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On Multiplicities of the Lyashko–Looijenga Mapping on Discriminant Strata*

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1. Introduction

1.1. Lyashko–Looijenga mapping. Let M be a (complex) manifold and let $F\colon M \to \mathbb{CP}^1$ be an analytic function. Assume that the set of critical points of F is finite and the multiplicity of each critical point is also finite. The second of these assumptions means that any function $F_1\colon M \to \mathbb{CP}^1$ close to F also has a finite set of critical points. Then the *Lyashko–Looijenga* (LL) mapping associates with F the unimodal polynomial (i.e., the one with leading coefficient 1) in one variable t whose roots are the finite critical values of F,

$$\mathrm{LL}\colon F \mapsto (t-t_1)^{l_1}\cdots(t-t_k)^{l_k},$$

and the multiplicity of each root is equal to the sum of the multiplicities of all its preimages.

We study the LL mapping mainly for the case in which F is a polynomial or a family of polynomials in one variable. The LL mapping takes a polynomial P of degree $n+1$ to a polynomial of degree n since the number of critical values of P (counting multiplicities) coincides with that of the roots of its derivative, i.e., is equal to n.

Our main goal is to study the LL mapping on the space $\mathscr{P}$ of polynomials

$$P(x) = x^{n+1} + p_2x^{n-1} + \cdots + p_{n+1}. \tag{1}$$

This family is a universal unfolding of the A_n singularity (see [8]). It can also be treated as the family of rational mappings $P\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ with a single pole of order $n+1$ considered to within linear fractional transformations in the preimage. Indeed, under an appropriate transformation of coordinates, any such mapping takes the form (1).

The LL mapping associates with a polynomial (1) the unimodal polynomial

$$D(t) = t^n + d_1t^{n-1} + \cdots + d_n, \tag{2}$$

whose roots are the critical values of P. Hence, the LL mapping maps the space $\mathscr{P} \cong \mathbb{C}^n$ of polynomials P with coordinates $p_2,\dots,p_{n+1}$ to the space $\mathscr{D} \cong \mathbb{C}^n$ of polynomials D with coordinates $d_1,\dots,d_n$.

1.2. Discriminant and stratification of the space of polynomials. The LL mapping is polynomial and *finite* (see [1, 8]), i.e., the set of preimages of an arbitrary point in the target space is finite. The *multiplicity* of the LL mapping, i.e., the number of preimages of a generic point $D \in \mathscr{D}$, is equal to $(n+1)^{n-1}$.

A nongeneric point has fewer preimages than a generic one. All preimages of nongeneric points form the *discriminant* $\Sigma \subset \mathscr{P}$ of the LL mapping in the source space $\mathscr{P}$. The discriminant consists of polynomials with multiple critical values. These polynomials form a hypersurface in the space $\mathscr{P}$. Below we describe a natural stratification of the space $\mathscr{P}$. In this stratification, the discriminant is the union of all strata of codimension at least 1, and its complement consisting of generic polynomials is the only stratum of codimension zero.

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It is shown in [8] that the LL mapping is nondegenerate on the complement of the discriminant. We generalize this assertion and prove that the restriction of the LL mapping to each discriminant stratum is also nondegenerate.

Associate with each critical value $t = \bar{t}$ of a polynomial P the following partition of its degeneracy. Suppose the equation $P(x) = \bar{t}$ has k_0 simple roots, k_1 roots of multiplicity $2, \dots,$ and k_n roots of multiplicity $n+1$. The *partition associated with the critical value* $\bar{t}$ is the partition $X(\bar{t}) = 1^{k_1}2^{k_2}\cdots n^{k_n}$, where k_i is the number of occurrences of i. In other words, the critical value $\bar{t}$ is attained at k_1 simple critical points, $\dots$, and k_n critical points of multiplicity n. If, for example, $\bar{t}$ is a generic critical value (P takes the value $\bar{t}$ at a single critical point, and the multiplicity of this point is one), then $X(\bar{t}) = 1^1$. The *degeneracy* of $\bar{t}$ is

$$A(X(\bar{t})) = 1 \cdot k_1 + 2 \cdot k_2 + \cdots + n \cdot k_n.$$

The automorphism group $\mathrm{Aut}(X)$ of a partition $X = 1^{k_1}\cdots n^{k_n}$ consists of permutations preserving the values of the parts of this partition. It contains

$$\#\mathrm{Aut}(X) = k_1! \cdots k_n!$$

elements.

Following [7], we call the (unordered) finite set of partitions associated with all critical values of a polynomial P the *passport* of this polynomial. The passport of a generic polynomial consists of n copies of the partition 1^1. The Riemann–Hurwitz formula implies that if a set $\mathbf{X} = \{X_1, \dots, X_c\}$ of partitions is the passport of a polynomial, then $A(X_1) + \cdots + A(X_c) = n$. It can be shown (e.g., see [9]) that the converse is also true.

The automorphism group $\mathrm{Aut}(\mathbf{X})$ of a passport $\mathbf{X} = \{X_1, \dots, X_c\}$ consists of all permutations σ of the partitions X_i preserving their type (i.e., such that the partition $\sigma(X_i)$ coincides with X_i for all $i = 1, \dots, c$). The number of elements in this group is denoted by $\#\mathrm{Aut}(\mathbf{X})$.

We associate a *stratum* $\Sigma_{\mathbf{X}} \subset \mathscr{P}$ with each set $\mathbf{X} = \{X_1, \dots, X_c\}$ of partitions. This stratum consists of all polynomials whose passport coincides with $\mathbf{X}$. Each stratum is a cylinder with a one-dimensional element because a change in the free term does not affect the passport of the polynomial. Generic polynomials form the "improper stratum" in $\mathscr{P}$.

