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TORSION OF RIGID CIRCULAR SHAFT OF VARYING DIAMETER EMBEDDED IN AN ELASTIC HALF SPACE

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Abstract

The axially symmetric torsion of rigid circular shaft of varying diameter embedded in an elastic half space is studied by line-loaded integral equation method (LLIEM), where the problem is formulated by distributions of ficitious fundamental loads PRCHS (point ring couple in half space) along the axis of symmetry in interval of the shaft and is reduced to a one-dimensional and non-singular Fredholm integral equation of the first kind and is easily solved numerically. Numerical examples of torsin of rigid conic, cylinder, conical-cylinder embedded in an elastic half space are given and compared with the known result obtained by the others. The exact solution of torsion of rigid half sphere embedded in an elastic half space is also presented.

I. Introduction

The problem of torsion of a rigid cylinder embedded in an elastic half space was studied by Luco[1]. However, his work is concerned only with cylinder and the method, the extended Hankel transform he used is quite complicated. Besides, his assumptions (two more assumptions (8), (9) of [1], in addition to the bonding between the cylinder and the half space were used), the obtained integral equations (two equations) are also complicated. And, the torsion of rigid circular shalf of varying diameter embedded in an elastic half space has not been studied yet. In this paper, this problem is studied by line-loaded integral equation method (LLIEM) with only one assumption of perfect binding between the shaft and the surrounding material. The LLIEM has been used for the study of serveral problems by the author[2-8]. The advantage of the obtained integral equation is one-dimensional and non-singular. Here, the problem is again reduced to a one-dimensional and non-singular Fredholm integral equation of the first kind and is easily solved numerically. In Section 2, the solution of the fundamental load PRCHS (Point ring couple in half space) is found and the integral equation is formulated. In Section 3, the exact solution of torsion of rigid half sphere embedded in an elastic half space is presented. In Section 4, numerical examples of torsion of rigid conic, cylinder and conical-cylinder embedded in an elastic half space are given. Numerical result of torsion of rigid cylinder is compared with Lüco's [1]. In paper [9], the author proved theoretically that the numerical results of torsion of rigid cylinder shown in this paper are correct, while the result shown in Luco's paper [1] is incorrect.

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II. Solution of Prchs and Formulation of Integral Equation

Let r, θ, and z be the cylindrical coordinates. Due to the symmetry of the problem, there is no dependence upon θ, and the only nonvanishing component of the displacement vector is v, the displacement in the θ -direction. The nonzero components of stress are

$$\tau_{r\theta}=G(\partial v/\partial r-v/r),\quad \tau_{\theta z}=G\partial v/\partial z \tag{2.1}$$

where G is the shear modulus of the half space. In absence of body forces, v satisfies the equation of equilibrium

$$\partial^2 v/\partial r^2+r^{-1}\partial v/\partial r-r^{-2}v+\partial^2 v/\partial z^2=0 \tag{2.2}$$

For the LLIEM, at first, a fundamental solution of a fictitious load, where the governing linear differential equation (2.2) is satisfied, is needed. Here, a PRCHS, the simplest one of fictitious loads suitable for our problem is chosen. The definition of a PRCHS is:

Definition — The effect of a ring couple $M=2\pi af\cdot a$ formed by force per unit length f uniformly distributed on, tangent to, the circumference of a circle on $z=c$ of the half space, centered at the z-axis with radius a, is called a "Point ring couple in half space" (PRCHS) M applied on $z=c$ if the radius a approaches zero (Fig. 1). i.e., $M=\lim\limits_{a\to 0} 2\pi af\cdot a$

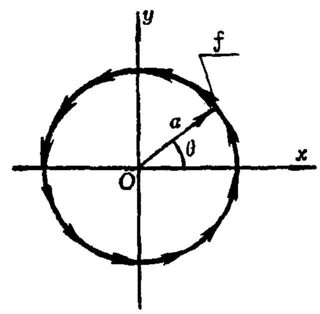

Fig. 1 Scheme of a PRCHS (a→0)

The solution of a PRCHS M applied on $z=c$ can be obtained by integration of Mindlin's[10] solution of horizontal force at a point in the interior of a half space. The solution of a PRCHS M applied on $z=c$ is now listed as follows.

$$\left.\begin{aligned}
&v(r,z)=Mr(8\pi G)^{-1}(R_{1r}^{-3}+R_{2r}^{-3})\\
&\tau_{r\theta}(r,z)=-3Mr^2(8\pi)^{-1}(R_{1r}^{-5}+R_{2r}^{-5})\\
&\tau_{\theta z}(r,z)=-3Mr(8\pi)^{-1}[(z-c)R_{1r}^{-5}+(z+c)R_{2r}^{-5}]\\
&R_{1r}=[r^2+(z-c)^2]^{1/2},\quad R_{2r}=[r^2+(z+c)^2]^{1/2}
\end{aligned}\right\} \tag{2.3}$$

where (r,z) is an observation point, $(0,c)$ is the applied point of a PRCHS, M coincides with the θ-direction.

