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The Absence of an Additional Real-Analytic First Integral in Some Problems of Dynamics

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1. Statement of the results. It was proved in [1, 2] that in some dynamic problems in the complexified phase space, there is no additional meromorphic first integral (functionally independent of the known integrals).

In the present paper, we prove that there are no meromorphic (in particular, analytic) first integrals for these problems in the real phase space as well. The following theorems hold.

Theorem 1. *In the problem on the motion of a heavy rigid body about a fixed point, the existence of a general additional real-meromorphic* first integral takes place only in the well-known Euler, Lagrange, and Kowalewski cases. The existence of a particular integral (for the area constant equal to zero) takes place only in the above three cases and in the Goryachev–Chaplygin case.*

Remark. For a dynamically nonsymmetric body, the assertion of Theorem 1 follows from [3, 4]. For the problem on the motion of a dynamically symmetric heavy rigid body about a fixed point, the absence of an additional (general or particular) real-meromorphic first integral was proved in [5, 6] (also see [7]) for the case in which one of the principal moments of inertia is much smaller than the other two (a perturbed spherical pendulum), in [8, 9] (also see [7, 10]) for the perturbed Lagrange case, and in [11] for the case in which the center of gravity lies in the equatorial plane and the ratio of the principal moments of inertia is greater than 4.

Theorem 2. *The Henon–Heiles system* [12] *has no additional real-meromorphic first integral.*

Theorem 3. *In the Suslov problem* [13] *on the motion of a rigid body about a fixed point with the nonholonomic constraint prescribing that the angular velocity is orthogonal to a direction fixed in the reference frame attached to the body, for the case in which this direction coincides with one of the principal inertia axes at the suspension point, the body is in a uniform gravitational field, and the center of gravity lies on the above-mentioned axis, the existence of an additional real-meromorphic first integral takes place only if the body is dynamically symmetric (an analog of the Lagrange case of the motion of a heavy rigid body about a fixed point without additional constraints).*

2. Some facts to be used. We use the following definitions from [15]. Let M be a complex manifold, let v be an analytic vector field on M, and let Γ be a complex phase curve of v other than an equilibrium.

Let f be an analytic function on M such that f is constant on Γ. By f' we denote the function on the normal bundle $N\Gamma = T_\Gamma M/T\Gamma$ of Γ in M generated by the function df on $T_\Gamma M$. Obviously, f' is linear on the fibers of $N\Gamma$.

Let $H = (H_1, \dots, H_l)$, $l \geq 0$, be analytic first integrals of v whose differentials are linearly independent on Γ. The *reduced phase space* of the system in normal variations along Γ is defined as the level surface $N_p\Gamma = \{\xi \in N\Gamma \mid H'(\xi) = p\}$, $p \in \mathbb{C}^l$, of the first integrals H' of this system; the *reduced system in variations* is the restriction of the system in normal variations to this surface.

Obviously, $N_p\Gamma$ is a holomorphic affine bundle (for $p = 0$, a holomorphic vector bundle) over Γ, and the reduced system in variations is linear (for $p = 0$, homogeneous linear).

The *monodromy group* of a linear system in a holomorphic affine (vector) bundle over a Riemann surface is the image of the natural antirepresentation of the fundamental group of this surface at some point in the group of affine (linear) transformations of the fiber over that point.

*That is, representable as a ratio of real-analytic functions in the neighborhood of any point.

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Proposition [15]. *If the differential equation corresponding to* v *has an additional meromorphic first integral in a domain* $U \subset M$ *such that the fundamental group of* Γ *(at some point) can be represented by loops entirely lying in* U, *then the monodromy group of the reduced system in variations along* Γ *has a rational first integral for any* $p \in \mathbb{C}^l$.

We use the expression "almost all" to mean "all except for possibly countably many."

3. Proof of Theorems 1–3. We assume that the body in the problem on the motion of a heavy rigid body about a fixed point is dynamically symmetric. As was mentioned, for a dynamically nonsymmetric body the result follows from [3, 4].