1.3. Statement of the main theorem. We associate with each passport $\mathbf{X} = \{X_1, \dots, X_c\}$ the partition

$$T = T(\mathbf{X}) = (A(X_1), \dots, A(X_c)) = 1^{m_1} \cdots n^{m_n}$$

of n, where $m_i = \#\{X_j \in \mathbf{X},\ A(X_j) = i\}$, i.e., the partition $T(\mathbf{X})$ is determined by the degeneracies of all critical values. The automorphism group of this partition contains

$$\#\mathrm{Aut}(T) = m_1! \cdots m_n!$$

elements.

Theorem 1.1. *The multiplicity $\mu_{\mathbf{X}}$ of the restriction of the* LL *mapping to the stratum $\Sigma_{\mathbf{X}}$ determined by a passport $\mathbf{X} = \{X_1, \dots, X_c\}$ is given by the formula*

$$\mu_{\mathbf{X}} = (n+1)^{c-1} \frac{\#\mathrm{Aut}(T(\mathbf{X}))}{\#\mathrm{Aut}(\mathbf{X}) \prod_{i=1}^{c} \#\mathrm{Aut}(X_i)} \cdot \prod_{i=1}^{c} \frac{(n - A(X_i))!}{s(X_i)!},$$

where $s(X_i)$ is the number of noncritical preimages of a critical value with partition X_i.

Note that the number $s(X)$ for a partition X is given by the relation

$$s(X) = n + 1 - A(X) - l(X),$$

where $l(X)$ is the number of elements in $X = 1^{k_1} \cdots n^{k_n}$, i.e., $l(X) = k_1 + \cdots + k_n$.

The passport of a generic polynomial consists of n copies of the partition 1^1, and the theorem gives the value $\mu(\mathrm{LL}) = (n+1)^{n-1}$ for the multiplicity of the LL mapping at a generic point.

1.4. Brief history. The Lyashko–Looijenga mapping was introduced by Lyashko in 1973 (unpublished) and, independently, by Looijenga [8]. They proved that for some families of functions this mapping is a ramified finite covering and calculated its multiplicity for the so-called universal unfoldings of simple singularities (the statement of the main theorem of Lyashko coinciding with that of Looijenga is presented in [2]). Moreover, Looijenga established a correspondence between generic polynomial coverings of the complex sphere and the trees with indexed edges and thus gave a new proof of the Cayley enumeration theorem for marked trees.

Developing their approach, Arnold [1] suggested a way of enumerating some classes of graphs associated with rational rather than polynomial coverings of the complex sphere. Close enumeration formulas appear in a different context in the papers of Goulden and Jackson [5, 6]. D. Zvonkine [10] showed that numbers appearing in formulas in [6] can be interpreted as the multiplicities of the LL mapping restricted to the discriminant strata in the space of polynomial coverings. The proof of Theorem 1.1 below presents an independent derivation of the Goulden–Jackson formulas based on the ideas of Lyashko, Looijenga, and Arnold.

The paper has the following structure. In Sec. 2, we briefly describe a relationship between the multiplicities of the restrictions of the LL mapping to the discriminant strata and the enumeration of cacti and sets of permutations possessing some specific properties. Section 3 is devoted to calculating the multiplicity of the LL mapping on some special strata using the quasihomogeneous mapping technique. Section 4 contains the proof of the main theorem based on studying the geometry of the strata. Here we lift the LL mapping to the space of ordered sets of the critical points of polynomials and prove that the restriction of the lifted mapping to the "standard" planes is nondegenerate. This assertion generalizes the nondegeneracy of the LL mapping in the complement of the discriminant to arbitrary strata.

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2. Multiplicities, Cacti, and Permutations

Classification problems for ramified coverings of the complex sphere can usually be reduced to purely combinatorial problems of enumeration of embedded graphs. The principles of this reduction are well known. Below we present a brief description of such a reduction for the case of degenerate polynomial coverings. It consists in establishing the correspondence (first introduced in [9]) between these coverings and graphs of a special form called cacti and also sets of permutations satisfying some specific conditions. For a more detailed presentation, see [4].

As usual, all plane embeddings of graphs are considered to within a plane isotopy.

2.1. The cactus of a polynomial. Let $P\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ be a polynomial of the form (1) and let $t_1, \ldots, t_c$ be its critical values. We choose an arbitrary noncritical value $t_0 \in \mathbb{C}$ and join the point t_0 to all critical values by smooth nonintersecting and non-self-intersecting curve segments such that the cyclic order of the segments determined by the counterclockwise rotation at t_0 coincides with that determined by the indexing of the critical values. The resulting pattern is called a *c-star on the set* $t_1, \ldots, t_c$. The vertices $t_1, \ldots, t_c$ are referred to as *black* and the vertex t_0 as *white*.

The preimage of the star under the mapping P is a graph embedded in the source sphere $\mathbb{CP}^1$. The vertices of the graph are the preimages of t_i ($0 \leqslant i \leqslant c$) and its edges are those of the star rays. The preimages of t_0 are white vertices of the graph. The preimages of the other points t_i are black vertices, and they are marked by the index i. This graph is composed of $n+1$ stars glued over some of the black vertices. It can be proved that the resulting graph is always a tree.

Definition 2.1. A *c-cactus* is a plane tree with black and white vertices that is obtained by gluing together a number of c-stars over black vertices with the same marks.

Thus, every polynomial P determines a cactus. The theorem below states the converse.

