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FOURIER SERIES OF RATIONAL FRACTIONS OF JACOBIAN ELLIPTIC FUNCTIONS*

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Abstract

In this paper more than ninety of the Fourier series of rational fractions of Jacobian elliptic functions sn(u,k), cn(u,k) and dn(u,k) are listed, which cannot be found in the handbood[1] and Ref. [2]. For the detection and study of chaotic behavior and subharmonic bifurcations in a two-dimensional Hamiltonian system subject to external periodic forcing by Melnikov's method, and for study of some problems of physical science and engineering, these formulas can be used.

I. Introduction and Notations

In recent years, the study of a strange attractor and chaotic motions in nonlinear dynamical systems is as ref. [3] pointed out: "Indeed, there is an 18th or 19th century flavor to some of this work that is refreshingly concrete in this era of staggering abstraction." Except for physical and numerical experiments, Melnikov's method is one of the few analytical methods available for the detection and study of chaotic motions in time-periodic perturbed integrable systems. The equations of a continuous family of periodic orbits of an unperturbed integrable system can usually be expressed as rational fractions of Jacobian elliptic functions. For example, we can calculate the parameter equations of all cubic closed curves and some quartic circuits in terms of elliptic functions (see refs. [5], [6]). Thus to determine Melnikov's functions, we must calculate integrals of combinations of functions consisting of rational fractions of elliptic functions with products of the triangle functions $\sin n\omega t$ and $\cos n\omega t$. This implies that the Fourier coefficients of rational fractions of elliptic functions need to be evaluated. In ref. [1], these results can not been found. Considering the new interest in earth satellite orbit theory, earth-moon trajectory theory and close binary star system, in 1980, paper [2] gave some Fourier expansions. Unfortunately, the fundamental parameters α^2 and β in elliptic functions were confined within certain ranges. The formulas given in [2] did not completely suit the needs of applications. In the present paper we give more than ninety Fourier series of Jacobian elliptic functions differing from those in [1] and [2], in order to avoid unnecessary repetition of the amount of labour. These formulas can be provided for various application.

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We note that the notations and definitions of the elliptic integrals of the first, second, third kind, and of the Jacobian elliptic functions follow the convention in [1] and [2]. For the sake of simplicity, we write

$$W_0=\frac{\pi K'}{2K},\quad W=\frac{\pi(K'-u_0)}{2K},\quad W_1=\frac{\pi(K'-u_0/2)}{2K} \tag{1.1}$$

where u_0 depends on the following formulas. We define the further notations:

$$A(z,n)=\exp[nz]+(-1)^n\exp[-nz],\quad B(z,n)=\exp[nz]+(-1)^{n+1}\exp[-nz] \tag{1.2}$$

II. Expansions of Fourier Series

According to five different kinds, we give the expansions as follows.

(I) The power functions of sn(u,k), cn(u,k) and dn(u,k)

I. 1.1 $$\mathrm{sn}^2u=\frac{K-E}{k^2K}-\frac{\pi^2}{k^2K^2}\sum_{n=1}^{\infty}n\,\mathrm{csch}\,2nW_0\cos\frac{n\pi u}{K}$$

I. 1.2 $$\mathrm{sn}^3u=\frac{\pi}{8k^3K^3}\sum_{n=0}^{\infty}[4K^2(1+k^2)-(2n+1)^2\pi^2]\mathrm{csch}(2n+1)W_0\cdot\sin\frac{(2n+1)\pi u}{2K}$$

I. 1.3 $$\mathrm{sn}^4u=\frac{(2+k^2)K-2(1+k^2)E}{3k^4K}+\frac{\pi^2}{3k^4K^2}\sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{K}\right)^2+6n+(4-2k^2)\right]\cdot\mathrm{csch}\,2nW_0\cdot\cos\frac{n\pi u}{K}$$

I. 2.1 $$\mathrm{cn}^2u=\frac{E-k'^2K}{k^2K}+\frac{\pi}{k^2K^2}\sum_{n=1}^{\infty}n\,\mathrm{csch}\,2nW_0\cos\frac{n\pi u}{K}$$

I. 2.2 $$\mathrm{cn}^3u=\frac{\pi}{8k^3K^3}\sum_{n=0}^{\infty}[(2n+1)^2\pi^2-4K^2(1-2k^2)]\mathrm{sech}(2n+1)W_0\cdot\cos\frac{(2n+1)\pi u}{2K}$$

I. 2.3 $$\mathrm{cn}^4u=\frac{(2-3k^2)k'^2K+2(2k^2-1)E}{3k^4K}+\frac{\pi^2}{3k^4K^2}\sum_{n=1}^{\infty}\left[\left(\frac{n\pi}{K}\right)^2+4(2-k^2)\right]\cdot\mathrm{csch}\,2nW_0\cdot\cos\frac{n\pi u}{K}$$

I. 3.1 $$\mathrm{dn}^2u=\frac{E}{K}+\frac{\pi^2}{K^2}\sum_{n=1}^{\infty}n\,\mathrm{csch}\,2nW_0\cos\frac{n\pi u}{K}$$

I. 3.2 $$\mathrm{dn}^3u=\frac{\pi(2-k^2)}{4K}+\frac{\pi}{2K}\sum_{n=1}^{\infty}\left(\left(\frac{n\pi}{K}\right)^2+2-k^2\right)\mathrm{sech}\,2nW_0\cos\frac{n\pi u}{K}$$

I . 3.3 $$\mathrm{dn}^4 u=\frac{2(2-k^2)E-(1-k^2)K}{3K}+\frac{\pi^2}{6K^2}\sum_{n=1}^{\infty}n\left[\left(\frac{n\pi}{K}\right)^2+4(2-k^2)\right]\cdot\mathrm{csch}2nW_0\cos\frac{n\pi u}{K}$$

I . 3.4 $$\frac{1}{\mathrm{dn}^2 u}=\frac{E}{k'^2K}+\frac{\pi^2}{k'^2K^2}\sum_{n=1}^{\infty}(-1)^n n\,\mathrm{csch}2nW_0\cos\frac{n\pi u}{K}$$

I . 3.5 $$\frac{1}{\mathrm{dn}^3 u}=\frac{(2-k^2)\pi}{4k'^3K}+\frac{\pi}{2k'^3K}\sum_{n=1}^{\infty}(-1)^n\left[\left(\frac{n\pi}{K}\right)^2+2-k^2\right]\mathrm{sech}2nW_0\cos\frac{n\pi u}{K}$$

I . 3.6 $$\frac{1}{\mathrm{dn}^4 u}=\frac{2(2-k^2)E-(1-k^2)K}{3k'^4K}+\frac{\pi^2}{6k'^4K^2}\sum_{n=1}^{\infty}(-1)^n n\left[\left(\frac{n\pi}{K}\right)^2+4(2-k^2)\right]\mathrm{csch}2nW_0\cos\frac{n\pi u}{K}$$

(II) The functions of $(1\pm\beta\mathrm{sn}u)^{-l}$, $(l=1,2)$

Case 1 $0<\beta<k$ Define u_0 by $\mathrm{dn}(u_0,k)=k'/\sqrt{1-\beta^2}$, $0<u_0<K$.

