\Big[\int_0^T (\langle R^2 x_t^v, x_t^v\rangle + \langle N^2 v_t^2, v_t^2\rangle)\mathrm{d}t + \langle Q^2 x_T^v, x_T^v\rangle\Big]. \end{cases} \tag{5}$$

Here $Q^1$, $Q^2$, $R^1$ and $R^2$ are $n \times n$ nonnegative symmetric bounded matrices, and $N^1$ and $N^2$ are $k \times k$ positive symmetric bounded matrices and the inverse $(N^1)^{-1}$, $(N^2)^{-1}$ are also bounded.

The problem is to look for $(u^1(\cdot), u^2(\cdot)) \in \mathbf{R}^k \times \mathbf{R}^k$ which is called the Nash equilibrium point for the game, such that

$$\begin{cases} J^1(u^1(\cdot), u^2(\cdot)) \leqslant J^1(v^1(\cdot), u^2(\cdot)), & \forall v^1(\cdot) \in \mathbf{R}^k; \\ J^2(u^1(\cdot), u^2(\cdot)) \leqslant J^2(u^1(\cdot), v^2(\cdot)), & \forall v^2(\cdot) \in \mathbf{R}^k. \end{cases} \tag{6}$$

Here the actions of the two players are disturbed by the random jump process, and the optimal solutions are discontinuous processes. This kind of differential games problem has the practical application background. We need the following assumption:

$$\begin{cases} B^i(N^i)^{-1}(B^i)^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^i(N^i)^{-1}(B^i)^{\mathrm{T}}, \\ B^i(N^i)^{-1}(B^i)^{\mathrm{T}}C^{\mathrm{T}} = C^{\mathrm{T}}B^i(N^i)^{-1}(B^i)^{\mathrm{T}}, \qquad i = 1,2, \text{ a.s.} \\ B^i(N^i)^{-1}(B^i)^{\mathrm{T}}E^{\mathrm{T}} = E^{\mathrm{T}}B^i(N^i)^{-1}(B^i)^{\mathrm{T}}, \end{cases} \tag{7}$$

Now, using the solution of FBSDEP, we can give the explicit form of the Nash equilibrium point for this problem.

**Theorem 2** *The function* $(u_t^1, u_t^2) = (-(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1, -(N^2)^{-1}(B^2)^{\mathrm{T}}p_t^2), t \in [0,T]$, *is a Nash equilibrium point for the above games problem, where* $(x_t, p_t^1, p_t^2, q_t^1, q_t^2, k_t^1, k_t^2)$ *is the solution of the following different dimensional FBSDEP*:

$$\begin{cases} \mathrm{d}x_t = (Ax_t - B^1(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1 - B^2(N^2)^{-1}(B^2)^{\mathrm{T}}p_t^2 + \alpha_t)\mathrm{d}t \\ \qquad + (Cx_t + \beta_t)\mathrm{d}B_t + \int_Z (Ex_{t_-} + \gamma_{t_-}(z))\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ -\mathrm{d}p_t^1 = (A^{\mathrm{T}}p_t^1 + C^{\mathrm{T}}q_t^1 + E^{\mathrm{T}}k_t^1 + R^1x_t)\mathrm{d}t - q_t^1\mathrm{d}B_t - \int_Z k_t^1(z)\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ -\mathrm{d}p_t^2 = (A^{\mathrm{T}}p_t^2 + C^{\mathrm{T}}q_t^2 + E^{\mathrm{T}}k_t^2 + R^2x_t)\mathrm{d}t - q_t^2\mathrm{d}B_t - \int_Z k_t^2(z)\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ x_0 = a, \quad p_T^1 = Q^1x_T, \quad p_T^2 = Q^2x_T. \end{cases} \tag{8}$$