Theorem 2.2. *Consider c points $t_1, \dots, t_c \in \mathbb{C}$, a c-star on these points, and a c-cactus formed by $n+1$ glued stars. Then there is an $(n+1)$th degree polynomial P such that it is unique to within a linear change of variable $P(x) \mapsto P(ax+b)$ $(a, b \in \mathbb{C},\ a \neq 0)$, $t_1, \dots, t_c$ are its critical values, and the preimage of the star with respect to P and the original cactus are isotopic.*

This theorem immediately follows from the Riemann existence theorem. The number of distinct polynomials corresponding to the same cactus is equal to the ratio of $n+1$ to the number of automorphisms of the cactus. This assertion follows from the fact that if P is a polynomial of the form (1), then the only polynomials of the form (1) that can be obtained from P by transformations of the form $P(x) \mapsto P(ax+b)$ are $P_i = P(\varepsilon^i x)$, $i = 0, \dots, n$, where ε is an $(n+1)$th primitive root of unity.

For $1 \leqslant i \leqslant c$, the set of the ith cactus black-vertex valences each of which is decreased by unity is a partition. The ordered list of these partitions is called the *passport* of the cactus. The passport of a polynomial realizing a given cactus is obtained from that of the cactus by forgetting the order of the partitions.

Theorem 2.2 together with the main theorem implies the following assertion, first proved in [6].

Corollary 2.3. *The number of different cacti with a picked star that have a passport $\mathbf{X} = [X_1, \dots, X_c]$ is equal to*

$$(n+1)^{c-1} \frac{1}{\prod_{i=1}^{c} \#\mathrm{Aut}(X_i)} \prod_{i=1}^{c} \frac{(n - A(X_i))!}{s(X_i)!}.$$

For a Morse polynomial P, the valences of all black vertices of the cactus corresponding to P are equal to 1 or 2. In this case, the enumeration of the cacti reduces to that of the trees with indexed edges (see [8, 1]).

2.2. Permutations associated with cacti. Consider a cactus and *pick up a star* in it. Suppose that the white vertices of the cactus are indexed by the numbers from 1 to $n+1$. Let P be a polynomial realizing this cactus. We take a star on its critical values $t_1, \dots, t_c$.

On the image sphere, consider a closed path starting at the noncritical value $t_0 \in \mathbb{CP}^1$, going along the ith ray of the star, passing around the ith critical value t_i in the positive direction, and then returning to t_0 along the same ray. This path induces a permutation σ_i on the $(n+1)$-element set of white vertices of the cactus.

This permutation is defined as follows. Take one of the white vertices. It belongs to the preimage of the noncritical value t_0. As the point moves along the chosen path in the image sphere, its preimage moves continuously along a path in the preimage sphere and returns to a white vertex (which is generally distinct from the original one). The index of the new white vertex is the image of the index of the original vertex under the mapping σ_i. The mapping σ_i is in fact a permutation since it is reversible.

The permutation σ_i can also be described as the "rotation in the cactus around the black vertices marked by i," namely, it realizes a cyclic permutation of white vertices joined to the ith black vertex according to the cyclic order on these vertices. In particular, the permutation σ_i depends only on the cactus (and on the indexing of its white vertices) rather than on the polynomial realizing the cactus.

The set of lengths of cycles of the permutation σ_i each of which is decreased by one determines a partition called the *type* of the permutation. The previous remark implies that the list of such partitions for $1 \leqslant i \leqslant c$ coincides with the passport of the cactus.

The product $\sigma_c \circ \cdots \circ \sigma_1$ of permutations σ_i taken in the indicated order is a cyclic permutation. Indeed, the path in the image sphere corresponding to this product passes around all critical values. In other words, this is a path around the ramification point at infinity. The polynomial P has a ramification point of order $n+1$ at infinity, and therefore the corresponding permutation is simply a cyclic permutation.

For definiteness, assume that the white vertex of the selected star is marked by 1 and that the permutation $\sigma_c \circ \cdots \circ \sigma_1$ is the cycle $(1, \dots, n+1)$.

Then the following theorem takes place (see [6]).

Theorem 2.4. *There is a natural one-to-one correspondence between the set of c-cacti with a selected star that are formed of $n+1$ glued stars and have the passport $[X_1, \ldots, X_c]$ and the set of tuples of c permutations $\sigma_1, \ldots, \sigma_c$ on an $(n+1)$-element set such that the type of the permutation σ_i coincides with X_i for all i and $\sigma_c \circ \cdots \circ \sigma_1 = (1, \ldots, n+1)$.*

Hence, the enumeration of cacti is equivalent to that of c-tuples of permutations satisfying the conditions of the theorem.

3. Quasihomogeneous Mappings

3.1. Quasihomogeneous Bezout theorem. The notion of a quasihomogeneous mapping generalizes that of a homogeneous one. Let $x_1, \ldots, x_n$ be coordinates in $\mathbb{C}^n$. Suppose a positive integer $w(x_i) = w_i$ called the weight is associated with each coordinate x_i.

A function $f\colon \mathbb{C}^n \to \mathbb{C}$ is said to be *quasihomogeneous* of weight $W \in \mathbb{N}$ (with respect to the set of weights w_i) if

$$f(\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n) = \lambda^W f(x_1, \ldots, x_n)$$

for any $\lambda \in \mathbb{C}$.

The main example of a quasihomogeneous function is a monomial. The weight of a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is $\alpha_1 w_1 + \cdots + \alpha_n w_n$. A sum of monomials of weight W is a polynomial of weight W.