II . 1.1 $$\frac{1}{1\pm\beta\mathrm{sn}u}=\frac{\Pi(\beta^2,k)}{K}+\frac{\beta\pi}{K\sqrt{(1-\beta^2)(k^2-\beta^2)}}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{2K}\cdot\mathrm{csch}nW_0\left(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

II . 2.1 $$\frac{1}{(1\pm\beta\mathrm{sn}u)^2}=\frac{c_0}{4K}-\frac{\beta^2\pi}{K(1-\beta^2)(k^2-\beta^2)}\sum_{n=1}^{\infty}\mathrm{csch}\,nW_0\left[\frac{n\pi}{2K}\cos\frac{n\pi u_0}{2K}-\frac{(2k^2-\beta^2-k^2\beta^2)}{\beta\sqrt{(1-\beta^2)(k^2-\beta^2)}}\sin\frac{n\pi u_0}{2K}\right]\left(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

where

$$c_0=\frac{4}{(1-\beta^2)(k^2-\beta^2)}[(1-\beta^2)(2k^2-\beta^2-k^2\beta^2)\Pi(\beta^2,k)-\beta^2E+(\beta^2-k^2)F]$$

II . 3.1 $$\frac{\mathrm{cn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{K\sqrt{k^2-\beta^2}}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{2K}\mathrm{sech}W_0\left(\sin\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\pm\cos\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

II . 4.1 $$\frac{\mathrm{cn}u}{(1\pm\beta\mathrm{sn}u)^2}=-\frac{\beta\pi}{K\sqrt{1-\beta^2}(k^2-\beta^2)}\sum_{n=1}^{\infty}\mathrm{sech}nW_0\left(\frac{n\pi}{2K}\cos\frac{n\pi u_0}{2K}-\frac{k^2\sqrt{1-\beta^2}}{\beta\sqrt{k^2-\beta^2}}\sin\frac{n\pi u_0}{2K}\right)\left(\sin\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\right.$$

$$\pm\cos\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

II. 5.1 $$\frac{\mathrm{dn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{K\sqrt{1-\beta^2}}\sum_{n=0}^{\infty}\cos\frac{n\pi u_0}{2K}\mathrm{sech}nW_0\Big(\cos\frac{n\pi}{2}$$
$$\cdot\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

II. 6.1 $$\frac{\mathrm{dn}u}{(1\pm\beta\mathrm{sn}u)^2}=\frac{\beta\pi}{K(1-\beta^2)\sqrt{k^2-\beta^2}}\sum_{n=0}^{\infty}\mathrm{sech}nW_0\Big(\frac{n\pi}{2K}\sin\frac{n\pi u_0}{2K}+\frac{\sqrt{k^2-\beta^2}}{\beta\sqrt{1-\beta^2}}$$
$$\cdot\cos\frac{n\pi u_0}{2K}\Big)\Big(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

II. 7.1 $$\frac{\mathrm{cn}u\mathrm{dn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{2\beta K}\sum_{n=1}^{\infty}\mathrm{csch}nW_0\Big\{2\cos\frac{n\pi u_0}{2K}\sin\frac{n\pi}{2}\cos\frac{n\pi u}{2K}$$
$$\mp\Big[1+(-1)^n-2\cos\frac{n\pi u_0}{2K}\cos\frac{n\pi}{2}\Big]\sin\frac{n\pi u}{2K}\Big\}$$

II. 8.1 $$\frac{\mathrm{cn}u\mathrm{dn}u}{(1\pm\beta\mathrm{sn}u)^2}=\frac{\pi}{K\sqrt{(1-\beta^2)(k^2-\beta^2)}}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{2K}\mathrm{csch}nW_0$$
$$\cdot\Big(\pm\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}-\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

Case 2 $k<\beta<1$. Define u_0 by $\mathrm{dn}(u_0,k')=k/\beta$. $0<u_0<K'$.

II. 1.2 $$\frac{1}{1\pm\beta\mathrm{sn}u}=\frac{\Pi(\beta^2,k)}{K}+\frac{\beta\pi}{K\sqrt{(1-\beta^2)(\beta^2-k^2)}}\sum_{n=1}^{\infty}\mathrm{sh}\frac{n\pi u_0}{2K}$$
$$\cdot\mathrm{csch}nW_0\Big(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

II. 2.2 $$\frac{1}{(1\pm\beta\mathrm{sn}u)^2}=\frac{c_0}{4K}+\frac{\beta^2\pi}{K(1-\beta^2)(k^2-\beta^2)}\sum_{n=1}^{\infty}\mathrm{ch}\frac{n\pi u_0}{2K}\mathrm{csch}nW_0$$
$$\cdot\Big[\frac{n\pi}{2K}+\frac{(\beta^2-2k^2+k^2\beta^2)}{\beta\sqrt{(1-\beta^2)(\beta^2-k^2)}}\Big]\Big(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

where c_0 is the same as II. 2.1.

II. 3.2 $$\frac{\mathrm{cn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{K\sqrt{\beta^2-k^2}}\sum_{n=1}^{\infty}\mathrm{sh}\frac{n\pi u_0}{2K}\mathrm{sech}nW_0\Big(\sin\frac{n\pi}{2}$$
$$\cdot\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

II. 4.2 $$\frac{\mathrm{cn}u}{(1\pm\beta\mathrm{sn}u)^2}=\frac{\beta\pi}{K(\beta^2-k^2)\sqrt{1-\beta^2}}\sum_{n=1}^{\infty}\mathrm{sech}nW_0\Big[\frac{n\pi}{2K}\mathrm{ch}\frac{n\pi u_0}{2K}$$
$$-\frac{k\sqrt{1-\beta^2}}{\beta\sqrt{\beta^2-k^2}}\mathrm{sh}\frac{n\pi u_0}{2K}\Big]\Big(\sin\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\cos\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\Big)$$

II. 5.2 $$\frac{\mathrm{dn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{K\sqrt{1-\beta^2}}\sum_{n=0}^{\infty}{}'\mathrm{ch}\frac{n\pi u_0}{2K}\mathrm{sech}nW_0\left(\cos\frac{n\pi}{2}\cdot\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

II. 6.2 $$\frac{\mathrm{dn}u}{(1\pm\beta\mathrm{sn}u)^2}=\frac{\beta\pi}{K(1-\beta^2)\sqrt{\beta^2-k^2}}\sum_{n=0}^{\infty}{}'\mathrm{sech}nW_0\left[\frac{n\pi}{2K}\mathrm{sh}\frac{n\pi u_0}{2K}+\frac{\sqrt{\beta^2-k^2}}{\beta\sqrt{1-\beta^2}}\mathrm{ch}\frac{n\pi u_0}{2K}\right]\left[\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right]$$

