Note that one can consider differential equations on the space $(N)_{\mu}^{-1}$ that are adjoint to equations of the type (3), and an analog of Theorem 2 holds.

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Basic Functions Associated with a Two-Dimensional Dirac System*

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In this paper, a special basis is constructed in the space of smooth integrable functions of two real variables. The basis functions are associated with the solution of the two-dimensional Dirac system

$$\begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix} \varphi = \frac{1}{2} \begin{pmatrix} 0 & q \\ -\bar q & 0 \end{pmatrix} \varphi, \qquad E(-kz)\varphi|_{|z|\to\infty} = I. \tag{1}$$

Here $z, k \in \mathbb{C}$, the bar stands for complex conjugation, $E(kz) = \operatorname{diag}(\exp(kz), \exp(\overline{kz}))$, and I is the identity matrix.

We introduce the following notation for sesquilinear forms with weight functions $f(z)$ and $h(k)$:

$$(\varphi^{(i)}, \psi^{(j)})_f = \iint_{\mathbb{C}} dz \wedge d\bar z\, (\bar f(z)\, \varphi_{1i}(z,k)\overline{\psi_{1j}}(z,k) + f(z)\varphi_{2i}(z,k)\overline{\psi_{2j}}(z,k)), \tag{2}$$

$$\langle \varphi_{(i)}, \psi_{(j)} \rangle_h = \iint_{\mathbb{C}} dk \wedge d\bar k (\overline{\varphi_{i1}}(z,k)\psi_{j1}(z,k)\bar h(k) - \overline{\varphi_{i2}}(z,k)\psi_{j2}(z,k)\, h(k)). \tag{3}$$

Here and in what follows, $f(z)$ and $h(k)$ are nonanalytic functions with respect to the complex variables z and k respectively.

Theorem 1. *Let q, the partial derivatives of q with respect to z and $\bar z$ of order ≤ 2, and $|q|^2$ be smooth integrable functions, and let the following condition be satisfied*:

$$\frac{1}{2\pi} \sup_{z\in\mathbb{C}} \iint dx\, dy \frac{|q(x+iy)|}{|z-(x+iy)|} < 1. \tag{4}$$

Then each function $u(z) \in C^1 \cap L_1$ can be represented in the form

$$u(z) = \frac{1}{\pi i} \langle \varphi_{(2)}, \psi_{(2)} \rangle_{\hat u}, \tag{5}$$

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where $\hat{u}(k)$ *is defined by the formula* $\hat{u} = -(\varphi^{(2)}, \psi^{(2)})_u/(4\pi i)$. *Here* ψ *is the solution of the problem adjoint to problem* (1) *with respect to the sesquilinear form* (2) *with weight function* q.

The results on the basis property of the squared solutions of the one-dimensional Dirac system are well known [1–3]. Similar results on the basis property of functions associated with a nonstationary Schrödinger equation with periodic potential were obtained in [4] for the two-dimensional case.

Problem (1) is used for integration of the Davey–Stewartson-2 system (DS-2) [5, 6]

$$i\partial_t Q + (\partial_z^2 - \partial_{\bar{z}}^2)Q + (g+\bar{g})Q = 0, \qquad \partial_{\bar{z}} g = \partial_z |Q|^2. \tag{6}$$

If the potential q in problem (1) is equal to Q, then, using formula (5), one can separate the dependence on the space and time variables in the solution of DS-2 linearized on the functions Q, g as a background:

$$i\partial_t U + 2(\partial_z^2 + \partial_{\bar{z}}^2)U + (g+\bar{g})U + (V+\bar{V})Q = F(z,t), \qquad \partial_{\bar{z}} V = \partial_z(Q\bar{U} + \bar{Q}U). \tag{7}$$

Theorem 2. *Suppose that* $q = Q$, *the function* Q *from the solution of system* (6) *satisfies the assumptions of Theorem* 1 *for* $0 \le t \le T_0$, *and* $F(z,t) \in C_1 \cap L_1$ *for any* $t \in [0, T_0]$. *Then the solution* U *of the initial boundary value problem for system* (7) *with the conditions* $U|_{t=0} = U_0(z)$, $V|_{|z|\to\infty} = 0$ *such that* $U, \partial_t U, \partial_z^\alpha U, \partial_{\bar{z}}^\alpha U, \partial_z\partial_{\bar{z}} U \in C^1 \cap L_1$, $\alpha = 1, 2$, *for any* $t \in [0, T_0]$, *has the form* $U(z,t) = -i\langle \varphi_{(2)}, \psi_{(2)}\rangle_{\widehat{U}}/\pi$, *where* $\widehat{U}$ *is the solution of the Cauchy problem*

$$\partial_t \widehat{U} - 2i(k^2 + \bar{k}^2)\widehat{U} = \widehat{F}(k,t), \qquad \widehat{U}|_{t=0} = -\frac{1}{4\pi i}(\varphi^{(2)}, \psi^{(2)})_{U_0}; \tag{8}$$

here $\widehat{F}(k,t) = -(\varphi^{(2)}, \psi^{(2)})_F/(4\pi i)$.