As a simple illustration of the notion of a quasihomogeneous function, we present an explicit formula for the LL mapping on family (1). The discriminant of a polynomial $P(x) \in \mathcal{P}$ (i.e., the product $\prod_{i \neq j}(x_i - x_j)$ of the pairwise differences of its roots) is a polynomial in the coefficients $p_2, \ldots, p_{n+1}$ of $P(x)$. Let t be a parameter. Consider the discriminant of the polynomial $P(x) - t$ as a polynomial in t with coefficients depending on the parameters $p_2, \ldots, p_n$.

Lemma 3.1. *The* LL *mapping takes a polynomial $P = P(x)$ to the discriminant of the polynomial $P(x) - t$ to within a constant factor.*

Proof. A value $t = \bar{t}$ is a root of the discriminant of the polynomial $P(x) - t$ if and only if the polynomial $P(x) - \bar{t}$ has multiple roots, i.e., if and only if $\bar{t}$ is a critical value of P. A generic polynomial in family (1) has n distinct critical values. Therefore, the degree with respect to t of the discriminant of the polynomial $P(x) - t$ is at least n. Let us set the weight of x equal to 1. If we take $w(p_i) = i$, then P is a quasihomogeneous polynomial of weight $n+1$. For the polynomial $P(x) - t$ to be also quasihomogeneous we must set $w(t) = n+1$. The weight of the discriminant is equal to $n(n+1)$ since the discriminant is the product of $n(n+1)$ terms of weight 1. On the other hand, the weight of the monomial t^n is $n(n+1)$, whence the weight of the coefficient before t^n in the discriminant must be zero, i.e., it is a constant. The lemma is proved.

Suppose now that the ith component f_i of a mapping $F\colon \mathbb{C}^n \to \mathbb{C}^n$, $F = (f_1, \ldots, f_n)$, is a polynomial of weight W_i and that F is finite. It is well known that a quasihomogeneous mapping is finite if and only if the only preimage of the origin is the origin (for example, see [3]).

The following statement generalizes the Bezout theorem.

Theorem 3.2 (see [3]). *The multiplicity of a finite quasihomogeneous mapping F is given by the formula*

$$\mu(F) = \frac{W_1 \cdots W_n}{w_1 \cdots w_n}.$$

Lemma 3.1 implies that the mapping LL: $\mathcal{P} \to \mathcal{D}$ is polynomial and finite. If we set $w(d_i) = i(n+1)$, then it becomes quasihomogeneous. The Bezout theorem now permits easily calculating its multiplicity at a generic point:

$$\mu(\mathrm{LL}) = \frac{w(d_1) \cdots w(d_n)}{w(p_2) \cdots w(p_{n+1})} = \frac{1(n+1) \cdots n(n+1)}{2 \cdots (n+1)} = (n+1)^{n-1}.$$

3.2. Primitive strata. We start by calculating the multiplicity of the LL mapping restricted to a stratum in which all critical values except one are nondegenerate. These strata are termed *primitive.* The simplest primitive stratum is called the *Maxwell stratum.* It corresponds to the case of two distinct critical points with a common critical value. The passport $\mathbf{X} = \{X_1, \ldots, X_{n-1}\}$ of the Maxwell stratum consists of $n-1$ partitions, where the first partition is $X_1 = 1^2$ and the others are Morse partitions, $X_i = 1^1$, $i = 2, \ldots, n-1$.

A generic polynomial in the Maxwell stratum has the form

$$P(x) = (x^2 + p_1 x + p_2)^2 (x^{n-3} + p'_1 x^{n-2} + \cdots + p'_{n-3}) + p_{n+1},$$

where $p'_1 + 2p_1 = 0$. Setting

$$w(p_i) = i, \quad i = 1, 2, n+1, \qquad w(p'_i) = i, \quad i = 1, \ldots, n-3,$$

for the weights of the coordinates in the preimage we conclude that the total weight of coordinates in the preimage is equal to

$$2(n+1)(n-3)!\,.$$

On the other hand, we have $T(\mathbf{X}) = 1^{n-2}2^1$, and a generic polynomial in the image has the form

$$D(t) = (t - t_1)^2 D_1(t),$$

where $t_1 = p_{n+1}$ is a multiple critical value and

$$D_1(t) = t^{n-2} + d_1 t^{n-1} + d_2 t^{n-2} + \cdots + d_{n-2}$$

is a polynomial of the $(n-2)$th degree. This representation dictates the following choice of weights of coordinates in the image: $w(t_1) = n+1$, $w(d_i) = i(n+1)$, $i = 1, \ldots, n-2$, and the total weight of coordinates in the image equals $(n-2)!\,(n+1)^{n-1}$.

We can now calculate the multiplicity of the restriction of the LL mapping to the Maxwell stratum,

$$\mu = \frac{(n-2)!\,(n+1)^{n-1}}{2(n+1)(n-3)!} = \frac{(n-2)(n+1)^{n-2}}{2},$$

which coincides with Arnold's results [1].

The above calculation can easily be generalized to an arbitrary primitive stratum. Namely, suppose that the partition corresponding to the first critical value has the form $X_1 = 1^{k_1} \cdots n^{k_n}$ and that the others are Morse critical values, $X_j = 1^1$, $j = 2, \ldots, c$. Then $c = n + 1 - A(X_1)$. A generic polynomial with the degenerate critical level p_{n+1} of the above type can be represented in the form

$$P(x) = P_0(x) P_1^2(x) \cdots P_n^{n+1}(x) + p_{n+1},$$

where

$$P_i(x) = x^{k_i} + p_{i,1} x^{k_i - 1} + \cdots + p_{i,k_i}$$

is a polynomial of degree k_i, $k_0 = s(X_1)$ is the number of simple roots of the polynomial P, $k_0 + 2k_1 + \cdots + (n+1)k_n = n+1$, and $p_{0,1} + 2p_{1,1} + \cdots + (n+1)p_{n,1} = 0$. In other words, we have parametrized the closure of the primitive stratum by the space of coefficients p_{ij} of the polynomials $P_0, \ldots, P_n$.