Obviously, (2.3) satisfies the equation of equilibrium (2.2) and the boundary condition of free surface ($z=0$) of the half space embedded by a twisted shaft and possesses a simplest form. Therefore (2.3) is chosen as a fundamental solution of our problem.

The second step of LLIEM is to satisfy the boundary condition by the distributions of the fundamental loads. The boundary condition for the torsion of a rigid shaft of revolution with generator $p=p(z)$, under the assumption of perfect bonding between the shaft and the half space, is:

$$\Phi=v(p,z)/p=K,\ \text{for}\ (r,z)\in S=\{(p,z)\,|\,p=p(z),\ z\in[0,L]\} \tag{2.4}$$

where Φ is the twisting angle of a point (p,z) on the surface of the shaft, K is a given constant, S is the intersurface between the shaft and the half space, L is the embedded length.

Let PRCHS with unknown intensity function $x(c)$ be distributed along the $c(=z)$-axis on $[0,h]$

(let $h<L$, otherwise, singularity occurs at point or points of the z-axis of the half space and this is not the case of our problem), i.e., outside the region of the half space occupied, so as to make the boundary condition (2.4) to be satisfied. The components of displacement and stress of any point (r,z) in the half space due to these distributions of PRCHS can be obtained by integration of (2.3) and is listed:

$$\left.\begin{aligned} v(r,\ z)&=r(8\pi G)^{-1}\int_0^h (R_{1r}^{-3}+R_{2r}^{-3})x(c)dc \\ \tau_{r\theta}(r,z)&=-3r^2(8\pi)^{-1}\int_0^h (R_{1r}^{-5}+R_{2r}^{-5})x(c)dc \\ \tau_{\theta z}(r,\ z)&=-3r(8\pi)^{-1}\int_0^h [(z-c)R_{1r}^{-5}+(z+c)R_{2r}^{-5}]x(c)dc \end{aligned}\right\} \quad (2.5)$$

Substituting (2.5) into (2.4), we have:

$$\int_0^h (R_{1p}^{-3}+R_{2p}^{-3})x(c)dc=8\pi GK \qquad z\in[0,\ L] \quad (2.6)$$

where $R_{1p}=[p^2+(z-c)^2]^{1/2},\ R_{2p}=[p^2+(z+c)^2]^{1/2},\ p=p(z).$

(2.6) is a one-dimensional and non-singular Fredholm integral equation of the first kind and is easily solved numerically. Once $x(c)$ has been solved the components of displacement and stress of any point (r,z) in the half space can be obtained from (2.5).

The relationship between the total torque T and $x(c)$ is obtained by the equilibrium of the rigid shaft, i.e.,

$$T=\int_0^h x(c)dc=2\pi\int_0^L p^2\tau_{\theta n}(p,z)[1+(dp/dz)^2]^{-1/2}dz \quad (2.7)$$

where T is the total applied torque corresponding to the given twisted angle K.

$$\tau_{\theta n}(p,z)=\tau_{r\theta}(p,z)\cos\alpha-\tau_{\theta z}(p,z)\sin\alpha \quad (2.8)$$

$$\tan\alpha=dp/dz \quad (2.9)$$

III. Exact Solution of Torsion of Rigid Half Sphere Embedded in a Half Space

The boundary condition for torsion of a rigid half sphere with radius R_0 embedded in an elastic half space is:

$$\Phi=v(p,z)/p=K \text{ on } S=\{(p,z)\,|\,p^2+z^2=R_0^2,\ z\in[0,R_0]\} \quad (3.1)$$

If we put a PRCHS on the origin, i.e., $c=0$, then, by (2.3), we have:

$$v(r,z)/r=M(4\pi GR^3)^{-1} \quad (3.2)$$

where $R=(r^2+z^2)^{1/2}$

Obviously, (3.2) satisfies (3.1) if we let $R=R_0=$constant. Therefore (3.2) is the solution of this problem. From the equilibrium of the half sphere we get the applied torque $T=M$. Hence, the solution can be written as:

$$\left.\begin{aligned} v(r,z)&=Kr(R_0/R)^3,\ K=T(4\pi GR_0^3)^{-1} \\ \tau_{r\theta}(r,z)&=-3Tr^2(4\pi R^5)^{-1} \\ \tau_{\theta z}(r,z)&=-3Trz(4\pi R^5)^{-1} \end{aligned}\right\} \quad (3.3)$$

IV. Numerical Solution and Examples

The numerical method used here for Fredholm integral equation of the first kind (2.6) is the method of direct discretization. Let $A_x = b$ be the discrete form of $\int_0^h A(z,c)x(c)dc = F(z)$

$$Ax = b. \tag{4.1}$$

where $A = [a_{ij}]$ is an $n \times n$ matrix $b = \{b_i\}$ and $x = \{x_j\}$ are n columns.

$$a_{ij} = \int_{(j-1)t}^{jt} A(z_i;c)dc, \quad b_i = F(z_i), \quad x_j = x(c_j) \qquad (i,j=1,2,\cdots,n) \tag{4.2}$$

$t = (h-0)/n$, n is the number of linear equations to be solved.