II. 7.2 $$\frac{\mathrm{cn}u\mathrm{dn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{2\beta K}\sum_{n=1}^{\infty}\mathrm{csch}nW_0\left\{2\mathrm{ch}\frac{n\pi u_0}{2K}\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi u}{2K}\mp\left[1+(-1)^n-2\mathrm{ch}\frac{n\pi u_0}{2K}\cos\frac{n\pi}{2}\right]\sin\frac{n\pi u}{2K}\right\}$$

II. 8.2 $$\frac{\mathrm{cn}u\mathrm{dn}u}{(1\pm\beta\mathrm{sn}u)^2}=\frac{\pi}{K\sqrt{(1-\beta^2)(\beta^2-k^2)}}\sum_{n=1}^{\infty}\mathrm{csch}nW_0\mathrm{sh}\frac{n\pi u_0}{2K}\cdot\left[\pm\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}-\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right]$$

Case 3 $1<\beta<+\infty$. Define u_0 by $\mathrm{cn}(u_0,k)=k'/\sqrt{\beta^2-k^2}$, $0<u_0<K$.

II. 1.3 $$\frac{1}{1\pm\beta\mathrm{sn}u}=\frac{\Pi(\beta^2,k)}{K}-\frac{\beta\pi}{K\sqrt{(\beta^2-1)(\beta^2-k^2)}}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{2K}\cdot\mathrm{cth}nW_0\left(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

II. 3.3 $$\frac{\mathrm{cn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{K\sqrt{\beta^2-k^2}}\sum_{n=1}^{\infty}\cos\frac{n\pi u_0}{2K}\mathrm{th}\,nW_0\left(\sin\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\pm\cos\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

II. 5.3 $$\frac{\mathrm{dn}u}{1\pm\beta\mathrm{sn}u}=-\frac{\pi}{K\sqrt{\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{2K}\mathrm{th}\,nW_0\left(\cos\frac{n\pi}{2}\cos\frac{n\pi u}{2K}\mp\sin\frac{n\pi}{2}\sin\frac{n\pi u}{2K}\right)$$

II. 7.3 $$\frac{\mathrm{cn}u\mathrm{dn}u}{1\pm\beta\mathrm{sn}u}=\frac{\pi}{2\beta K}\sum_{n=1}^{\infty}\mathrm{csch}nW_0\left\{2\cos\frac{n\pi u_0}{2K}\mathrm{ch}nW_0\sin\frac{n\pi}{2}\cdot\cos\frac{n\pi u}{2K}\mp\left[(-1)^n+1-2\cos\frac{n\pi u_0}{2K}\mathrm{ch}nW_0\cos\frac{n\pi}{2}\right]\sin\frac{n\pi u}{2K}\right\}$$

(III) The functions with $(1\pm\beta\mathrm{cn}u)^{-l}$, $(l=1,2)$

Case 1 $0<\beta<1$. Define u_0 by $\mathrm{cn}(u_0,k')=\beta$, $0<u_0<K'$. The Fourier series of the functions $1/(1\pm\beta\mathrm{cn}u)$, $1/(1\pm\beta\mathrm{cn}u)^2$, $\mathrm{sn}u/(1\pm\beta\mathrm{cn}u)$, $\mathrm{dn}u/(1\pm\beta\mathrm{cn}u)$, $\mathrm{cn}u\mathrm{dn}u/(1\pm\beta\mathrm{cn}u)$ can be found in ref. [2].

Ⅲ. 4.1
$$\frac{\mathrm{sn}u}{(1\pm\beta\mathrm{cn}u)^2}=\frac{\mp\pi}{K(1-\beta^2)(k^2+k'^2\beta^2)}\sum_{n=1}^{\infty}\frac{1}{A(W_0,n)}\left\{\frac{n\pi\beta\sqrt{1-\beta^2}}{2K}A(W;\mp n)-\frac{k^2(1-\beta^2)^2}{\sqrt{k^2+k'^2\beta^2}}B(W;\mp n)\right\}\sin\frac{n\pi u}{2K}$$

Ⅲ. 6.1
$$\frac{\mathrm{dn}u}{(1\pm\beta\mathrm{cn}u)^2}=\frac{\pi}{2K(1-\beta^2)^{3/2}(k^2+k'^2\beta^2)^{\frac{1}{2}}}\mp\frac{\pi}{K(1-\beta^2)(k^2+k'^2\beta)}\cdot\sum_{n=1}^{\infty}\left[\frac{n\pi\beta\sqrt{k^2+k'^2\beta^2}}{2K}B(W;\mp n)\mp\frac{(k^2+k'^2\beta^2)}{1-\beta^2}A(W;\mp n)\right]\frac{\cos(n\pi u/2K)}{A(W_0,n)}$$

Ⅲ. 8.1
$$\frac{\mathrm{sn}u\mathrm{dn}u}{(1\pm\beta\mathrm{cn}u)^2}=\frac{\pi^2}{2K^2\sqrt{(1-\beta^2)(k^2+k'^2\beta^2)}}\sum_{n=1}^{\infty}\frac{nB(W,\mp n)}{B(W_0,n)}\sin\frac{n\pi u}{2K}$$

Case 2 $1<\beta<+\infty$. Define u_0 by $\mathrm{cn}(u_0,k)=1/\beta$, $0<u_0<K$.

Ⅲ. 1.2
$$\frac{1}{1\pm\beta\mathrm{cn}u}=\frac{\pi}{(1-\beta^2)K}-\frac{\beta\pi}{K\sqrt{(\beta^2-1)(k^2+k'^2\beta^2)}}\sum_{n=1}^{\infty}\frac{A(W_0,\mp n)}{B(W_0,n)}\cdot\sin\frac{n\pi u_0}{2K}\cos\frac{n\pi u}{2K}$$

Ⅲ. 3.2
$$\frac{\mathrm{sn}u}{1\pm\beta\mathrm{cn}u}=\pm\frac{\pi}{K\sqrt{k^2+k'^2\beta^2}}\sum_{n=1}^{\infty}\frac{B(W,\mp n)}{A(W_0,n)}\cos\frac{n\pi u_0}{2K}\sin\frac{n\pi u}{2K}$$

Ⅲ. 5.2
$$\frac{\mathrm{dn}u}{1\pm\beta\mathrm{cn}u}=\pm\frac{\pi}{K\sqrt{\beta^2-1}}\sum_{n=1}^{\infty}\frac{B(W,\mp n)}{A(W_0,n)}\cos\frac{n\pi u_0}{2K}\sin\frac{n\pi u}{2K}$$

Ⅲ. 7.2
$$\frac{\mathrm{sn}u\mathrm{dn}u}{1\pm\beta\mathrm{cn}u}=\pm\frac{\pi}{K\beta}\sum_{n=1}^{\infty}\frac{[1+(-1)^n-A(W,\mp n)]}{B(W_0,n)}\cos\frac{n\pi u_0}{2K}\sin\frac{n\pi u}{2K}$$

(IV) The functions with $(1\pm\beta\mathrm{dn}u)^{-l}$, $(l=1,2)$

Case 1 $0<\beta<1$. Define u_0 by $\mathrm{dn}(u_0,k')=k/\sqrt{1-k'^2\beta^2}$, $0<u_0<K'$.