After the weights $w(p_{i,j}) = i$ and $w(p_{n+1}) = n+1$ are chosen, the total weight of coordinates in the preimage becomes

$$(n+1)\, s(X_1)!\, k_1! \cdots k_n! = (n+1) \#\mathrm{Aut}(X_1)\, s(X_1)!\,.$$

On the other hand, we have $T(\mathbf{X}) = 1^{n-A(X_1)} A(X_1)^1$, and a generic polynomial in the image has the form

$$D(t) = (t - t_1)^{A(X_1)} D_1(t),$$

where $t_1 = p_{n+1}$ is the multiple critical value and D_1 is a generic polynomial of degree $n - A(X_1)$. Hence, the total weight of the coordinates in the image is equal to

$$(n+1)^c (n - A(X_1))!,$$

and the multiplicity takes the form

$$\mu_{\mathbf{X}}(LL) = (n+1)^{c-1} \frac{1}{\#\mathrm{Aut}(X_1)} \cdot \frac{(n-A(X_1))!}{s(X_1)!}.$$

Note that this formula is in complete agreement with the predictions of the main theorem because, for a primitive stratum, we have $\#\mathrm{Aut}(\mathbf{X}) = \#\mathrm{Aut}(T(\mathbf{X})) = (n-A(X_1))!$, $\#\mathrm{Aut}(X_i) = 1$ for $i = 2, \dots, c$, and $A(X_i) = s(X_i) = n-1$ for a nondegenerate partition $X_i = 1^1$.

This result will be used below in the proof of the main theorem in the general case.

4. Proof of the Main Theorem

4.1. Auxiliary mappings. To prove Theorem 1.1 we replace the LL mapping by its analog that associates with the (ordered) set of critical points of a polynomial the (ordered) set of its critical values. Namely, consider the commutative square diagram

$$\begin{array}{ccc} \mathscr{P} & \xrightarrow{\mathrm{LL}} & \mathscr{D} \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\tau} \\ \widehat{\mathscr{P}} & \xrightarrow{\widehat{\mathrm{LL}}} & \widehat{\mathscr{D}} \end{array} \tag{3}$$

Here $\widehat{\mathscr{P}}$ denotes the space $\mathbb{C}^n$, which is identified with the Cartesian product $\mathbb{C}^{n-1} \times \mathbb{C}^1$ in which the first factor is the hyperplane

$$a_1 + \dots + a_n = 0 \tag{4}$$

in the n-dimensional space of ordered sets of critical points of polynomials P and the other factor is the line of $(n+1)$th roots of the free term p_{n+1} of the polynomial P. Accordingly, the mapping α has the form

$$\alpha\colon (a_1, \dots, a_n; a) \mapsto (n+1) \int_0^x (\xi - a_1) \cdots (\xi - a_n)\, d\xi + a^{n+1}. \tag{5}$$

The space $\widehat{\mathscr{D}}$ is the space of ordered n-tuples of critical values of P. The mapping τ is simply the Vieta mapping

$$\tau\colon (t_1, \dots, t_n) \mapsto (t - t_1) \cdots (t - t_n), \tag{6}$$

associating with an n-tuple of numbers the unimodal polynomial whose roots are these numbers.

The $\widehat{\mathrm{LL}}$ mapping takes a point $A = (a_1, \dots, a_n, a) \in \widehat{\mathscr{P}}$ to the point $(t_1, \dots, t_n) \in \widehat{\mathscr{D}}$, where $t_i = P(a_i)$ is the value of $P = \alpha(A)$ at the point a_i. The mappings α, τ, and $\widehat{\mathrm{LL}}$ are obviously polynomial.

Lemma 4.1. *The mappings α, τ, and* LL *are quasihomogeneous and finite. The mapping $\widehat{\mathrm{LL}}$ is homogeneous of degree $n+1$ in each coordinate and finite.*

Proof. The (quasi)homogeneity of the mappings under consideration means the following. Consider the action of the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ on the space $\widehat{\mathscr{P}}$ by dilations, $\lambda\colon (a_1, \dots, a_n; a) \mapsto (\lambda a_1, \dots, \lambda a_n; \lambda a)$, $\lambda \in \mathbb{C}^*$. This action induces the action of the group $\mathbb{C}^*$ on the spaces $\mathscr{P}$, $\widehat{\mathscr{D}}$, and $\mathscr{D}$, which commutes with the mappings of the square diagram (3). In particular, the corresponding action on $\widehat{\mathscr{D}}$ is the dilation $\lambda\colon (t_1, \dots, t_n) \mapsto (\lambda^{n+1} t_1, \dots, \lambda^{n+1} t_n)$ since the λ-fold dilation of the critical points of the polynomial and the λ^{n+1}-fold dilation of its free term induce the λ^{n+1}-fold dilation of the critical values.

The mappings are finite since for each of them the preimage of the origin consists only of the origin.

4.2. Calculating multiplicities. The stratification of the space $\mathscr{P}$ described in Sec. 1.2 induces stratifications of the other spaces at the vertices of the square diagram (3). To calculate the multiplicities of the LL mapping restricted to the discriminant strata it suffices to calculate the multiplicities of the restriction of the mappings α, $\widehat{\mathrm{LL}}$, and τ to the corresponding strata.