**Proof** We first prove the existence of the solution of Eq. (8). We consider the following FBSDEP:

$$\begin{cases} \mathrm{d}X_t = (AX_t - P_t + \alpha_t)\mathrm{d}t + (CX_t + \beta_t)\mathrm{d}B_t + \int_Z (EX_{t_-} + \gamma_{t_-}(z))\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ -\mathrm{d}P_t = [A^{\mathrm{T}}P_t + (B^1(N^1)^{-1}(B^1)^{\mathrm{T}}R^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}R^2)X_t \\ \qquad + C^{\mathrm{T}}Q_t + E^{\mathrm{T}}K_t]\mathrm{d}t - Q_t\mathrm{d}B_t - \int_Z K_{t_-}(z)\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ X_0 = a, \ P_T = (B^1(N^1)^{-1}(B^1)^{\mathrm{T}}Q^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}Q^2)X_T. \end{cases} \tag{9}$$

From the assumption (7), we notice that if $(x_t, p_t^1, p_t^2, q_t^1, q_t^2, k_t^1, k_t^2)$ is the solution of Eq. (8), $(X_t, P_t, Q_t, K_t)$ solves the FBSDEP (9), here

$$X_t = x_t, \quad P_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}p_t^2,$$
$$Q_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}q_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}q_t^2,$$
$$K_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}k_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}k_t^2.$$

On the other hand, we can first use FBSDEP(9) to get solution $X_t$ which is the forward solution $x_t$ of Eq. (8), then obtain $(p_t^1, q_t^1, k_t^1)$ and $(p_t^2, q_t^2, k_t^2)$.

It is easy to check that FBSDEP(9) satisfies the assumptions (2) and (3). According to Theorem 1, there exists a unique solution $(X_t, P_t, Q_t, K_t)$ of Eq. (9). Now, from the existence and uniqueness result of BSDEP in Ref. [5], we can let $(p_t^1, q_t^1, k_t^1)$ and $(p_t^2, q_t^2, k_t^2)$ be the solutions of the following BSDEP:

$$\begin{cases} -\mathrm{d}p_t^1 = (A^{\mathrm{T}}p_t^1 + C^{\mathrm{T}}q_t^1 + E^{\mathrm{T}}k_t^1 + R^1X_t)\mathrm{d}t - q_t^1\mathrm{d}B_t - \int_Z k_{t_-}^1(z)\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ -\mathrm{d}p_t^2 = (A^{\mathrm{T}}p_t^2 + C^{\mathrm{T}}q_t^2 + E^{\mathrm{T}}k_t^2 + R^2X_t)\mathrm{d}t - q_t^2\mathrm{d}B_t - \int_Z k_{t_-}^2(z)\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ p_T^1 = Q^1X_T, \quad p_T^2 = Q^2X_T. \end{cases}$$

We let

$$\overline{P}_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}p_t^2,$$
$$\overline{Q}_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}q_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}q_t^2,$$
$$\overline{K}_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}k_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}k_t^2.$$

Then we get

$$\begin{cases} -\mathrm{d}\overline{P}_t = [A^{\mathrm{T}}\overline{P}_t + (B^1(N^1)^{-1}(B^1)^{\mathrm{T}}R^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}R^2)X_t \\ \qquad + C^{\mathrm{T}}\overline{Q}_t + E^{\mathrm{T}}\overline{K}_t]\mathrm{d}t - \overline{Q}_t\mathrm{d}B_t - \int_Z \overline{K}_{t-}(z)\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ \overline{P}_T = [B^1(N^1)^{-1}(B^1)^{\mathrm{T}}Q^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}Q^2]x_T. \end{cases}$$

For fixed $\{X_t\}_{t\geq 0}$, because of the existence and uniqueness of the BSDEP, we have

$$P_t = \overline{P}_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}p_t^2,$$
$$Q_t = \overline{Q}_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}q_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}q_t^2,$$
$$K_t = \overline{K}_t = B^1(N^1)^{-1}(B^1)^{\mathrm{T}}k_t^1 + B^2(N^2)^{-1}(B^2)^{\mathrm{T}}k_t^2.$$

Then $(X_t, P_t, Q_t, K_t)$ is the unique solution of FBSDEP (9) and $(X_t, p_t^1, p_t^2, q_t^1, q_t^2, k_t^1, k_t^2)$ is the solution of FBSDEP (8).