Ⅳ. 1.1
$$\frac{1}{1\pm\beta\mathrm{dn}u}=\frac{c_0}{2K}+K\sqrt{\frac{2\beta\pi}{(1-\beta^2)(1-k'^2\beta^2)}}\sum_{n=1}^{\infty}\mathrm{sh}\frac{n\pi u_0}{K}\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2}{1-\beta^2}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)-\frac{\beta\pi}{\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}$$

IV. 2.1 $$\frac{1}{(1+\beta \mathrm{dn} u)^2}=\frac{c_0}{2K}+\frac{2\beta\pi}{K(1-\beta^2)(1-k'^2\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{K}\mathrm{ch}\frac{n\pi u_0}{K}-\frac{2-\beta^2-k'^2\beta^2}{\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}\mathrm{sh}\frac{n\pi u_0}{K}\right\}\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2(\beta^2E-K)}{(1-\beta^2)(1-k'^2\beta^2)}+\frac{2(2-2\beta^2+k^2\beta^2)}{(1-\beta^2)^2(1-k'^2\beta^2)}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)+\frac{\beta\pi(2-2\beta^2+\beta^2k^2)}{[(1-\beta^2)(1-k'^2\beta^2)]^{3/2}}-\frac{2k\beta^2}{(1-\beta^2)^{3/2}(1-k'^2\beta^2)}\mathrm{arc\,tg}\frac{\beta k}{\sqrt{1-\beta^2}}$$

IV. 3.1 $$\frac{1}{1-\beta \mathrm{dn} u}=\frac{c_0}{2K}+\frac{2\beta\pi}{K\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}\sum_{n=1}^{\infty}\mathrm{sh}4nW_1\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2}{1-\beta^2}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)+\frac{\beta\pi}{\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}$$

IV. 4.1 $$\frac{1}{(1-\beta \mathrm{dn} u)^2}=\frac{c_0}{2K}+\frac{2\beta\pi}{K\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{K}\mathrm{ch}4nW_1+\frac{2-\beta^2-k'^2\beta^2}{\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}\mathrm{sh}4nW_1\right\}\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2(\beta^2E-K)}{(1-\beta^2)(1-k'^2\beta^2)}+\frac{2(2-2\beta^2+\beta^2k^2)}{(1-\beta^2)^2(1-k'^2\beta^2)}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)-\frac{\beta\pi(2-2\beta^2+\beta^2k^2)}{[(1-\beta^2)(1-k'^2\beta^2)]^{3/2}}+\frac{2k\beta^2}{(1-\beta^2)^{3/2}(1-k'^2\beta^2)}\mathrm{arc\,tg}\frac{\beta k}{\sqrt{1-\beta^2}}$$

IV. 5.1 $$\frac{\mathrm{sn} u}{1+\beta \mathrm{dn} u}=\frac{2\pi}{kK\sqrt{1-k'^2\beta^2}}\sum_{n=1}^{\infty}\mathrm{ch}\frac{(2n-1)\pi u_0}{2K}\cdot\mathrm{csch}2(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

IV. 6.1 $$\frac{\mathrm{sn} u}{(1+\beta \mathrm{dn} u)^2}=\frac{\pi}{kK(1-k'^2\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{[(-1)^n-1]n\beta\pi}{2K\sqrt{1-\beta^2}}\mathrm{sh}\frac{n\pi u_0}{2K}+\frac{[(-1)^n+1]}{\sqrt{1-k'^2\beta^2}}\mathrm{ch}\frac{n\pi u_0}{2K}\right\}\mathrm{csch}2nW_0\sin\frac{n\pi u}{2K}$$

Ⅳ. 7.1
$$\frac{\operatorname{sn}u}{1-\beta\operatorname{dn}u}=\frac{2\pi}{kK\sqrt{1-k'^2\beta^2}}\sum_{n=1}^{\infty}\operatorname{ch}2(2n-1)W_1\operatorname{csch}2(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

Ⅳ. 8.1
$$\frac{\operatorname{sn}u}{(1-\beta\operatorname{dn}u)^2}=\frac{\pi}{kK(1-k'^2\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{[1-(-1)^n]n\beta\pi}{2K\sqrt{1-\beta^2}}\operatorname{sh}2nW_1+\frac{[1+(-1)^n]}{\sqrt{1-k'^2\beta^2}}\operatorname{ch}2nW_1\right\}\operatorname{csch}2nW_0\sin\frac{n\pi u}{2K}$$

Ⅳ. 9.1
$$\frac{\operatorname{cn}u}{(1+\beta\operatorname{dn}u)}=\frac{2\pi}{kK\sqrt{1-\beta^2}}\sum_{n=1}^{\infty}\operatorname{sh}\frac{(2n-1)\pi u_0}{2K}\cdot\operatorname{csch}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

Ⅳ.10.1
$$\frac{\operatorname{cn}u}{(1+\beta\operatorname{dn}u)^2}=-\frac{2\pi}{kK(1-\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{(2n-1)\beta\pi}{2K\sqrt{1-k'^2\beta^2}}\operatorname{ch}\frac{(2n-1)\pi u_0}{2K}+\frac{1}{\sqrt{1-\beta^2}}\operatorname{sh}\frac{(2n-1)\pi u_0}{2K}\right\}\operatorname{csch}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

Ⅳ.11.1
$$\frac{\operatorname{cn}u}{1-\beta\operatorname{dn}u}=\frac{2\pi}{kK\sqrt{1-\beta^2}}\sum_{n=1}^{\infty}\operatorname{sh}2(2n-1)W_1\operatorname{csch}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

Ⅳ.12.1
$$\frac{\operatorname{cn}u}{(1-\beta\operatorname{dn}u)^2}=\frac{2\pi}{kK(1-\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{(2n-1)\beta\pi}{2K\sqrt{1-k'^2\beta^2}}\operatorname{ch}2(2n-1)W_1-\frac{1}{\sqrt{1-\beta^2}}\operatorname{sh}2(2n-1)W_1\right\}\operatorname{csch}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

Ⅳ.13.1
$$\frac{\operatorname{sn}u\operatorname{cn}u}{1+\beta\operatorname{dn}u}=\frac{2\pi}{\beta k^2K}\sum_{n=1}^{\infty}\left[\operatorname{sh}2nW_0-\operatorname{ch}\frac{n\pi u_0}{K}\right]\operatorname{csch}4nW_0\sin\frac{n\pi u}{K}$$