For each of the systems in question, in [1, 2] an invariant complex real-analytic manifold $\widetilde{M}^2$ is indicated on which the system has a real-analytic first integral (the energy integral) and is integrable by quadratures. Moreover, in [1, 2] a one-parameter family Γ_k, $0 < k \leq 1$, of complex phase curves of this system is indicated which are not equilibria, on which the differentials of the known first integrals $H = (H_1, \ldots, H_l)$ are linearly independent, and for which the following assertion holds.

Proposition 1. *For almost all* k, *there exists a* $p \in \mathbb{C}^l$ *such that the monodromy group of the reduced system in variations along the phase curve* Γ_k *on the reduced phase space* $N_p\Gamma_k = \{\xi \in N\Gamma_k \mid H'(\xi) = p\}$ * *has no rational first integral.*

Proposition 1 together with the proposition stated in Sec. 2 implies the following assertion.

Proposition 2. *None of the systems in question has an additional meromorphic first integral in a domain* U *of the complexified phase space such that the fundamental group of the phase curve* Γ_k *can be represented by loops entirely lying in* U *for more than countably many values of* k.

Proof of Proposition 1. Let us show that Proposition 1 holds with $p = 0$ for all systems in question except for the problem on the motion of a heavy rigid body about a fixed point in the Goryachev–Chaplygin case.

Indeed, for the Henon–Heiles system it was proved in [1] that the desired first integral can exist only for the values of k such that the eigenvalues of a two-dimensional linear symplectic transformation depending analytically on k, which are not identically equal to $\pm i$, assume the values $\pm i$. Obviously, there are at most countably many such values of k.

For the problem on the motion of a heavy rigid body about a fixed point for the case in which the center of gravity does not lie in the equatorial plane, it was proved in [1] that the above-mentioned first integral cannot exist throughout the interval $0 < k < 1$; however, the proof given there also remains valid for any subset of this interval that has an accumulation point, in particular, for any uncountable subset.

For the problem on the motion of a heavy rigid body about a fixed point for the case in which the center of gravity lies in the equatorial plane, as well as for the Suslov problem, the results of [1, 2] imply the following.

(I) The reduced system in variations is invariant with respect to an involutive diffeomorphism that has no fixed points and is linear in the fibers of the reduced phase space (this diffeomorphism is generated by an involutive diffeomorphism of the original phase space, which is written out explicitly in [1, 2]).

(II) The monodromy group of the factor system of the reduced system by the action of the above-mentioned diffeomorphism can have a rational first integral only for the values of k such that the eigenvalues of a two-dimensional linear symplectic transformation analytically depending on k, which are not roots of unity identically, are roots of unity. Clearly, the set of such k is at most countable.

Since a linear system in a holomorphic vector bundle over a Riemann surface obviously has a meromorphic first integral if and only if its monodromy group has a rational first integral, property (II) implies the following.

(III) For almost all k, the factor system has no meromorphic first integral.

* Rigorously speaking, in the notation for the operation $H \to H'$, one should indicate the value of the parameter k of the curve Γ_k; to simplify the notation, we agree not to do so.

By a lemma in [15, Sec. 1.5], it follows from (I) and (III) that for these k the reduced system in variations also has no meromorphic first integral, whence, in turn, it follows that its monodromy group has no rational first integral, which completes the proof.

It remains to prove Proposition 1 for the Goryachev–Chaplygin case. In this situation, the Euler–Poisson system describing the motion of a heavy rigid body about a fixed point has the following form for an appropriate choice of the units of measurement and the directions of the principal inertia axes at the suspension point [1]:

$$\begin{gathered} \dot{M}_1 = 3M_2M_3\,, \quad \dot{M}_2 = -3M_1M_3 + \gamma_3\,, \quad \dot{M}_3 = -\gamma_2\,, \\ \dot{\gamma}_1 = 4M_3\gamma_2 - M_2\gamma_3\,, \quad \dot{\gamma}_2 = -4M_3\gamma_1 + M_1\gamma_3\,, \quad \dot{\gamma}_3 = M_2\gamma_1 - M_1\gamma_2\,. \end{gathered} \tag{1}$$

Here $\overrightarrow{M} = (M_1, M_2, M_3)$ and $\vec{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ are the kinetic moment of the body and the unit vector in the direction opposite to the gravitational force, respectively, defined by their projections on the principal inertia axes at the suspension point.

