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EXPLICIT SOLITARY-WAVE SOLUTIONS TO GENERALIZED POCHHAMMER-CHREE EQUATIONS *

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Abstract: *For the solitary-wave solution* $u(\xi) = u(x - vt + \xi_0)$ *to the generalized Pochhammer-Chree equation (PC equation)*

$$u_{tt} - u_{ttxx} + ru_{xxt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0, \quad r, a_i = \text{consts}(r \neq 0), \tag{I}$$

the formula $\int_{-\infty}^{+\infty} [u'(\xi)]^2 d\xi = \frac{1}{12rv}(C_+ - C_-)^3[3a_3(C_+ + C_-) + 2a_2]$, $C_\pm = \lim_{\xi \to \pm\infty} u(\xi)$, *is established, by which it is shown that the generalized PC equation* (I) *does not have bell profile solitary-wave solutions but may have kink profile solitary-wave solutions. However a special generalized PC equation*

$$u_{tt} - u_{ttxx} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0, \quad a_i = \text{consts} \tag{II}$$

may have not only bell profile solitary-wave solutions, but also kink profile solitary-wave solutions whose asymptotic values satisfy $3a_3(C_+ + C_-) + 2a_2 = 0$. *Furthermore all expected solitary-wave solutions are given. Finally some explicit bell profile solitary-wave solutions to another generalized PC equation*

$$u_{tt} - u_{ttxx} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0, \quad a_i = \text{const} \tag{III}$$

are proposed.

Key words: nonlinear evolution equation; generalized Pochhammer-Chree equation; solitary-wave solution; exact solution

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Introduction

Pochhammer-Chree equation (PC equation in short)

$$u_{tt} - u_{ttxx} - u_{xx} - \frac{1}{p}(u^p)_{xx} = 0, \tag{1}$$

is used to describe the propagation of longitudinal deformation waves in an elastic rod, where $p = 3$ or $p = 5$ reflects two possible constitutive choices for the material[1,2]. Bogolubsky[1] and

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Clarkson et al.[2] gave some solitary-wave solutions of (1) with $p = 2$, $p = 3$, $p = 5$ and studied the interaction of two solitary-waves numerically. Moreover Clarkson et al.[2] considered the Painlevé property for the generalized PC equation

$$u_{tt} - u_{ttxx} - (\sigma(u))_{xx} = 0, \tag{2}$$

and pointed out that in order that (2) be of Painlevé type, $\sigma(w)$ must necessarily be of the form

$$\sigma(w) = a_0 + a_1 w + a_2 w^2 + a_3 w^3.$$

In this paper, we would like to present solitary-wave solutions of the following generalized PC equation

$$u_{tt} - u_{ttxx} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0, \quad a_i = \text{const}, \tag{II}$$

when $a_2 \neq 0$ or $a_3 \neq 0$. Because the dissipation usually needs to be considered in practical problems, we would also seek solitary-wave solutions of

$$u_{tt} - u_{ttxx} + r u_{xxt} - (a_1 u + a_2 u^2 + a_3 u^3)_{xx} = 0, \quad r,\ a_i = \text{const}\ (r \neq 0), \tag{I}$$

which is more important. There have existed some studies on Cauchy problem of the equation describing the propagation of longitudinal deformation waves in an elastic rod and possessing the dissipative term u_{xxt} (e.g. [3]), but it hasn't been found that there exists any result on explicit exact solitary-wave solutions of the generalized PC equation (I), as far as we know. Finally we shall propose some explicit solitary-wave solutions to the following generalized PC equation

$$u_{tt} - u_{ttxx} - (a_1 u + a_3 u^3 + a_5 u^5)_{xx} = 0, \quad a_i = \text{const}. \tag{III}$$

1 Exact Solitary-Wave Solutions of (I)

Traveling-wave solutions $u(x,t) = u(x - vt) = u(\xi)$ to the first generalized PC equation (I) must satisfy

$$v^2 u''(\xi) - v^2 u^{(4)}(\xi) - rvu'''(\xi) - (a_1 u + a_2 u^2 + a_3 u^3)_{\xi\xi} = 0. \tag{3}$$

If we seek solitary-wave solutions of (I) to satisfy

$$u'(\xi),\ u''(\xi),\ u'''(\xi) \to 0,\ |\xi| \to +\infty, \tag{4}$$

then after integrating (3) twice, we find that $u(\xi)$ has to satisfy

$$u''(\xi) + \frac{r}{v}u'(\xi) + \frac{a_1 - v^2}{v^2}u(\xi) + \frac{a_2}{v^2}u^2(\xi) + \frac{a_3}{v^2}u^3(\xi) = \frac{C}{v^2}, \tag{5}$$

where C is an integration constant.