Ⅳ.14.1
$$\frac{\operatorname{sn}u\operatorname{cn}u}{(1+\beta\operatorname{dn}u)^2}=\frac{2\pi^2}{k^2K\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}\sum_{n=1}^{\infty}n\operatorname{sh}\frac{n\pi u_0}{K}\operatorname{csch}4nW_0\sin\frac{n\pi u}{K}$$

Ⅳ.15.1
$$\frac{\operatorname{sn}u\operatorname{cn}u}{1-\beta\operatorname{dn}u}=\frac{2\pi}{\beta k^2K}\sum_{n=1}^{\infty}[\operatorname{ch}2nW_1-\operatorname{sh}2nW_0]\operatorname{csch}4nW_0\sin\frac{n\pi u}{K}$$

Ⅳ.16.1
$$\frac{\operatorname{sn}u\operatorname{cn}u}{(1-\beta\operatorname{dn}u)^2}=\frac{2\pi^2}{k^2K\sqrt{(1-\beta^2)(1-k'^2\beta^2)}}\sum_{n=1}^{\infty}n\operatorname{sh}4nW_1\operatorname{csch}4nW_0\sin\frac{n\pi u}{K}$$

Case 2 $1<\beta<1/k'$. Define u_0 by $\operatorname{dn}(u_0,k)=1/\beta$, $0<u_0<K$.

Ⅳ. 1.2 $$\frac{1}{1+\beta \mathrm{dn}u}=\frac{c_0}{2K}-\frac{2\beta\pi}{K\sqrt{(\beta^2-1)(1-k'^2\beta^2)}}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{K}\cdot\operatorname{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2}{1-\beta^2}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)$$

Ⅳ. 2.2 $$\frac{1}{(1+\beta \mathrm{dn}u)^2}=\frac{c_0}{2K}-\frac{2\beta\pi}{K(\beta^2-1)(1-k'^2\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{K}\cos\frac{n\pi u_0}{K}-\frac{(2-\beta^2-k'^2\beta^2)}{\sqrt{(\beta^2-1)(1-k'^2\beta^2)}}\sin\frac{n\pi u_0}{K}\right\}\operatorname{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2(\beta^2E-K)}{(1-\beta^2)(1-k'^2\beta^2)}+\frac{2(2-2\beta^2+\beta^2k^2)}{(1-\beta^2)^2(1-k'^2\beta^2)}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)-\frac{k\beta^2}{(\beta^2-1)^{3/2}(1-k'^2\beta^2)}\ln\frac{\beta k+\sqrt{\beta^2-1}}{\beta k+\sqrt{\beta^2-1}}$$

Ⅳ. 3.2 $$\frac{1}{1-\beta \mathrm{dn}u}=\frac{c_0}{2K}-\frac{2\pi}{K}\sum_{n=1}^{\infty}\sin\frac{n\pi u_0}{K}\operatorname{cth}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2}{1-\beta^2}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)$$

Ⅳ. 5.2 $$\frac{\mathrm{sn}u}{1+\beta \mathrm{dn}u}=\frac{2\pi}{kK\sqrt{1-k'^2\beta^2}}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi u_0}{2K}\cdot\operatorname{csch}2(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

Ⅳ. 6.2 $$\frac{\mathrm{sn}u}{(1+\beta \mathrm{dn}u)^2}=-\frac{2\pi}{kK(1-k'^2\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{(2n-1)\beta\pi}{2K\sqrt{\beta^2-1}}\sin\frac{(2n-1)\pi u_0}{2K}-\frac{1}{\sqrt{1-k'^2\beta^2}}\cos\frac{(2n-1)\pi u_0}{2K}\right\}\operatorname{csch}2(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

Ⅳ. 7.2 $$\frac{\mathrm{sn}u}{1-\beta \mathrm{dn}u}=\frac{2\pi}{kK\sqrt{1-k'^2\beta^2}}\sum_{n=1}^{\infty}\cos\frac{(2n-1)\pi u_0}{2K}\cdot\operatorname{cth}2(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

IV. 9.2 $$\frac{\mathrm{cn}u}{1+\beta\mathrm{dn}u}=\frac{2\pi}{kK\sqrt{\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi u_0}{2K}\cdot\mathrm{csch}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

IV.10.2 $$\frac{\mathrm{cn}u}{(1+\beta\mathrm{dn}u)^2}=\frac{2\pi}{kK(\beta^2-1)}\sum_{n=1}^{\infty}\left\{\frac{(2n-1)\beta\pi}{2K\sqrt{1-k'^2\beta^2}}\cos\frac{(2n-1)\pi u_0}{2K}-\frac{1}{\sqrt{\beta^2-1}}\sin\frac{(2n-1)\pi u_0}{2K}\right\}\mathrm{csch}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

IV.11.2 $$\frac{\mathrm{cn}u}{1-\beta\mathrm{dn}u}=-\frac{2\pi}{kK\sqrt{\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{(2n-1)\pi u_0}{2K}\cdot\mathrm{cth}2(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

IV.13.2 $$\frac{\mathrm{sn}u\mathrm{cn}u}{1+\beta\mathrm{dn}u}=\frac{2\pi}{\beta k^2K}\sum_{n=1}^{\infty}\left\{\mathrm{sh}2nW_0-\cos\frac{n\pi u_0}{K}\right\}\mathrm{csch}4nW_0\sin\frac{n\pi u}{K}$$

IV.14.2 $$\frac{\mathrm{sn}u\mathrm{cn}u}{(1+\beta\mathrm{dn}u)^2}=\frac{2\pi^2}{k^2K\sqrt{(\beta^2-1)(1-k'^2\beta^2)}}\sum_{n=1}n\sin\frac{n\pi u_0}{K}\cdot\mathrm{csch}4nW_0\sin\frac{n\pi u}{K}$$

IV.15.2 $$\frac{\mathrm{sn}u\mathrm{cn}u}{1-\beta\mathrm{dn}u}=\frac{2\pi}{\beta k^2K}\sum_{n=1}^{\infty}\left\{\cos\frac{n\pi u_0}{K}\mathrm{ch}4nW_0-\mathrm{sh}2nW_0\right\}\cdot\mathrm{csch}4nW_0\sin\frac{n\pi u}{2K}$$

Case 3 $1/k'<\beta<+\infty$. Define u_0 by $\mathrm{dn}(u_0,k')=k\beta/\sqrt{\beta^2-1}$, $0<u_0<K'$

IV.1.3 $$\frac{1}{1+\beta\mathrm{dn}u}=\frac{c_0}{2K}+\frac{2\beta\pi}{K\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}\sum_{n=1}^{\infty}\mathrm{sh}\frac{n\pi u_0}{K}\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2}{1-\beta^2}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)+\frac{\beta\pi}{\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}$$