Examples of torsion of rigid conic, cylinder, conical-cylinder have been calculated. Some significant results are listed as follows.

4.1 Torsion of embedded rigid cylinder

We are concerned most with the study of the bond stresses acting on the contact of the rigid cylinder with surrounding medium. Fig. 2 shows the distributions of stresses along the boundary. Four features should be mentioned:

1. $\tau_{r\theta}(r_0,z)$ keeps exactly constant value along the lateral surface for $0 \leqslant z < L$
2. $\tau_{\theta z}(r,L)$ linearly varies with r at end-surface for $0 \leqslant r \leqslant r_0$; the contributions of $\tau_{\theta z}(r,L)$ to the torque T is small and can be neglected especially for longer embedding ($L/r_0 > 4$). Therefore we suggest an approximate formula for evaluating $\tau_{r\theta}(r_0,z)$:

$$\tau_{r\theta}(r_0,z) = T(2\pi r_0^2 L)^{-1} \quad \text{for} \quad L/r_0 > 4 \tag{4.3}$$

3. The distribution of stress along the boundary has a discontinuous point at the corner (r_0, L).
4. The above features are independent of L/r_0.

It can be seen from Fig. 2 that there is quite a difference between our result and Luco's result (dotted line in Fig. 2). A theoretical analysis shown in another paper of the author[9] shows that our numerical result exactly coincides with the result of theoretical analysis while Luco's result is incorrect.

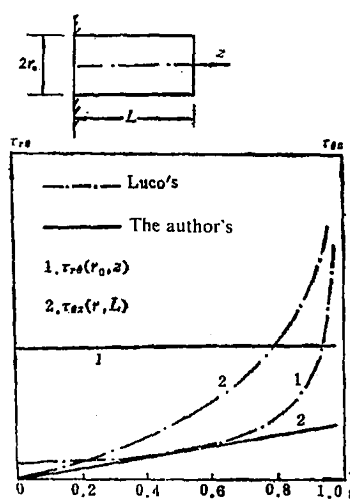

Fig. 2 Scheme of stress distribution on surfaces of a twisted rigid cylinder embedded in an elastic half space.

4.2 Torsion of embedded rigid conic

Fig. 3 shows that the stress $\tau_{\theta n}$ (p, z) linearly varies with r along the boundary. The maximum stress occurs at ($L\tan\alpha$, 0) . Since $\tau_{\theta n}(p,\ z)$ linearly varies along the boundary, we can find an expression for the maximum stress $\tau_{\theta n}(r_0,0)$ by (2.7), i.e.,

$$\tau_{\theta n}(r_0,\ 0) = \pi r_0^3 T/(2\sin\alpha) \tag{4.4}$$

where $r_0 = L\tan\alpha$.

4.3 Torsion of embedded rigid conical-cylinder

Fig. 4 shows that the features for rigid cylindrical and conical shafts are kept for the combination of cylinder and conic.

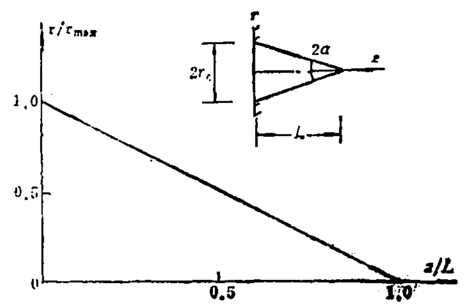

Fig. 3 Scheme of stress distribution on surface of a twisted rigid conic embedded in an elastic half space.

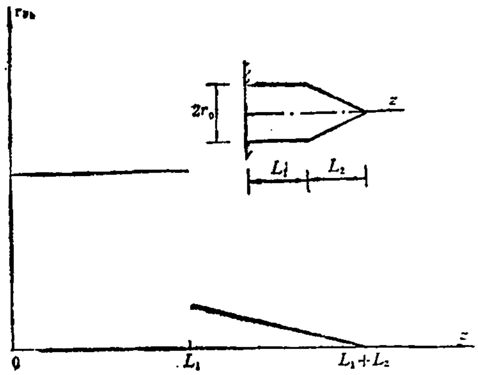

Fig. 4 Scheme of stress distribution on surfaces of a twisted rigid conical cylinder embedded in an elastic half space.

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