For PC equation (1) with $p = 2$, $p = 3$, Bogolubsky[1] and Clarkson et al.[2] gave bell profile solitary-wave solutions with the same asymptotic values. Does there exist any bell profile solitary-wave solution to the generalized PC equation (I)? In order to answer this question, we first make the following analysis. Let

$$C_+ = \lim_{\xi \to +\infty} u(\xi), \quad C_- = \lim_{\xi \to -\infty} u(\xi). \tag{6}$$

Multiplying (5) by $u'(\xi)$ and integration over $(-\infty, \xi)$ leads to

$$\frac{1}{2}[u'(\xi)]^2 + \frac{r}{v}\int_{-\infty}^{\xi}[u'(\xi)]^2 d\xi + \frac{a_1 - v^2}{2v^2}u^2(\xi) + \frac{a_2}{3v^2}u^3(\xi) + \frac{a_3}{4v^2}u^4(\xi) = \frac{C}{v^2}u(\xi) + C_1. \tag{7}$$

Making the limit of (7) at $\xi \to -\infty$ gives

$$C_1 = \frac{a_1 - v^2}{2v^2}C_-^2 + \frac{a_2}{3v^2}C_-^3 + \frac{a_3}{4v^2}C_-^4 - \frac{C}{v^2}C_-, \tag{8}$$

and then substituting (8) into (7) and making the limit of (7) at $\xi \to +\infty$ yields

$$\frac{r}{v}\int_{-\infty}^{+\infty}[u'(\xi)]^2 d\xi = \frac{a_3}{4v^2}(C_-^4 - C_+^4) + \frac{a_2}{3v^2}(C_-^3 - C_+^3) + \frac{a_1 - v^2}{2v^2}(C_-^2 - C_+^2) - \frac{C}{v^2}(C_- - C_+). \tag{9}$$

Now let us set $\xi \to -\infty$, $\xi \to +\infty$ in (5) respectively and then we have

$$\begin{cases} C = (a_1 - v^2)C_- + a_2 C_-^2 + a_3 C_-^3, \\ C = (a_1 - v^2)C_+ + a_2 C_+^2 + a_3 C_+^3, \end{cases} \tag{10}$$

which leads to

$$\begin{cases} C = -[a_2 C_+ C_- + a_3 C_+ C_- (C_+ + C_-)], \\ a_1 - v^2 = -[a_2(C_- + C_+) + a_3(C_- C_+ + C_-^2 + C_+^2)]. \end{cases}$$

Substituting these quantities into (9) and then we obtain upon simplification

$$\int_{-\infty}^{+\infty}[u'(\xi)]^2 d\xi = \frac{1}{12rv}(C_+ - C_-)^3[3a_3(C_+ + C_-) + 2a_2]. \tag{11}$$

By (11), we obtain the following results on the generalized PC equation (I):

(i) $u'(\xi)$ is square integrable over $(-\infty, +\infty)$, i.e. $\int_{-\infty}^{+\infty}[u'(\xi)]^2 d\xi$ exists.

(ii) When C_+, C_- and r are fixed, the smaller $|v|$ is, the steeper the solitary-wave generally becomes. When C_+, C_- and $|v|$ are fixed, the smaller $|r|$ is, the steeper the solitary wave generally becomes.

(iii) The generalized PC equation (I) with $r \neq 0$ does not have any bell profile solitary-wave solution (because when $C_+ = C_-$, we have $u(\xi) = C_+ = C_-$, which is not a solitary-wave solution), but it may have kink profile solitary-wave solutions (with different asymptotic values). However the generalized PC equation (II) with $r = 0$ may have not only bell profile solitary-wave solutions, but also kink profile solitary-wave solutions possessing $3a_3(C_+ + C_-) + 2a_2 = 0$.

In what follows, we would like to present solitary-wave solutions expected in the third result above. In order to obtain kink profile solitary-wave solutions of the generalized PC equation (I), we assume that (5) has solutions of the form

$$u(\xi) = \frac{A e^{\alpha(\xi+\xi_0)}}{1 + e^{\alpha(\xi+\xi_0)}} + D, \tag{12}$$

where A, α, D are constants to be determined and ξ_0 is an arbitrary phase shift. We easily have

$$u'(\xi) = \frac{A\alpha e^{\alpha(\xi+\xi_0)}}{(1 + e^{\alpha(\xi+\xi_0)})^2}, \quad u''(\xi) = \frac{A\alpha^2(e^{\alpha(\xi+\xi_0)} - e^{2\alpha(\xi+\xi_0)})}{(1 + e^{\alpha(\xi+\xi_0)})^3}. \tag{13}$$

Substitution of (12) and (13) into (5) may yield

$$\begin{cases} a_3 D^3 + a_2 D^2 + (a_1 - v^2) D - C = 0, \\ v^2\alpha^2 + vr\alpha + 2a_2 D + 3a_3 D^2 + (a_1 - v^2) = 0, \\ a_3 A^2 + (a_2 + 3a_3 D) A - v^2\alpha^2 - vr\alpha = 0, \\ (a_2 + 3a_3 D) A = 3v^2\alpha^2 + vr\alpha. \end{cases} \tag{14}$$