Let us consider system (1) in the complexified phase space $M^5 = \{x = (\overrightarrow{M}, \vec{\gamma}) \in \mathbb{C}^6 \mid \vec{\gamma}^2 = 1\}$. The system has the functionally independent general first integrals

$$H = \tfrac{1}{2}(M_1^2 + M_2^2) + 2M_3^2 + \gamma_1$$

(the energy integral) and

$$P = \langle \overrightarrow{M}, \vec{\gamma} \rangle$$

(the area integral); on the manifold $M_0^4 = \{x \in M^5 \mid P(x) = 0\}$, the system has the additional particular first integral

$$F = M_3(M_1^2 + M_2^2) - M_1\gamma_3\,.$$

System (1) has the invariant two-dimensional manifold $\widetilde{M}^2 \subset M_0^4$ defined in M^5 by the equations $M_1 = M_3 = \gamma_2 = 0$.

On $\widetilde{M}^2$, the system has the one-parameter family of solutions $x = \tilde{x}(t, k)$, $0 < k \leq 1$, given by the formulas

$$\begin{gathered} M_1 = M_3 = \gamma_2 = 0, \qquad M_2 = \widetilde{M}_2(t, k) = -2k\,\mathrm{cn}(t, k), \\ \gamma_1 = \tilde{\gamma}_1(t, k) = 2k^2\,\mathrm{sn}^2(t, k) - 1, \qquad \gamma_3 = \tilde{\gamma}_3(t, k) = 2k\,\mathrm{sn}(t, k)\,\mathrm{dn}(t, k), \qquad k = \sqrt{(1 + h)/2}, \end{gathered} \tag{2}$$

where $\mathrm{sn}(t, k)$, $\mathrm{cn}(t, k)$, and $\mathrm{dn}(t, k)$ are the Jacobi elliptic functions of modulus k and h is the energy integral constant.

These solutions are single-valued and meromorphic; if $0 < k < 1$ (which is assumed in the sequel), then they are doubly periodic in t with periods $T_{1,2}(k) = 2K(k) \pm 2iK'(k)$, where $K(k)$ is the complete elliptic integral of the first kind with modulus k, $K'(k) = K(k')$, and $k' = \sqrt{1 - k^2}$; in the periodicity cells, each of the solutions has the two poles $a_{1,2}(k) = \pm iK'(k) \pmod{T_{1,2}(k)}$. We denote the corresponding phase curves by Γ_k.

The reduced system in variations along Γ_k on the reduced phase space $N_p\Gamma_k = \{\xi \in N\Gamma_k \mid H'(\xi) = 0, P'(\xi) = p\}$, $p \in \mathbb{C}$, has the form

$$\begin{gathered} \dot{M}_1' = 3\widetilde{M}_2(t, k)M_3'\,, \qquad \dot{M}_3' = -\gamma_2'\,, \\ \dot{\gamma}_2' = \tilde{\gamma}_3(t, k)M_1' - 4\tilde{\gamma}_1(t, k)M_3'\,, \qquad \tilde{\gamma}_1(t, k)M_1' + \tilde{\gamma}_3(t, k)M_3' + \widetilde{M}_2(t, k)\gamma_2' = p\,. \end{gathered} \tag{3}$$

Let $\alpha_{i,k}\colon [0, 1] \to \Gamma_k$ $(i = 1, 2)$ be smooth loops with a common marked point such that their lifts $\beta_{i,k}\colon [0, 1] \to \mathbb{C}$ with respect to $\tilde{x}(\,\cdot\,, k)$ are, respectively, a rectilinear path such that $\beta_{1,k}(1) - \beta_{1,k}(0) = 4K(k)$ and a loop that goes around the pole $a_1(k)$ of the solution $x = \tilde{x}(t, k)$ once in the positive direction. Let $g_{i,k,p}$ be the corresponding monodromy transformations.

By a *Jordan* affine transformation we mean one having a fixed point and a Jordan block as the matrix of its linear part.

The eigenvalues and eigenvectors of an affine transformation are defined as those of its linear part.

Let $F'_{k,p}$ be the restriction of the function

$$F' = \widetilde{M}_2^2(t,k)M_3' - \tilde{\gamma}_3(t,k)M_1' \colon N\Gamma_k \to \mathbb{C}$$

to the fiber of $N_p\Gamma_k$ over $\alpha_{i,k}(0)$.

