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On the Nonoscillation of Elliptic Integrals

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To V. I. Arnold on the occasion of his sixtieth birthday

Consider the elliptic integral

$$\oint P(x,y)\,dx + Q(x,y)\,dy \tag{1}$$

of a real polynomial 1-form over the real ovals $y^2 + x^2 + x^4 = t$, $t > 0$, on the plane.

In this paper we estimate the number of zeros of the integral (1) regarded as a function of t. The estimation is based on a new assertion concerning the nonoscillation of a linear combination of eigenfunctions of a second-order linear differential operator (see Lemma 1 below).

The problem of estimating the number of limit cycles generated under a perturbation of a Hamiltonian polynomial vector field can be reduced to the problem of estimating the number of zeros of a real polynomial form over the level ovals of a real polynomial.

Let $\mathcal{J}_n$ be the space of integrals of the type (1) for real polynomial 1-forms of degree $\le n$.

The main result of the paper is the following.

Theorem 1. *The space $\mathcal{J}_n$ has the Chebyshev property on the ray $t > 0$. This means that the number of zeros (counted according to their multiplicities) of every integral in $\mathcal{J}_n$ that is not identically zero is less than the dimension of the space $\mathcal{J}_n$, which is equal to $2[(n-1)/2] + 1$.*

An analog of this theorem for integrals over the ovals $y^2 + x^4 - x^2 = t$ was proved earlier in [1]. This proof can be applied to the integrals in Theorem 1 as well. Here we give a different (real) version of the proof.

1. Scheme of the proof of Theorem 1. The proof of Theorem 1 is based on the following assertion.

Lemma 1. *Let f and g be continuous functions on a (possibly (semi)infinite) interval $I = (a,b)$ and let the function f be positive. Suppose that $\lambda_1 < \dots < \lambda_k$ and there are solutions $y_1, \dots, y_k$ of the equations $\ddot y_i = (\lambda_i f(t) + g(t))y_i$ with the following properties:*

1) *y_1 vanishes nowhere on I;*
2) *$y_i(t) \to 0$ as $t \to a$;*
3) *the derivatives of the functions y_i are bounded in a neighborhood of a.*

Then the space of functions of the form $\sum_{i=1}^k c_i y_i$ is a k-dimensional Chebyshev space.

Lemma 1 is proved at the end of the paper.

This lemma was first stated and proved by the author for the case in which $g = 0$ and $f(t) = 1/(t(4t+1))$. It is this special case of the lemma that is used in the proof of Theorem 1. Khovanskii noticed that the author's proof remains completely valid under the assumptions of Lemma 1 in the general case. Note that if $k = 2$, then Lemma 1 is a special case of the Sturm comparison theorem (see Sec. 4) under the assumption that the functions f and g can be continuously extended to the point a.

We shall show that the space $\mathcal{J}_n$ belongs to the class of Chebyshev spaces in Lemma 1.

Theorem 1 is a consequence of Lemma 1 and the following three assertions.

Lemma 2. *There is a basis $\{y_i\}$ in $\mathcal{J}_n$ formed by solutions of the second-order equations*

$$\ddot y_i = \frac{\lambda_i}{t(4t+1)}\, y_i, \qquad \lambda_1 < 0 < \lambda_2 < \dots. \tag{2}$$

The solution y_1 is everywhere positive.

Lemma 2 is proved in the next section.

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Lemma 3. *The space $\mathscr{I}_n$ is just the space of polynomial linear combinations*

$$I_0 P_0 + I_1 P_1$$

of the special elliptic integrals I_0 and I_1 of the forms $\omega_0 = y\,dx$ and $\omega_1 = x^2 y\,dx$ with real polynomials P_0 and P_1 of degrees not exceeding $[(n-1)/2]$ and $[(n-1)/2]-1$, respectively.

Proposition 1. *All integrals* (1) *can be analytically continued by zero to the point $t = 0$.*

Analogs of Lemma 3 and Proposition 1 for integrals over the ovals $y^2 + x^4 - x^2 = t$ were considered in [1]. The proof can be applied to the integrals from the space $\mathscr{I}_n$.

To prove Theorem 1, it remains to note that the basis functions y_i in Lemma 2 satisfy the assumptions of Lemma 1 (see Lemma 2 and Proposition 1). By Lemma 1, $\mathscr{I}_n$ is a Chebyshev space. The assertion of Theorem 1 concerning the dimension of this space follows from Lemma 3.

2. Proof of Lemma 2. We seek a basis $\{y_i\}$ in $\mathscr{I}_n$ for which the assertions of Lemma 2 hold. The equations for y_i mean that the y_i are eigenfunctions of the operator $L = t(4t+1)\,d^2/dt^2$.

Lemma 2 follows from the lemma below.