IV 2.3 $$\frac{1}{(1+\beta\mathrm{dn}u)^2}=\frac{c_0}{2K}+\frac{2\beta\pi}{K(\beta^2-1)(k'^2\beta^2-1)}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{n\beta\pi}{K}\mathrm{ch}\frac{n\pi u_0}{K}+\frac{(2-\beta^2-k'^2\beta^2)}{\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}\mathrm{sh}\frac{n\pi u_0}{K}\right\}\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2(\beta^2E-K)}{(1-\beta^2)(1-k'^2\beta^2)}+\frac{2(2-2\beta^2+\beta^2k^2)}{(1-\beta^2)^2(1-k'^2\beta^2)}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},k\right)$$
$$-\frac{\beta(2\beta k+\sqrt{\beta^2-1})}{(\beta^2-1)^{3/2}(k'^2\beta^2-1)}-\frac{1}{2k(\beta^2-1)^{3/2}}\ln\frac{\sqrt{\beta^2-1}-\beta k}{\sqrt{\beta^2-1}+\beta k}$$
$$+\frac{\beta\pi(4-4\beta^2+3\beta^2k^2)}{2[(\beta^2-1)(k'^2\beta^2-1)]^{3/2}}-\frac{2\beta^3k^2}{[(\beta^2-1)(k'^2\beta^2-1)]^{3/2}}\operatorname{arctg}\frac{\beta k}{\sqrt{\beta^2-1}}$$

Ⅳ.3.3 $$\frac{1}{1-\beta\,\mathrm{dn}u}=\frac{c_0}{2K}+\frac{2\beta\pi}{K\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{sh}4nW_1$$
$$\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2}{1-\beta^2}\Pi\left(\frac{\beta^2k^2}{\beta^2-1},\ k\right)-\frac{\beta\pi}{\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}$$

Ⅳ.4.3 $$\frac{1}{(1-\beta\,\mathrm{dn}u)^2}=\frac{c_0}{2K}+\frac{2\beta\pi}{K(\beta^2-1)(k'^2\beta^2-1)}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{n\beta\pi}{K}\mathrm{ch}4nW_1\right.$$
$$\left.-\frac{2-\beta^2-k'^2\beta^2}{\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}\mathrm{sh}4nW\right\}\mathrm{csch}4nW_0\cos\frac{n\pi u}{K}$$

where

$$c_0=\frac{2(\beta^2E-K)}{(1-\beta^2)(1-k'^2\beta^2)}+\frac{2(2-2\beta^2+\beta^2k^2)}{(1-\beta^2)^2(1-k'^2\beta^2)}\Pi\left(\frac{\beta^2k^2}{\beta^2-1};\ k\right)$$
$$+\frac{\beta(2\beta k+\sqrt{\beta^2-1})}{(\beta^2-1)^{3/2}(k'^2\beta^2-1)}+\frac{1}{2k(\beta^2-1)^{3/2}}\ln\frac{\sqrt{\beta^2-1}-\beta k}{\sqrt{\beta^2-1}+\beta k}$$
$$-\frac{\beta\pi(4-4\beta^2+3\beta^2k^2)}{2[(\beta^2-1)(k'^2\beta^2-1)]^{3/2}}+\frac{2\beta^3k^2}{[(\beta^2-1)(k'^2\beta^2-1)]^{3/2}}\operatorname{arctg}\frac{\beta k}{\sqrt{\beta^2-1}}$$

Ⅳ.5.3 $$\frac{\mathrm{sn}u}{1+\beta\,\mathrm{dn}u}=\frac{2\pi}{kK\sqrt{k'^2\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\mathrm{sh}\frac{n\pi u_0}{2K}\mathrm{csch}2nW_0\sin\frac{n\pi u}{2K}$$

Ⅳ.6.3 $$\frac{\mathrm{sn}u}{(1+\beta\,\mathrm{dn}u)^2}=\frac{2\pi}{kK(k'^2\beta^2-1)}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{2K\sqrt{\beta^2-1}}\mathrm{ch}\frac{n\pi u_0}{2K}\right.$$
$$\left.-\frac{1}{\sqrt{k'^2\beta^2-1}}\mathrm{sh}\frac{n\pi u_0}{2K}\right\}\sin\frac{n\pi}{2}\mathrm{csch}2nW_0\sin\frac{n\pi u}{2K}$$

Ⅳ.7.3 $$\frac{\mathrm{sn}u}{1-\beta\mathrm{dn}u}=-\frac{2\pi}{kK\sqrt{k'^2\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\mathrm{sh}2nW_1\mathrm{csch}2nW_0\sin\frac{n\pi u}{2K}$$

Ⅳ.8.3 $$\frac{\mathrm{sn}u}{(1-\beta\mathrm{dn}u)^2}=\frac{2\pi}{kK(k'^2\beta^2-1)}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{2K\sqrt{\beta^2-1}}\mathrm{ch}2nW_1\right.$$
$$\left.+\frac{1}{\sqrt{k'^2\beta^2-1}}\mathrm{sh}2nW_1\right\}\cdot\sin\frac{n\pi}{2}\mathrm{csch}2nW_0\sin\frac{n\pi u}{2K}$$

Ⅳ.9.3 $$\frac{\mathrm{cn}u}{1+\beta\mathrm{dn}u}=\frac{2\pi}{kK\sqrt{\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\mathrm{ch}\frac{n\pi u_0}{2K}\mathrm{csch}2nW_0\cos\frac{n\pi u}{2K}$$

Ⅳ.10.3. $$\frac{\mathrm{cn}u}{(1+\beta\mathrm{dn}u)^2}=\frac{2\pi}{kK(\beta^2-1)}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{2K\sqrt{k'^2\beta^2-1}}\mathrm{sh}\frac{n\pi u_0}{2K}\right.$$
$$\left.-\frac{1}{\sqrt{\beta^2-1}}\mathrm{ch}\frac{n\pi u_0}{2K}\right\}\sin\frac{n\pi}{2}\mathrm{csch}2nW_0\cos\frac{n\pi u}{2K}$$

Ⅳ.11.3 $$\frac{\mathrm{cn}u}{1-\beta\mathrm{dn}u}=-\frac{2\pi}{kK\sqrt{\beta^2-1}}\sum_{n=1}^{\infty}\sin\frac{n\pi}{2}\mathrm{ch}2nW_1\mathrm{csch}2nW_0\cos\frac{n\pi u}{2K}$$

Ⅳ.12.3 $$\frac{\mathrm{cn}u}{(1-\beta\mathrm{dn}u)^2}=\frac{2\pi}{kK(\beta^2-1)}\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi}{2K\sqrt{k'^2\beta^2-1}}\mathrm{sh}2nW_1\right.$$
$$\left.+\frac{1}{\sqrt{\beta^2-1}}\mathrm{ch}2nW_1\right\}\sin\frac{n\pi}{2}\mathrm{csch}2nW_0\cos\frac{n\pi u}{2K}$$