Lemma. *For almost all k and for all p, $g_{1,k,p}$ is a Jordan transformation with unit eigenvalue and with eigenvector annihilated by the 1-form $F'_{k,0}$. For all k and all $p \neq 0$, the transformation $g_{2,k,p}$ is a translation by a vector that is not annihilated by this 1-form ($g_{2,k,p}$ is the identity transformation for $p = 0$).*

It follows from the lemma (cf. [16, 17]) that the monodromy group has no rational first integral for almost all k and all $p \neq 0$. Indeed, the integral must be constant both on all lines parallel to the eigenvector of $g_{1,k,p}$ and on all lines parallel to the translation vector of $g_{2,k,p}$; hence, it is identically constant, which is what we had to prove.

Proof of the lemma. (A) *The case $p = 0$.* For $p = 0$, system (3) has the first integral F' and is integrable by quadratures. The general solution has the form

$$\begin{gathered} M_1' = \widetilde{M}_2^3(t,k)(c + 3fJ(t,k)), \qquad M_3' = \widetilde{M}_2(t,k)\tilde{\gamma}_3(t,k)(c + 3fJ(t,k)) + f\widetilde{M}_2^{-2}(t,k), \\ \gamma_2' = -(\widetilde{M}_2^2(t,k)\tilde{\gamma}_1(t,k) + \tilde{\gamma}_3^2(t,k))(c + 3fJ(t,k)) - f\widetilde{M}_2^{-3}(t,k)\tilde{\gamma}_3(t,k), \\ J(t,k) = \int_{t_0(k)}^{t} \widetilde{M}_2^{-4}(\tau,k)\,d\tau. \end{gathered} \tag{4}$$

Here c and f are arbitrary constants (f is the constant of the integral F'), $t_0 \colon (0,1) \to \mathbb{C}$ is an arbitrary function such that $t_0(k) \neq b_{m,n}(k)$, where the $b_{m,n}(k) = nK(k) + i(m+2n-1)K'(k)$ ($m, n \in \mathbb{Z}$) are the zeros and the poles of $M_2 = \widetilde{M}_2(t,k)$, and the integral is taken over an arbitrary piecewise smooth path avoiding the points $b_{m,n}(k)$.

Formula (4) implies the following.

(I) The solution is single-valued, and consequently, $g_{2,k,0} = \mathrm{id}$. Indeed, since the matrix of the right-hand side of system (3) has singularities only at the poles of the solution $x = \tilde{x}(t,k)$, it follows that the solution (4) can have a branching only at these points. However, it follows from (2) that the integrand in $J(t,k)$ is regular at these points, and consequently, the integral, as well as the entire solution (4), is single-valued.

(II) For $f = 0$, the solution is $4K(k)$-periodic, and therefore, the transformation $g_{1,k,0}$ has a fixed vector.

To prove the lemma for the case $p = 0$, it remains to show that for $f \neq 0$ and for almost all k the number $4K(k)$ is not a period of this solution, and consequently, $g_{1,k,0}$ is a Jordan transformation.

Let $\beta_k \colon [0,1] \to \mathbb{C}$ be a rectilinear path such that $\beta_k(1) - \beta_k(0) = 4K(k)$ and $\operatorname{Im}\beta_k(0) \neq nK'(k)$, $n \in \mathbb{Z}$. Set

$$\Delta(k) = J(\beta_k(1), k) - J(\beta_k(0), k).$$

We have

$$\Delta(k) = \int_{\beta_k} \widetilde{M}_2^{-4}(t,k)\,dt.$$

On passing to the new integration variable θ according to the formulas $\tilde{\gamma}_1(t,k) = -\sin\theta$, $\tilde{\gamma}_3(t,k) = \cos\theta$ (θ coincides with the Euler nutation angle for $0 \leq \theta < \pi$), we obtain

$$\Delta(k) = 2^{-5/2}\int_{\delta_k} (h + \sin\theta)^{-5/2}\,d\theta, \qquad h = 2k^2 - 1,$$

where $\delta_k \colon [0,1] \to \mathrm{S}^1_{\mathbb{C}}$ is a loop on the complexified circle $\mathrm{S}^1_{\mathbb{C}} = \mathbb{C}/2\pi\mathbb{Z}$ with angular coordinate θ (mod 2π) such that δ_k goes around the segment $I_k = [-\arcsin h, \pi + \arcsin h]$ once. The direction in which the loop is passed is related to the branch of the integrand as follows: as the loop δ_k contracts to I_k, the real part of the integrand is positive on the side of I_k on which the motion along the loop is in the direction of increase of $\operatorname{Re}\theta$.