Lemma 4. *The space $\mathscr{I}_n$ is invariant with respect to the operator L. The function I_0 is its eigenfunction with a negative eigenvalue. The other eigenvalues of the operator L on the space $\mathscr{I}_n$ are positive and pairwise distinct.*

Lemma 4 is proved in Sec. 3.

The desired basis $\{y_i\}$ consists of eigenfunctions of the operator L with $y_1 = I_0$. Its existence follows from the fact that the eigenvalues of this operator are pairwise distinct. To complete the proof of Lemma 2 it remains to note that the function $I_0(t)$ is positive because it is equal to the area of the domain bounded by the oval at the level t. This proves Lemma 2.

3. Proof of Lemma 4. Let us show that the operator L is represented in the basis of the functions $I_0, I_1, tI_0, tI_1, \dots$ by an upper triangular matrix whose first diagonal entry is negative and the others are positive and pairwise distinct. This will prove Lemma 4 because the diagonal entries of an upper triangular matrix are its eigenvalues and I_0 is an eigenfunction.

The proof of the assertion in the foregoing paragraph will be based on the following property of the integrals I_0 and I_1.

Lemma 5.

$$I_0 = \frac{4}{3}\,tI_0' - \frac{2}{3}\,I_1', \qquad I_1 = -\frac{2}{15}\,tI_0' + \left(\frac{4}{5}\,t + \frac{4}{15}\right)I_1'. \tag{3}$$

Similar equations for the integrals over the ovals $y^2 + x^4 - x^2 = t$ are given in [1] and equations remain valid for integrals over an arbitrary family of ovals on complex level surfaces (for instance, over the ovals that are intersections of a level surface with the plane $y \in \mathbb{R}$, $x \in i\mathbb{R}$). The equations in Lemma 5 can be derived from these general equations by substituting $x = ix'$.

We first show that the operator L preserves the space of linear combinations of the integrals I_0 and I_1 and is represented by the matrix

$$\begin{pmatrix} -3/4 & 1/2 \\ 0 & 5/4 \end{pmatrix} \tag{4}$$

in the basis I_0, I_1. The proof is reduced to a straightforward calculation of the values of the operator L on I_0 and I_1. On differentiating Eqs. (3) and transposing the terms containing I_j', $j = 0, 1$, to the left-hand side we obtain

$$I_0' = -4tI_0'' + 2I_1'', \qquad \frac{2}{15}\,I_0' + \frac{1}{5}\,I_1' = -\frac{2}{15}\,tI_0'' + \left(\frac{4}{5}\,t + \frac{4}{15}\right)I_1''. \tag{5}$$

The substitution of the right-hand side of the first equation for I_0' into the second equation results in

$$I_1' = 2tI_0'' + 4tI_1''. \tag{6}$$

Now let us express the integrals I_0 and I_1 via their second-order derivatives. To this end, we substitute formulas (5) and (6) for I_0' and I_1' into Eqs. (3) to obtain

$$I_0 = -\frac{4}{3}\,t\,(4t+1)\,I_0'' = -\frac{4}{3}\,LI_0,$$
$$I_1 = \frac{8}{15}\,t\,(4t+1)\,I_0'' + \frac{4}{5}\,t\,(4t+1)\,I_1'' = \frac{8}{15}\,LI_0 + \frac{4}{5}\,LI_1.$$

This implies the desired assertion concerning the action of L on I_0 and I_1.

We now show that L preserves the flag of subspaces that corresponds to the basis $I_0, I_1, tI_0, tI_1, \dots$ (and thus the related matrix is upper triangular). We shall prove that the diagonal element associated with the basis function $t^k I_0$ ($t^k I_1$) is equal to $-3/4 + k(4k+2)$ ($5/4 + k(4k+6)$, respectively). All these numbers are positive (except for $-3/4$) and, as can readily be seen, pairwise distinct. This proves the assertion in the beginning of Sec. 3.

To find these diagonal elements, we calculate the value of the operator L on the function $t^k I_0$ ($t^k I_1$) up to a linear combination of the preceding basis functions. We have

$$L(t^k I_j) = t^k L I_j + k(k-1)\,t\,(4t+1)\,t^{k-2} I_j + 2kt^k(4t+1)\,I_j'. \tag{7}$$

Let us perform the calculation for $j = 0$. The sum of the first two terms on the right-hand side of (7) is equal (with the indicated accuracy) to $(-3/4 + 4k(k-1))(t^k I_0)$ (see (4)). To calculate the third term we express $t(4t+1)I_0'$ via I_0 and I_1 using the first equation in (5). On multiplying this equation by $t(4t+1)$ we obtain $t(4t+1)I_0' = -4tLI_0 + 2LI_1$. This is equal to $3tI_0$ modulo a linear combination of the integrals I_0 and I_1 (see (4)). Hence, up to the accuracy indicated at the beginning of the paragraph, the third term on the right-hand side of (7) is equal to $6kt^k I_0$, and therefore $L(t^k I_0)$ is equal to $(-3/4 + k(4k+2))\,t^k I_0$. We can similarly prove that, up to the corresponding accuracy, $L(t^k I_1)$ is equal to $(5/4 + k(4k+6))\,t^k I_1$. This proves the assertions in the foregoing paragraph. The proof of Lemma 4 is complete.