We start by describing the induced stratifications.

The strata in the space $\mathscr{D}$ are numbered by the partitions $T = 1^{m_1} \cdots n^{m_n}$ of the number n, $n = 1 \cdot m_1 + \cdots + n \cdot m_n$. The stratum $\Delta_T \subset \mathscr{D}$ consists of polynomials having m_1 roots of multiplicity $1, \ldots,$ and m_n roots of multiplicity n.

For a partition T of n, let us fix a partition of the set of indices $\{1, \ldots, n\}$ into disjoint subsets such that there are precisely m_1 1-element subsets, ..., and m_n n-element subsets. Associate with such a partition the stratum $\widehat{\Delta}_T \subset \widehat{\mathscr{D}}$ of $\widehat{\mathscr{D}}$ consisting of the points $(t_1, \ldots, t_n) \in \widehat{\mathscr{D}}$ such that $t_i = t_j$ if and only if the indices i and j belong to the same subset of the partition. (For simplicity, we omit the indication of the chosen partition of the set of indices in the notation of a stratum.)

Similarly, we fix for a passport $\mathbf{X} = \{X_1, \ldots, X_c\}$ a partition $\mathbf{I} = \{I_1, \ldots, I_c\}$ of the set of indices $\{1, \ldots, n\}$ into c tuples $I_1, \ldots, I_c$ of pairwise disjoint subsets such that the ith tuple I_i of this partition contains precisely k_{i1} 1-element subsets, ..., and k_{in} n-element subsets, where $X_i = 1^{k_{i1}} \cdots n^{k_{in}}$. The stratum $\widehat{\Sigma}_\mathbf{I} \subset \widehat{\mathscr{P}}$ consists of all points $A = (a_1, \ldots, a_n; a) \in \widehat{\mathscr{P}}$ such that $a_i = a_j$ if and only if the indices i and j belong to the same subset of the partition and the values of the polynomial $\alpha(A)$ at the points a_i and a_j coincide if and only if the indices i and j belong to subsets of the same tuple of the partition.

Given a partition $\mathbf{I}$ of the set of indices for a passport $\mathbf{X}$, one can construct in a unique manner a partition $T(\mathbf{I})$ compatible with the former, which is obtained by taking the union of all subsets of indices inside each tuple.

The restrictions of the mappings of the square diagram (3) to the corresponding strata form the commutative square diagram

$$\begin{array}{ccc} \Sigma_\mathbf{X} & \xrightarrow{\mathrm{LL}} & \Delta_{T(\mathbf{X})} \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\tau} \\ \widehat{\Sigma}_\mathbf{I} & \xrightarrow{\widehat{\mathrm{LL}}} & \widehat{\Delta}_{T(\mathbf{I})} \end{array} \tag{7}$$

where the strata $\widehat{\Sigma}_\mathbf{I}$ and $\widehat{\Delta}_{T(\mathbf{I})}$ correspond to the compatible partitions of the set of indices $\{1, \ldots, n\}$.

The multiplicities of all mappings in this square diagram are constant, and they do not depend on the choice of either a point in the image or a partition $\mathbf{I}$. Accordingly, denote these multiplicities by $\mu_\mathbf{X}(\mathrm{LL})$, $\mu_\mathbf{X}(\alpha)$, $\mu_\mathbf{X}(\widehat{\mathrm{LL}})$, and $\mu_{T(\mathbf{X})}(\tau)$. The commutativity of the square diagram (7) implies

$$\mu_\mathbf{X}(\mathrm{LL}) = \frac{\mu_\mathbf{X}(\widehat{\mathrm{LL}}) \cdot \mu_{T(\mathbf{X})}(\tau)}{\mu_\mathbf{X}(\alpha)}, \tag{8}$$

and to prove the theorem it suffices to calculate all the multiplicities entering the right-hand side.

Lemma 4.2. *Let* $T = 1^{m_1} \cdots n^{m_n}$. *Then*

$$\mu_T(\tau) = m_1! \cdots m_n! = \#\mathrm{Aut}(T).$$

Indeed, a polynomial in the stratum Σ_T determines the (multi)set of its roots. If we also fix a partition of the set of indices $\{1, \ldots, n\}$ corresponding to T, then there remain $m_1!$ ways to index the roots of multiplicity $1, \ldots,$ and $m_n!$ ways to index the roots of multiplicity n.

The following assertion is also obvious.

Lemma 4.3. $\mu_\mathbf{X}(\alpha) = (n+1)\#\mathrm{Aut}(\mathbf{X}) \prod_{i=1}^{c} \#\mathrm{Aut}(X_i)$.

It is proved by analogy with the previous case; namely, there are $\#\mathrm{Aut}(\mathbf{X})$ ways to permute the tuples of sets of indices corresponding to equal partitions X_i and $\#\mathrm{Aut}(X_i)$ ways to permute sets of indices inside the ith tuple. The factor $n+1$ is due to the fact that there are $n+1$ $(n+1)$th roots of the free term.

The multiplicity of the mapping $\widehat{\mathrm{LL}}$ is calculated in the two concluding sections.

4.3. Multiplicity of the restriction of a projective mapping. Let $f\colon \mathbb{C}^n \to \mathbb{C}^n$, $f = (f_1, \dots, f_n)$, be a homogeneous mapping of a coordinatewise constant degree W. We also suppose that it is finite.