One can interpret the possibility of separating the variables in the solution of Eq. (7) as the result of the existence of the action–angle variables in the Hamiltonian approach to system DS-2 [7, 8]. Thus, the well-known approach to solving linearized integrable equations [1–4] can be applied to system (7).

The boundary value problem for more general systems than (1) and for small potentials of the Schwartz class was considered in [9]. In this paper, problem (1) is studied for a more general class of potentials than in [9].

Lemma 1. *Let* q *satisfy condition* (4); *then the solution of problem* (1) *exists for any* $k \in \mathbb{C}$.

To prove the lemma, we pass from problem (1) to the equivalent system of integral equations [6]

$$(I - G[q,k])\varphi = E(kz). \tag{9}$$

We shall solve system (9) in the space X' of continuous bounded matrix functions $\varphi(z,\bar{z})$ with the norm

$$\|\varphi\| = \max_{j=1,2} \sup_{z\in\mathbb{C}} (|E_{jj}(-kz)\varphi_{j1}(z,k)| + |E_{jj}(-kz)\varphi_{j2}(z,k)|).$$

The operator G is a contraction operator in X' if condition (4) is satisfied. The lemma is thereby proved.

We need the asymptotics of the solution of (1). If q satisfies the assumptions of Theorem 1, then the solution of problem (1) as $|k| \to \infty$ has the form

$$\begin{aligned} \varphi_{11}(z,k) &= \left(1 + \frac{1}{4k}\partial_{\bar{z}}^{-1}|q|^2 + O\left(\frac{1}{|k|^2}\right)\right)\exp(kz), \\ \varphi_{12}(z,k) &= \left(-\frac{1}{2\bar{k}}\bar{q} + O\left(\frac{1}{|k|^2}\right)\right)\exp(kz), \end{aligned} \tag{10}$$

where

$$\partial_{\bar{z}}^{-1}|q|^2 = \frac{1}{2\pi i}\iint_{\mathbb{C}} dw \wedge d\bar{w}\,\frac{|q(w,\bar{w})|^2}{z-w}.$$

The asymptotics of φ was used in [6] for expressing q via the scattering data of problem (1).

The matrix φ is simultaneously a solution of the so-called $\bar{D}$-problem [5]

$$\begin{pmatrix} \partial_{\bar{k}}\varphi_{11} & \partial_k\varphi_{12} \\ \partial_{\bar{k}}\varphi_{21} & \partial_k\varphi_{22} \end{pmatrix} = \varphi \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix}, \qquad \varphi E(-kz)|_{|k|\to\infty} = I. \tag{11}$$

The formula for $q(z)$ has the form [5, 6]

$$q(z) = \frac{1}{\pi i} \iint_{\mathbf{C}} dk \wedge d\bar{k}\, b(k)\, \varphi_{11}(z,k) \exp\left(\overline{-kz}\right), \tag{12}$$

where $b(k)$ is the scattering data for problem (1). It is given by the formula

$$b(k) = \frac{1}{4\pi i} \iint_{\mathbf{C}} dz \wedge d\bar{z}\, q(z)\, \varphi_{22}(z,k) \exp(-kz). \tag{13}$$

One can obtain formula (5) by computing the formal variation of the expression (12) with respect to δb and the expression (13) with respect to δq. A rigorous proof of formula (5) is given below.

Proof of Theorem 1. The right-hand side of (5) has the form

$$\begin{aligned} h = \frac{1}{\pi i} \iint_{\mathbf{C}} dk \wedge d\bar{k} \Bigg[\frac{1}{4\pi i} \iint_{\mathbf{C}} & dz' \wedge d\bar{z}' \big((u(z')\overline{\varphi_{12}}(z',k)\,\psi_{21}(z',k) \\ & + \bar{u}(z')\overline{\varphi_{22}}(z',k)\,\psi_{22}(z',k))\overline{\varphi_{21}}(z,k)\,\psi_{21}(z,k)\big) \\ - \frac{1}{4\pi i} \iint_{\mathbf{C}} & dz' \wedge d\bar{z}' \big((\bar{u}(z')\,\varphi_{12}(z',k)\overline{\psi_{12}}(z',k) \\ & + u(z')\,\psi_{22}(z',k)\overline{\psi_{22}}(z',k))\overline{\varphi_{22}}(z,k)\,\psi_{22}(z,k)\big) \Bigg]. \end{aligned}$$

The integrals with respect to z', $\bar{z}'$ converge, since the function f is integrable; one can rewrite the double integral with respect to k and $\bar{k}$ as an integral over the circle C_R of radius $R \to \infty$. Let us interchange the order of integration with respect to z', $\bar{z}'$ and k, $\bar{k}$. In the integrands, we replace the complex-conjugate entries of the matrices φ and ψ by formulas $\bar{\varphi}_{11} = \varphi_{22}$, $\bar{\varphi}_{21} = -\varphi_{12}$, $\bar{\psi}_{11} = \psi_{22}$, and $\bar{\psi}_{21} = -\psi_{12}$.