4. Proof of Lemma 1. We first show that each eigenfunction y_i vanishes nowhere on I. For $i = 1$, this is one of the assumptions of the lemma. For $i > 1$, this follows from the inequality $\lambda_i > \lambda_1$ and the following version of the Sturm comparison theorem appeared to y_1 and y_i.

The Sturm comparison theorem. *Let $F(t)$ and $G(t)$, $F(t) < G(t)$, be continuous functions on a (possibly (semi)infinite) interval $I = (a, b)$. Let the equations $\ddot{y}_1 = F(t)y_1$ and $\ddot{y}_2 = G(t)y_2$ have solutions y_1 and y_2 that are not identically zero. Suppose that the functions y_i can be continuously extended by zero to the point a and that their derivatives are bounded in a neighborhood of a. Then there is at least one zero of the function y_1 between any two zeros of the function y_2 on $I \cup a$.*

Since our version of the Sturm comparison theorem is somewhat more general than the classical statement [2], we present its proof in full.

Proof. Assume that there is a pair of zeros $c, d \in I \cup \{a\}$ of the function y_2 such that the function y_1 does not vanish on the interval (c, d). Without loss of generality, we can assume that $y_1, y_2 > 0$ on (c, d). Let $W = y_1\dot{y}_2 - y_2\dot{y}_1$ be the Wronskian of the pair of functions y_1, y_2. On one hand, the Wronskian $W(t)$ is strictly increasing on (c, d) because $\dot{W} = y_1\ddot{y}_2 - y_2\ddot{y}_1 = (G - F)y_1y_2 > 0$. On the other hand, as is shown below, $W(c) \geq 0 \geq W(d)$. (We assume that $W(a) = 0$ because the function W can be continuously extended by zero to the point a according to the condition of the theorem.) The resulting contradiction proves the theorem.

It remains to verify the inequalities at the end of the foregoing paragraph. By assumption, $y_2 = 0$ at the points c and d, and hence $W = y_1\dot{y}_2$. The function y_1 is positive on the interval (c, d), and, consequently, its endpoint values are nonnegative. This means that, at each of these points, the value of W either is zero or has the same sign as $\dot{y}_2$. (If $c = a$, then $W(c) = 0$.) We have $\dot{y}_2(d) < 0$ and the inequality $\dot{y}_2(c) > 0$ holds for $c \in I$, which follows from the fact that $y_2|_{(c,d)} > 0$ by assumption. This establishes the inequalities in the preceding paragraph and proves the Sturm comparison theorem.

For any smooth function q on an interval I, denote by $\#q|_I$ the number of zeros of q counted according to their multiplicities.

Lemma 6. 1. *The inequality* $\#q|_I \leq \#q'|_I + 1$ *holds for any smooth function* q *on the interval* I.

2. *If a function* q *vanishes as the argument tends to one of the endpoints of the interval* I, *then* $\#q|_I \leq \#q'|_I$.

Lemma 6 follows from the Rolle theorem according to which between any two zeros of a function there is at least one zero of its derivative.

We now return to the proof of Lemma 1. Let $Y = \sum_{i \leq k} c_i y_i$ be a nontrivial linear combination of the functions y_i. Denote by q the quotient

$$\frac{Y}{y_k} = c_k + \sum_{i \leq k} c_i \frac{y_i}{y_k}.$$

We must prove that $\#q|_I \leq k$. By assertion 1 in Lemma 6, it suffices to prove that $\#q'|_I < k - 1$. We have

$$q' = \frac{h}{y_k^2}, \quad \text{where } h = \sum_{i<k} c_i(y_i' y_j - y_i y_j').$$

Thus, it suffices to prove a similar inequality for $\#h|_I$. The function h can be continuously extended by zero to the point a (because the derivatives of y_i' are bounded). Therefore, by assertion 2 of Lemma 6, we obtain

$$\#h|_I \leq \#h'|_I. \tag{8}$$

Furthermore,

$$h' = \sum_{i<k} c_i(y_i'' y_j - y_i y_j'') = y_j f \sum_{i<k} c_i(\lambda_i - \lambda_j) y_i.$$

The sum on the right-hand side of this relation is a linear combination of $k - 1$ functions y_i, $i < k$. By induction, it is either identically zero, or has less than $k - 1$ zeros. This proves Lemma 1.

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