We obtain

$$\Delta(k) = \frac{1}{3\sqrt{2}}\frac{d^2}{dh^2}\int_{\delta_k}(h+\sin\theta)^{-1/2}d\theta = \frac{2\sqrt{2}}{3}\frac{d^2}{dh^2}\int_{-\arcsin h}^{\pi/2}(h+\sin\theta)^{-1/2}d\theta = \frac{4}{3}\frac{d^2}{dh^2}K(\sqrt{(1+h)/2}).$$

Since $K(k)\to\infty$ as $k\to 1$, it follows that $\Delta(k)\neq 0$ for almost all h and hence, as desired, for almost all k.

(B) *The case* $p\neq 0$. Since for any i, k, and p the linear part of the transformation $g_{i,k,p}$ coincides with $g_{i,k,0}$, it remains to show that

(i) for almost all k and all p, the transformation $g_{1,k,p}$ preserves the linear function $F'_{k,p}$;

(ii) for all k and all $p\neq 0$, the transformation $g_{2,k,p}$ does not preserve this function.

Indeed, it follows from (A) and (i) that for almost all k and all p the transformation $g_{1,k,p}$ has a Jordan linear part and preserves the lines parallel to the eigenvector of the linear part. It follows that $g_{1,k,p}$ has a fixed point and hence itself is a Jordan transformation.

It follows from (A) and (ii) that for all k and all $p\neq 0$ the transformation $g_{2,k,p}$ is a translation by a vector not annihilated by the 1-form $F'_{k,0}$. The proof is complete.

Let f_p be a function on Γ_k such that $f_p(x)$ is equal to the value of the derivative $\dot{F}'$ of F' along the trajectory of system (1), (3) on the fiber of $N_p\Gamma_k$ over x (the derivative is constant on the fiber, since $\dot{F}'$ is linear on the fibers of the normal bundle $N\Gamma_k$ and vanishes on the reduced phase space $N_0\Gamma_k$).

On setting

$$\Delta_i(k,p) = F'_{k,p}\circ g_{i,k,p} - F'_{k,p}, \qquad i=1,2,$$

we obtain

$$\Delta_i(k,p) = \int_{\beta_{i,k}} f_p(\tilde{x}(t,k))\,dt.$$

From (1), (3), and (2) we derive

$$f_p(\tilde{x}(t,k)) = -p\widetilde{M}_2(t,k) = 2p\operatorname{cn}(t,k),$$

whence $\Delta_1(k,p)=0$ and

$$\Delta_2(k,p) = 4pk\pi i\operatorname{Res}\operatorname{cn}(iK'(k),k) = 4\pi p\neq 0$$

for $p\neq 0$. The proof of the lemma and Proposition 1 is complete.

Each of the systems in question has a real hyperbolic fixed point x_0 on the manifold $\widetilde{M}^2$. The real part of the phase curve Γ_1 contains components that are real phase curves of solutions that tend to x_0 as $t\to\pm\infty$. (For the Henon–Heiles system there is only one such component; for each of the other systems there are two such components, and their union coincides with $\operatorname{Re}\Gamma_1$.) Let Ω be the closure of the union of these components.

Proposition 3. *For any complex neighborhood $U\subset\widetilde{M}^2$ of the set Ω, there exists an $\varepsilon>0$ such that for $0<|k-1|<\varepsilon$ the fundamental group of the phase curve Γ_k can be represented by loops lying in U.*

Remark. Proposition 3 is valid for $k=1$ as well, but this is not needed in the sequel.

Propositions 2 and 3 imply that the systems in question have no additional meromorphic first integrals in any complex (and hence real) domain of the phase space containing Ω, which proves Theorems 1–3.