Set

$$\begin{aligned} \omega_1 &= \varphi_{22}(z',k)\,\psi_{22}(z,k)\, dk - \varphi_{21}(z',k)\,\psi_{21}(z,k)\, d\bar{k}, \\ \omega_2 &= \varphi_{11}(z,k)\,\psi_{11}(z',k)\, d\bar{k} - \varphi_{12}(z,k)\,\psi_{12}(z',k)\, dk, \\ \omega_3 &= -\varphi_{11}(z',k)\,\psi_{21}(z,k)\, d\bar{k} + \varphi_{12}(z',k)\,\psi_{22}(z,k)\, dk, \\ \omega_4 &= -\varphi_{12}(z,k)\,\psi_{22}(z',k)\, dk + \varphi_{11}(z,k)\,\psi_{21}(z',k)\, d\bar{k}. \end{aligned}$$

In this notation, we obtain

$$h = \iint_{\mathbf{C}} dz' \wedge d\bar{z}'\, u(z') \iint_{C_R} \omega_1 \wedge \omega_2 + \iint_{\mathbf{C}} dz' \wedge d\bar{z}'\, \overline{u(z')} \iint_{C_R} \omega_4 \wedge \omega_3.$$

The forms ω_j, $j = 1, \dots, 4$, are closed; this follows from system (11) and the adjoint system for ψ (with respect to the sesquilinear form (3) with weight function $b(k)$). Using the Stokes formula, we pass from the integral over the large disk to the integral over the large circle. After substituting the asymptotics of the integrand as $|k| \to \infty$ and passing to the limit as $k \to \infty$, we obtain

$$h = -\frac{1}{4\pi^2} \lim_{R\to\infty} \iint_{\mathbf{C}} dz' \wedge d\bar{z}'\, u(z') \frac{\exp\left(k(z'-z) + \overline{k(z-z')}\right)}{(z'-z)\overline{(z-z')}} = u(z).$$

The proof of Theorem 2 is reduced to justifying formula (8):

$$\partial_t \widehat{U} = (\partial_t \varphi^{(2)}, \psi^{(2)})_U + (\varphi^{(2)}, \partial_t \psi^{(2)})_U + (\varphi^{(2)}, \psi^{(2)})_{\partial_t U}. \tag{14}$$

If $q = Q$, then the matrix φ satisfies the following evolution system with respect to t [5, 6]:

$$\partial_t\varphi = \varphi\begin{pmatrix} -2ik & 0 \\ 0 & 2i\bar{k} \end{pmatrix} + \begin{pmatrix} 2i\partial_z^2 + ig & i\partial_{\bar{z}}Q - iQ\partial_{\bar{z}} \\ i\partial_z\bar{Q} - i\bar{Q}\partial_z & -2i\partial_{\bar{z}}^2 - i\bar{g} \end{pmatrix}\varphi. \tag{15}$$

We use this system and the evolution system for ψ to evaluate the derivative (14). We express the derivative $\partial_t\varphi^{(2)}$ via system (15), the derivative $\partial_t\psi^{(2)}$ via the evolution system for $\psi^{(2)}$, and the derivative $\partial_t U$ via system (7). As a result, we get an expression that does not contain derivatives with respect to t. After some transformations, we obtain (8).

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Almost Euclidean Planes in ℓ_p^n

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We denote by ℓ_p^n $(p \geq 1)$ the space $\mathbb{R}^n$ equipped with the norm

$$\|x\| = \left(\sum_{k=1}^{n} |\xi_k|^p\right)^{1/p},$$

where the ξ_k are the canonical coordinates of x. The question on the existence of Euclidean subspaces in ℓ_p^n is natural from the geometric point of view and is related to various problems arising in spectral theory, numerical analysis, representation theory of groups, etc. (see [3, 4, 6, 7]). In this note, only 2-dimensional subspaces (planes) are considered.

Theorem 1 [3]. *If ℓ_p^n contains a Euclidean plane, then $p \in 2\mathbb{N} = \{2, 4, 6, \dots\}$.*

The converse is true if n is large enough.

Theorem 2 [4, 6]. *For $p \in 2\mathbb{N}$, the space ℓ_p^n contains a Euclidean plane if and only if $n \geq p/2 + 1$.*

In the case $n \leq p/2$, it is natural to consider ε-Euclidean planes, where ε is as small as possible. Now no constraint on p is needed, and the problem is related to the classical Dvoretzky theorem [2]. According to this theorem, for every $\varepsilon > 0$ there exists an $n(\varepsilon, p)$ such that for $n \geq n(\varepsilon, p)$ the space ℓ_p^n contains an ε-Euclidean plane. In other words, for $n \geq n(\varepsilon, p)$ there exists a linear embedding $f\colon \ell_2^2 \to \ell_p^n$ satisfying the inequality $\|f\| \cdot \|f^{-1}\| \leq 1 + \varepsilon$, where $f^{-1}\colon \operatorname{Im} f \to \ell_2^2$ is the left inverse of f. If $p \notin 2\mathbb{N}$, then the two-sided estimate

$$a(p)\,\varepsilon^{-1/(p+1)} \leq n(\varepsilon, p) \leq b(p)\,\varepsilon^{-1/(p+1)} \ln^4 \varepsilon$$

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