Lemma 4.4. *Let $V \subset \mathbb{C}^n$ be a homogeneous affine variety of pure dimension k in the preimage, and let the image $f(V)$ be irreducible. Then*

$$\deg V \cdot W^k = \deg f(V) \cdot \mu_V(f). \tag{9}$$

In this case, if f is nondegenerate at a generic point of V, then a generic point of the image $f(V)$ has precisely $\mu_V(f)$ geometrically distinct preimages.

Here $\deg V$ and $\deg f(V)$ are the degrees of these varieties and $\mu_V(f)$ is the multiplicity of the restriction of f to the variety V.

Proof. Consider the projectivization $\tilde{f}\colon \mathbb{CP}^{n-1} \to \mathbb{CP}^{n-1}$ of the mapping f and the corresponding projective varieties $\widetilde{V} \subset \mathbb{CP}^{n-1}$, $\tilde{f}(\widetilde{V}) = \widetilde{f(V)} \subset \mathbb{CP}^{n-1}$. We are first going to prove that

$$\deg \widetilde{V} \cdot W^{k-1} = \deg \tilde{f}(\widetilde{V}) \cdot \mu_{\widetilde{V}}(\tilde{f}). \tag{10}$$

Relation (9) follows from (10) because, by definition of the degree, $\deg V = \deg \widetilde{V}$ and $\deg f(V) = \deg \tilde{f}(\widetilde{V})$ and we have $\mu_V(f) = W \mu_{\widetilde{V}}(\tilde{f})$.

Let $\mathbb{CP}^{n-k}$ be a generic $(n-k)$-dimensional plane in the image. Then, by definition, $\#(\mathbb{CP}^{n-k} \cap \tilde{f}(\widetilde{V})) = \deg \tilde{f}(\widetilde{V})$. Now let us calculate in two different ways the number of preimages $\tilde{f}^{-1}(\mathbb{CP}^{n-k} \cap \tilde{f}(\widetilde{V}))$ belonging to $\widetilde{V}$.

Firstly, this number is equal to $\deg \tilde{f}(\widetilde{V}) \cdot \mu_{\widetilde{V}}(\tilde{f})$ since each point of transversal intersection has $\mu_{\widetilde{V}}(\tilde{f})$ preimages in $\widetilde{V}$.

On the other hand, the preimage $\tilde{f}^{-1}(\mathbb{CP}^{n-k})$ of an $(n-k)$-dimensional plane is a variety of degree W^{k-1} in the preimage. Indeed, such a plane is determined by $k-1$ linear equations in coordinates $y_1, \dots, y_n$ in the target space, and the substitution $y_i = f_i$ results in $k-1$ equations of degree W for $\tilde{f}^{-1}(\mathbb{CP}^{n-k})$. Therefore, the number of intersection points is equal to

$$\deg f^{-1}(\mathbb{CP}^{n-k}) \cdot \deg \widetilde{V} = W^{k-1} \deg \widetilde{V},$$

and relation (10) is thus proved. It is possible to choose an $(n-k)$-dimensional plane in the target space such that it does not pass through the images of degeneration points in V for the mapping $\tilde{f}$. All preimages of intersection points of such a plane with $\tilde{f}(V)$ have multiplicity 1, and therefore their number is precisely the multiplicity of the mapping.

We can now calculate the degree of the closure of a primitive stratum in $\widehat{\mathcal{P}}$ knowing the multiplicity of the restriction of the LL mapping to a primitive stratum in $\mathcal{P}$ (see Sec. 3.2). Let us fix a partition I of the subset of the set of indices corresponding to the degenerate partition X and denote the corresponding primitive stratum by $\widehat{\Sigma}_I$. The symbols $T(I)$ and μ_X (where the index is a partition instead of a passport) have the same meaning. The closure of a stratum $\widehat{\Sigma}_{\mathbf{I}} \subset \widehat{\mathcal{P}}$ will be denoted by $\mathrm{cl}(\widehat{\Sigma}_{\mathbf{I}})$.

Now the commutativity of the square diagram (7) for a primitive stratum $\widehat{\Sigma}_I \subset \widehat{\mathcal{P}}$ gives

$$\mu_X(\widehat{\mathrm{LL}}) = \frac{\mu_X(\alpha)\,\mu_X(\mathrm{LL})}{\mu_{T(X)}(\tau)} = \frac{(n+1)(n-A(X))!\,\#\mathrm{Aut}(X)\cdot(n+1)^{c-1}\,\frac{1}{\#\mathrm{Aut}(X)}\,\frac{(n-A(X))!}{s(X)!}}{(n-A(X))!}$$
$$= (n+1)^c\,\frac{(n-A(X))!}{s(X)!},$$

whence, by Lemma 4.4,

$$\deg \mathrm{cl}(\widehat{\Sigma}_I) = \frac{(n - A(X))!}{s(X)!}.$$

4.4. Degrees of strata. Let I be an arbitrary tuple of disjoint subsets of the set of indices $\{1, \dots, n\}$. The *standard plane* $\Pi_I \subset \widehat{\mathscr{P}}$ associated with the tuple I is defined by the set of equations $a_i = a_j$ for all pairs i, j belonging to the same subset of the tuple I.