Proof of Proposition 3. For each of the systems in question, in [1, 2] a one-parameter family of solutions $x=\varphi(t,k)$ corresponding to the phase curves Γ_k is indicated. For $0<k<1$ these solutions are single-valued, meromorphic, doubly periodic in t, and real for real t.

Let $T(k)$ and $T'(k)$ be, respectively, the minimal real and pure imaginary periods of the solution $x = \varphi(t, k)$. For the Henon–Heiles system, they are primitive periods (that is, any period is an integral linear combination of these periods), and in each cell the solution has the single pole $T'(k)/2 \pmod{T(k), T'(k)}$. For the other systems in question, the primitive periods are $T_{1,2}(k) = (T(k) \pm T'(k))/2$, and the solution has the two poles $\pm T'(k)/4 \pmod{T_{1,2}(k)}$ in each cell.

Let $t_0(k) = T(k)/2$ for the Henon–Heiles system, and let $t_0(k) = T(k)/4$ for the other systems. Let $\gamma_k, \gamma'_k\colon [0, 1] \to \Gamma_k$ be loops with a common marked point $x_k = \varphi(t_0(k), k)$ which correspond to the variation of time along the periods $T(k)$ and $T'(k)$ of the solution $x = \varphi(t, k)$. It follows from the preceding that for the Henon–Heiles system the homotopy classes of the loops γ_k and γ'_k are the generators of the fundamental group $\pi_1(\Gamma_k)$, whereas for the other systems these loops intersect at the second point $x'_k = \varphi(t_0(k) + T(k)/2, k) = \varphi(t_0(k) + T'(k)/2, k)$, and the fundamental group $\pi_1(\Gamma_k)$ can be represented by loops consisting of half-loops of these loops with ends at the points x_k and x'_k.

It follows from the formulas given in [1, 2] that x_k tends to x_0, the loop γ_k tends to Ω, and the loop γ'_k tends to x_0 as $k \to 1$ (the last assertion also follows from the normal form of the systems in question on $\widetilde{M}^2$ in a neighborhood of x_0 or from the continuous dependence of the solution of a differential equation on the initial data and the boundedness of the period $T'(k)$ on any interval $0 < k_0 < k < 1$). The proof of Proposition 3 is complete.

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We use this occasion to correct some inaccuracies in [1, 15].

In [1], the first term in the integrand in (13) must contain $\sin 2(\chi_0 + \omega x)$ and not $\sin(2\chi_0 + \omega x)$.

In the equation $\chi^1(\pi q, \chi_0) = 0$ after Eq. (19) the expression in the parentheses must be $-1/4 + \omega^2$ instead of $-1/4 + 1/\omega^2$; hence, it vanishes at $\omega = 1/2$ and not at $\omega = 2$.

The additional first integral for the Goryachev–Chaplygin case must have the form $F = M_3(M_1^2 + M_2^2) - M_1\gamma_3$, as in the present paper, and not $F = M_3(M_1^2 + M)_2^2 - M_1\gamma_3$.

Due to a proposition in [15, Sec. 1.2] (the proposition in Sec. 2 of the present paper), the proof in [1] of the absence of a particular additional meromorphic first integral in the problem on the motion of a heavy rigid body about a fixed point for all cases except for the known integrable cases is simultaneously a proof of the absence, in these cases, of a general additional meromorphic integral; hence, the separate proof of the latter assertion in [1], as well as the remark to the Kozlov theorem in [15], can be omitted.

In the algebraic lemma in [15, Sec. 1.2], the condition of commensurability for the weights $s_1, \dots, s_m$ is missed (that is, one must have $s_i = n_i s$, $n_i \in \mathbb{Z}$, $i = 1, \dots, m$, $s \in \mathbb{R}$); this condition is needed to ensure that the process of reducing the dependence index $\mu(\Phi_1, \dots, \Phi_r)$ of the original system of functions $\Phi_1, \dots, \Phi_r$ is finite.

At the same time, the condition that the weights $s_1, \dots, s_m$ are nonnegative is unnecessary in the case of the field of rational functions and is not used in the proof. Hence, the algebraic lemma in [18] is also unnecessary.

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