Now we try to prove $(u^1(\cdot), u^2(\cdot))$ is a Nash equilibrium point for our nonzero-sum game problem with random jump. We only prove

$$J^1(u^1(\cdot), u^2(\cdot)) \leqslant J^1(v^1(\cdot), u^2(\cdot)), \qquad \forall v^1(\cdot) \in R^k.$$

It is similar to get another inequality of Eq. (6). $x_t^{v^1}$ denotes the solution of the system:

$$\begin{cases} \mathrm{d}x_t^{v^1} = (Ax_t^{v^1} + B^1v_t^1 + B^2u_t^2 + \alpha_t)\mathrm{d}t + (Cx_t^{v^1} + \beta_t)\mathrm{d}B_t \\ \qquad + \int_Z (Ex_{t-}^{v^1} + \gamma_{t-}(z))\tilde{N}(\mathrm{d}z\mathrm{d}t), \\ x_0 = a; \end{cases} \tag{10}$$

$$\begin{aligned} J^1(v^1(\cdot), u^2(\cdot)) - J^1(u^1(\cdot), u^2(\cdot)) &= \frac{1}{2}E\Big[\int_0^T (\langle R^1x_t^{v^1}, x_t^{v^1}\rangle - \langle R^1x_t, x_t\rangle \\ &\quad + \langle N^1v_t^1, v_t^1\rangle - \langle N^1u_t^1, u_t^1\rangle)\mathrm{d}t + \langle Q^1x_T^{v^1}, x_T^{v^1}\rangle - \langle Q^1x_T, x_T\rangle\Big] \\ &= \frac{1}{2}E\Big[\int_0^T (\langle R^1(x_t^{v^1} - x_t), x_t^{v^1} - x_t\rangle + \langle N^1(v_t^1 - u_t^1), v_t^1 - u_t^1\rangle \\ &\quad + 2\langle R^1x_t, x_t^{v^1} - x_t\rangle + 2\langle N^1u_t^1, v_t^1 - u_t^1\rangle)\mathrm{d}t \\ &\quad + \langle Q^1(x_T^{v^1} - x_T), x_T^{v^1} - x_T\rangle + 2\langle Q^1x_T, x_T^{v^1} - x_T\rangle\Big]. \end{aligned}$$

From $Q^1x_T = p_T^1$, we use Itô's formula to $\langle x_T^{v^1} - x_T, p_T^1\rangle$ and get

$$E\langle x_T^{v^1} - x_T, p_T^1\rangle = E\int_0^T (-\langle R^1x_t, x_t^{v^1} - x_t\rangle + \langle B^1(v_t^1 - u_t^1), p_t^1\rangle)\mathrm{d}t.$$

So from $R^1$ and $Q^1$ are nonnegative, $N^1$ is positive:

$$\begin{aligned} &J^1(v^1(\cdot), u^2(\cdot)) - J^1(u^1(\cdot), u^2(\cdot)) \\ &\geqslant E\int_0^T (\langle N^1u_t^1, v_t^1 - u_t^1\rangle + \langle B^1(v_t^1 - u_t^1), p_t^1\rangle)\mathrm{d}t \\ &= E\int_0^T (\langle -N^1(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1, v_t^1 - u_t^1\rangle + \langle (B^1)^{\mathrm{T}}p_t^1, v_t^1 - u_t^1\rangle)\mathrm{d}t = 0. \end{aligned}$$

So $(u_t^1, u_t^2) = (-(N^1)^{-1}(B^1)^{\mathrm{T}}p_t^1, -(N^2)^{-1}(B^2)^{\mathrm{T}}p_t^2)$ is the Nash equilibrium point for our nonzero-sum games problem.

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