Ⅳ.13.3 $$\frac{\mathrm{sn}u\mathrm{cn}u}{1+\beta\mathrm{dn}u}=\frac{2\pi}{\beta k^2K}\sum_{n=1}^{\infty}\left[\mathrm{sh}2nW_0-(-1)^n\mathrm{ch}\frac{n\pi u_0}{K}\right]\mathrm{csch}4nW_0\sin\frac{n\pi u}{K}$$

Ⅳ.14.3 $$\frac{\mathrm{sn}u\mathrm{cn}u}{(1+\beta\mathrm{dn}u)^2}=\frac{2\pi^2}{k^2K\sqrt{(\beta^2-1)(k'^2\beta-1)}}\sum_{n=1}^{\infty}(-1)^{n+1}n$$
$$\cdot\mathrm{sh}\frac{n\pi u_0}{K}\mathrm{csch}4nW_0\sin\frac{n\pi u}{K}$$

Ⅳ.15.3 $$\frac{\mathrm{sn}u\mathrm{cn}u}{1-\beta\mathrm{dn}u}=-\frac{2\pi}{\beta k^2K}\sum_{n=1}^{\infty}\left[\mathrm{sh}2nW_0-(-1)^n\mathrm{ch}4nW_1\right]\mathrm{csch}4nW_0\sin\frac{n\pi u}{K}$$

Ⅳ.16.3 $$\frac{\mathrm{sn}u\mathrm{cn}u}{(1-\beta\mathrm{dn}u)^2}=\frac{2\pi^2}{k^2K\sqrt{(\beta^2-1)(k'^2\beta^2-1)}}\sum_{n=1}^{\infty}(-1)^{n+1}n\mathrm{sh}4nW_1$$
$$\cdot\mathrm{csch}4nW_0\sin\frac{n\pi u}{K}$$

(V) The functions with $(1\pm\beta^2\mathrm{sn}^2u)^{-l}$, $(l=1,2)$.

In the case $\beta>0$, the expansions of $(1+\beta^2\mathrm{sn}^2u)^{-l}$, $(l=1,2)$ were given by ref. [2]. The following formulas are new. We see that

$$1+\beta^2\mathrm{cn}^2u=(1+\beta^2)\left(1-\frac{\beta^2}{1+\beta^2}\mathrm{sn}^2u\right)$$

$$1+\beta^2\mathrm{dn}^2u=(1+\beta^2)\left(1-\frac{\beta^2k^2}{1+\beta^2}\mathrm{sn}^2u\right)$$

and

$$\frac{1}{1-\beta^2\mathrm{sn}^2u}=\frac{1}{2}\left[\frac{1}{1-\beta\mathrm{sn}u}+\frac{1}{1+\beta\mathrm{sn}u}\right]$$

Then using the formulas of [2] and II – IV in this paper, we can easily obtain the expansions of Fourier series about $(1\pm\beta^2\mathrm{cn}^2u)^{-l}$ and $(1\pm\beta^2\mathrm{dn}^2u)^{-l}$, $(l=1,2)$.

Next, we suppose that $\beta>0$, and define u_0 by $\mathrm{cn}(u_0,k')=\beta/\sqrt{1+\beta^2}$, $0<u_0<K'$.

V.1 $$\frac{\mathrm{sn}u}{1-\beta^2\mathrm{sn}^2u}=\frac{\pi}{K\sqrt{(1+\beta^2)(k^2+\beta^2)}}\sum_{n=1}^{\infty}\mathrm{sh}(2n-1)W\cdot\mathrm{csch}(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

V.2 $$\frac{\mathrm{sn}u}{(1+\beta^2\mathrm{sn}^2u)^2}=\frac{\pi}{2K(1+\beta^2)(k^2+\beta^2)}\sum_{n=0}^{\infty}\left[\frac{2k^2+(1+k^2)\beta^2}{\sqrt{(1+\beta^2)(k^2+\beta^2)}}\mathrm{ch}(2n-1)W-\frac{(2n-1)\beta\pi}{2K}\mathrm{sh}(2n-1)W\right]\mathrm{csch}(2n-1)W_0\sin\frac{(2n-1)\pi u}{2K}$$

V.3 $$\frac{\mathrm{cn}u}{(1+\beta^2\mathrm{sn}^2u)^2}=\frac{\pi}{K(1+\beta^2)(k^2+\beta^2)}\sum_{n=1}^{\infty}\left\{\frac{(2n-1)\pi\beta\sqrt{1+\beta^2}}{2K}\mathrm{sh}(2n-1)W+\frac{\beta^4+(1+2k^2)\beta^2+2k^2}{\sqrt{k^2+\beta^2}}\mathrm{ch}(2n-1)W\right\}\mathrm{sech}(2n-1)W_0\cos\frac{(2n-1)\pi u}{2K}$$

V.4 $$\frac{\mathrm{dn}u}{1+\beta^2\mathrm{sn}^2u}=\frac{\pi}{2K\sqrt{1+\beta^2}}+\frac{\pi}{K\sqrt{1+\beta^2}}\sum_{n=1}^{\infty}\mathrm{ch}2nW\mathrm{sech}2nW_0\cos\frac{n\pi u}{K}$$

V.6 $$\frac{\mathrm{dn}u}{(1+\beta^2\mathrm{sn}^2u)^2}=\frac{\pi[\beta^4+(2+k^2)\beta^2+2k^2]}{4K(k^2+\beta^2)\sqrt{(1+\beta^2)^3}}+\frac{\pi}{2K(1+\beta^2)(K^2+\beta^2)}\cdot\sum_{n=1}^{\infty}\left\{\frac{n\beta\pi\sqrt{k^2+\beta^2}}{K}\mathrm{sh}2nW+\frac{\beta^4+(2+k^2)\beta^2+2k^2}{\sqrt{1+\beta^2}}\mathrm{ch}2nW\right\}\cos\frac{n\pi u}{K}$$

V.7 $$\frac{\mathrm{sn}u\mathrm{cn}u}{1+\beta^2\mathrm{sn}^2u}=\frac{\pi}{K\beta\sqrt{k^2+\beta^2}}\sum_{n=1}^{\infty}\mathrm{sech}2nW_0\mathrm{sh}2nW\sin\frac{n\pi u}{K}$$

V.8 $$\frac{\mathrm{sn}u\mathrm{cn}u}{(1+\beta^2\mathrm{sn}^2u)^2}=\frac{\pi}{2K\beta(k^2+\beta^2)}\sum_{n=1}^{\infty}\mathrm{sech}2nW_0\left[\frac{\beta n\pi}{K\sqrt{1+\beta^2}}\mathrm{ch}2nW+\frac{k^2}{\sqrt{k^2+\beta^2}}\mathrm{sh}2nW\right]\sin\frac{n\pi u}{K}$$