Theorem 4.5. *The restriction of $\widehat{\mathrm{LL}}$ to the plane Π_I is nondegenerate in the complement of the intersection of this plane by other standard planes not containing Π_I and by the plane $a = 0$.*

Proof. Denote the critical points coinciding on the plane Π_I by $b_0, b_1, \dots, b_m$, and let β_i be the number of critical points merging at b_i, i.e., $\beta_0 b_0 + \dots + \beta_m b_m = 0$. We set

$$\Phi_I(x) = (n+1)(x - b_0)^{\beta_0} \cdots (x - b_m)^{\beta_m}.$$

Then the restriction of $\widehat{\mathrm{LL}}$ to the plane Π_I has the coordinatewise form

$$\widehat{\mathrm{LL}}_i \colon (b_0, b_1, \dots, b_m; a) \mapsto \int_0^{b_i} \Phi_I(\xi)\, d\xi + a^{n+1}.$$

For $i = 0, 1, \dots, m$, we now fix polynomial 1-forms on the line

$$\omega_i = -\beta_i \frac{\Phi_I(x)\, dx}{(x - b_i)}.$$

The cohomology classes of these forms in the relative cohomology space $H^1(\mathbb{C}^1, \{b_0, \dots, b_m\})$ satisfy the linear equation $\omega_0 + \dots + \omega_m = d\Phi_I(x) \equiv 0$. If all b_i are pairwise distinct, then there are no other relations between the 1-forms. Indeed, if a 1-form $\omega = \lambda_0 \omega_0 + \dots + \lambda_m \omega_m$ represents the zero relative cohomology class, then the degree and the multiplicities of zeros of the polynomial $\int_{b_0}^{x} \omega$ at the points b_i coincide with those of the polynomial $\Phi_I(x)$, whence these polynomials are proportional. Therefore, the cohomology classes of the 1-forms ω_i span the entire relative cohomology space.

Subtracting the row corresponding to the component $\widehat{\mathrm{LL}}_0$ from all other rows in the Jacobi matrix $(\partial \widehat{\mathrm{LL}}_j / \partial b_i)$ results in the $(m+1) \times (m+2)$-matrix

$$\begin{pmatrix} \int_0^{b_0} \omega_0 & \dots & \int_0^{b_0} \omega_m & (n+1)a^n \\ \int_{b_0}^{b_1} \omega_0 & \dots & \int_{b_0}^{b_1} \omega_m & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \int_{b_0}^{b_m} \omega_0 & \dots & \int_{b_0}^{b_m} \omega_m & 0 \end{pmatrix}.$$

Since the classes of the 1-forms ω_j span the entire cohomology space, the rank of the latter matrix is $m+1$ at the points at which all b_i are pairwise distinct and $a \neq 0$. The theorem is proved.

Now let us fix a passport $\mathbf{X} = \{X_1, \dots, X_c\}$ and one of the corresponding partition $\mathbf{I} = \{I_1, \dots, I_c\}$ of the set of indices. We say that an index i is *essential* (for the partition $\mathbf{I}$) if the number of subsets in the tuple of indices $I_k \in \mathbf{I}$ containing i is greater than one. The critical point and the critical value corresponding to this index are also termed essential. Theorem 4.5 implies the following assertion completing the proof of Theorem 1.1.

Corollary 4.6. *The closure $\mathrm{cl}(\widehat{\Sigma}_{\mathbf{I}})$ of a stratum $\widehat{\Sigma}_{\mathbf{I}}$ is the intersection of the closures $\mathrm{cl}(\widehat{\Sigma}_{I_i})$ of the primitive strata $\widehat{\Sigma}_{I_i}$. The intersection is transversal at almost all points in $\mathrm{cl}(\widehat{\Sigma}_{\mathbf{I}})$, i.e., everywhere except possibly the subvariety of codimension 1 consisting of points at which an essential critical point merges with another critical point. In particular, $\mathrm{cl}(\widehat{\Sigma}_{\mathbf{I}})$ is smooth outside this subvariety, and we have*

$$\deg \mathrm{cl}(\widehat{\Sigma}_{\mathbf{I}}) = \prod_{i=1}^{c} \deg \mathrm{cl}(\widehat{\Sigma}_{I_i}) = \prod_{i=1}^{c} \frac{(n - A(X_i))!}{s(X_i)!}.$$

Proof. Associate with the partition $\mathbf{I}$ of the set of indices into tuples of subsets the partition $I = I(\mathbf{I})$, $I = I_1 \cup \dots \cup I_k$, of the set of indices into subsets by omission of the intermediate hierarchy level. As above, let b_i denote the sets of pairwise coincident critical points on the standard plane Π_I. The stratum $\widehat{\Sigma}_{\mathbf{I}}$ is a subvariety in the plane Π_I. The restriction of $\widehat{\mathrm{LL}}$ to this plane is nondegenerate almost everywhere. Therefore the stratum $\widehat{\Sigma}_{\mathbf{I}}$, which is locally the preimage of a plane, is smooth at each of its points at which $b_i \neq b_j$. As two unessential critical points merge, the smoothness of the stratum closure is preserved.

The transversality of the intersection of primitive strata at a generic point also follows from the previous theorem. Indeed, the above argument shows that each of the primitive strata $\widehat{\Sigma}_{I_i}$ is smooth at its points of intersection with the plane Π_I, at which its essential critical points do not coincide with other critical points. The images of the strata Π_{I_i} under the mapping $\widehat{\mathrm{LL}}$ are transversally intersecting planes, and since the mapping is nondegenerate, the strata themselves also intersect transversally.

According to the complex analog of the Rolle lemma, if two critical points with the same critical values merge, then at least one more critical point with a different critical value merges with them. Hence, the image of the set of points in the closure of the stratum $\widehat{\Sigma}_{\mathbf{I}}$ corresponding to the merging critical points has codimension 1 in the image of the entire closure. Since the mapping $\widehat{\mathrm{LL}}$ is proper, this set also has codimension 1 in the closure of the stratum.

Corollary 4.6 and hence the main theorem are proved.

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