V 9. $$\frac{\mathrm{sn}u\mathrm{dn}u}{(1+\beta^2\mathrm{sn}^2u)^2}=\frac{\pi}{2K\beta(1+\beta^2)}\sum_{n=1}^{\infty}\mathrm{sech}(2n-1)W_0\left[\frac{n\beta\pi}{2K\sqrt{1+\beta^2}}\mathrm{ch}(2n-1)W+\frac{1}{\sqrt{1+\beta^2}}\mathrm{sh}(2n-1)W\right]\sin\frac{(2n-1)\pi u}{2K}$$

V.10 $$\frac{\mathrm{cn}u\mathrm{dn}u}{1+\beta^2\mathrm{sn}^2u}=\frac{\pi}{\beta K}\sum_{n=1}^{\infty}\mathrm{csch}(2n-1)W_0\mathrm{sh}(2n-1)W\cos\frac{(2n-1)\pi u}{2K}$$

V.11 $$\frac{\mathrm{cn}u\mathrm{dn}u}{(1+\beta^2\mathrm{sn}^2u)^2}=\frac{\pi}{2\beta K}\sum_{n=1}^{\infty}\mathrm{csch}(2n-1)W_0\left[\frac{n\beta\pi\mathrm{ch}(2n-1)W}{2K\sqrt{(1+\beta^2)(k^2+\beta^2)}}+\mathrm{sh}(2n-1)W\right]\cos\frac{(2n-1)\pi u}{2K}$$

III. Method of the Evaluation of the Fourier Coefficients

To obtain the preceding formulas, the fundamental method used was the evaluation of the Fourier coefficients by contour integration and computation of residues in complex plane. The contour integrals are taken around some fundamental parallelograms. The choice of a parallelogram is based on the property of the expansion function. Now let us give two examples to illustrate the details of evaluating the loop integrals.

Example 1 Consider the formula (IV.1.1), namely, in case $0<\beta<1$, and calculate the expansion formula of the function $(1+\beta\mathrm{dn}u)^{-1}$.

Write

$$f(z)=[1/(1+\beta\mathrm{dn}z)]\exp[in\pi z/K]$$

and take the contour as Fig. 1.

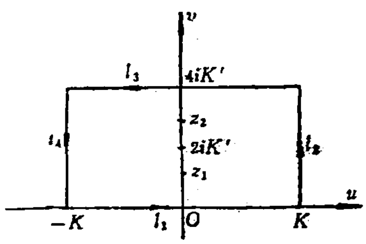

Fig. 1

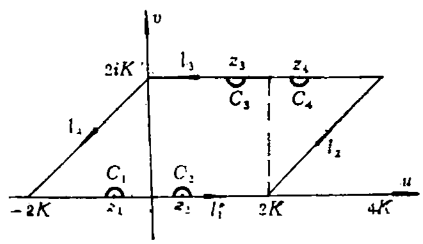

Fig. 2

By using the double periodicity of the elliptic functions, we see that inside the parallelogram, the function $f(z)$ has two simple poles at

$$z_1=i(2K'-u_0),\quad z_2=i(2K'+u_0)$$

where we define u_0 by $\mathrm{dn}(u_0,k')=k/\sqrt{1-k'^2\beta^2}$, $0<u_0<K'$. Computing residues at z_1 and z_2, we have

$$\mathrm{Res}[f(z),\ z_1]=-\beta\exp[-2n\pi K'/K]\cdot\exp[n\pi u_0/K]/(i\sqrt{(1-\beta^2)(1-k'^2\beta^2)})$$
$$\mathrm{Res}[f(z),z_2]=\beta\exp[-2n\pi K'/K]\cdot\exp[-n\pi u_0/K]/(i\sqrt{(1-\beta^2)(1-k'^2\beta^2)})$$

Around four sides of the parallelogram in Fig. 1 to express the integral of $f(z)$, we obtain

$$\int_{l_1}=\int_{-K}^{K}\frac{\exp[in\pi u/K]}{1+\beta\mathrm{dn}u}du,\quad \int_{l_2}+\int_{l_4}=0,\quad \int_{l_3}=-\exp[-4n\pi K'/K]\int_{l_1}$$

Then by the theory of residues, it follows that

$$(1-\exp[-4n\pi K'/K])\int_{-K}^{K}\frac{\exp[in\pi u/K]}{1+\beta\mathrm{dn}u}du$$
$$=\frac{4\beta\pi\mathrm{sh}(n\pi u_0/K)}{\sqrt{(1-\beta^2)(1-K'^2\beta^2)}}\cdot\exp[-2n\pi K'/K]$$

This gives the Fourier coefficients in the formula (IV.1.1).

Example 2 Consider the formula (III.1.2), namely, in case $1<\beta<+\infty$, and compute the Fourier series of the function $(1-\beta\mathrm{cn}u)^{-1}$.

Let

$$f(z)=(1-\beta\mathrm{cn}z)^{-1}\exp[in\pi z/2K]$$

and take the contour as Fig. 2

The function $f(z)$ has four simple poles at

$$z_1=-u_0,\quad z_2=u_0$$
$$z_3=2K-u_0+2iK',\quad z_4=2K+u_0+2iK'$$

From Fig. 2, we find that the function $f(z)$ has singularities at points $z_i\,(i=1-4)$ on two sides of the parallelogram. By introducing four small half-cycles $c_i(i=1-4)$, $f(z)$ can be made regularization (see Fig. 2).

Using the results of the complex analysis, we have

$$\int_{C_1}=-\int_{C_2}=\frac{\beta\pi i\exp[in\pi u_0/2K]}{\sqrt{(\beta^2-1)(k^2+k'^2\beta^2)}}$$
$$\int_{C_3}=-\int_{C_4}=\frac{(-1)^n\beta\pi i\exp[in\pi u_0/2K]\cdot\exp[-n\pi K'/K]}{\sqrt{(\beta^2-1)(k^2+k'^2\beta^2)}}$$

and

$$\int_{l_2}+\int_{l_4}=0$$

Thus from the Cauchy integral theorem, it follows that

$$\int_{-2K}^{2K}\frac{\exp[in\pi u/2K]}{1-\beta \mathrm{cn}u}du=\frac{-2\beta\pi\sin(n\pi u_0/2K)}{\sqrt{(\beta^2-1)(k^2+k'^2\beta^2)}}\cdot\frac{[1+(-1)^n\exp[-n\pi K'/K]]}{[1-(-1)^n\exp[-n\pi K'/K]]}$$

$$=-\frac{2\beta\pi\sin(n\pi u_0/2K)}{\sqrt{(\beta^2-1)(k^2+k'^2\beta^2)}}\cdot\frac{A(W_0,n)}{B(W_0,n)}\qquad(n\neq 0)$$

This immediately validates the formula (III.1.2).

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