By solving this system of algebraic equations, we obtain four sets of solutions

$$\alpha_1 = P,\ A_1 = Q,\ D_1 = -\frac{1}{2}Q - \frac{r\,|\,v\,|}{\sqrt{-18a_3 v}} - \frac{a_2}{3a_3}, \tag{15}$$

where $C = a_3 D_1^3 + a_2 D_1^2 + (a_1 - v^2) D_1$ is chosen;

$$\alpha_2 = P,\ A_2 = -Q,\ D_2 = \frac{1}{2}Q + \frac{r\,|\,v\,|}{\sqrt{-18a_3 v}} - \frac{a_2}{3a_3}, \tag{16}$$

where $C = a_3 D_2^3 + a_2 D_2^2 + (a_1 - v^2) D_2$ is chosen;

$$\alpha_3 = -P,\ A_3 = Q,\ D_3 = -\frac{1}{2}Q + \frac{r\,|\,v\,|}{\sqrt{-18a_3 v}} - \frac{a_2}{3a_3}, \tag{17}$$

where $C = a_3 D_3^3 + a_2 D_3^2 + (a_1 - v^2) D_3$ is chosen;

$$\alpha_4 = -P,\ A_4 = -Q,\ D_4 = \frac{1}{2}Q - \frac{r\,|\,v\,|}{\sqrt{-18a_3 v}} + \frac{a_2}{3a_3}, \tag{18}$$

where $C = a_3 D_4^3 + a_2 D_4^2 + (a_1 - v^2) D_4$ is chosen. In the above formulas, we have used the following

$$P = \sqrt{\frac{-1}{3v^2}\left[r^2 - 6(a_1 - v^2) + \frac{2a_2^2}{a_3}\right]},\quad Q = \sqrt{\frac{2}{3a_3}\left[r^2 - 6(a_1 - v^2) + \frac{2a_2^2}{a_3}\right]}. \tag{19}$$

Note that (12) may be written as

$$u(\xi) = \frac{A}{2}\mathrm{th}\left[\frac{\alpha}{2}(\xi + \xi_0)\right] + \left(D + \frac{A}{2}\right). \tag{20}$$

From the above four sets of A_i, α_i, D_i, $1 \leqslant i \leqslant 4$, we can generate four solutions $u_i(\xi)$, $1 \leqslant i \leqslant 4$ by (20). But $u_1(\xi) = u_4(\xi)$, $u_2(\xi) = u_3(\xi)$, since th$x$ is an odd function. Therefore we arrive at the following result.

Theorem 1 Let v be a constant wave speed and

$$a_3 < 0,\ r^2 - 6(a_1 - v^2) + \frac{2a_2^2}{a_3} < 0. \tag{21}$$

Then the generalized PC equation (I) has the kink profile solitary-wave solutions

$$u_{\mathrm{PC},r}^{\pm} = \pm\left\{\frac{1}{2}Q\cdot\mathrm{th}\left[\frac{1}{2}P(x - vt + \xi_0)\right] - \frac{r\,|\,v\,|}{\sqrt{-18a_3 v}}\right\} - \frac{a_2}{3a_3}, \tag{22}$$

where P, Q are defined by (19).

By now, we have obtained the kink profile solitary-wave solutions of the generalized PC equation (I) to satisfy (11).

The situation of $r = 0$ of the theorem above gives rise to a special result.

Theorem 2 Let v be a constant wave speed and

$$a_3 < 0,\ 6(v^2 - a_1) + \frac{2a_2^2}{a_3} < 0. \tag{23}$$

Then the generalized PC equation (II) with $r = 0$ has the kink profile solitary-wave solutions

$$u_{\mathrm{PC}}^{\pm} = \pm \frac{1}{2}\sqrt{\frac{2}{3a_3}\left[6(v^2 - a_1) + \frac{2a_2^2}{a_3}\right]} \times$$
$$\mathrm{th}\left\{\frac{1}{2}\sqrt{\frac{-1}{3v^2}\left[6(v^2 - a_1) + \frac{2a_2^2}{a_3}\right]}(x - vt + \xi_0)\right\} - \frac{a_2}{3a_3}. \tag{24}$$

It is easy to see that the solutions shown in the theorem above satisfy the relation of asymptotic values: $3a_3(C_+ + C_-) + 2a_2 = 0$.

2 Bell Profile Solitary-Wave Solutions of (II)

In this section, we want to give bell profile solitary-wave solutions of the generalized PC equation (II) expected in the last section, except the kink profile solitary-wave solutions in Theorem 2.

The traveling-wave solutions of the generalized PC equation (II) $u(x,t) = u(x - vt) = u(\xi)$ to possess the condition (4) must satisfy

$$u''(\xi) + \frac{a_1 - v^2}{v^2}u(\xi) + \frac{a_2}{v^2}u^2(\xi) + \frac{a_3}{v^2}u^3(\xi) = \frac{C}{v^2}, \tag{25}$$

where C is an integration constant. In order to search for bell profile solitary-wave solutions of (II), we assume that the equation (25) has solutions of the form

$$u(\xi) = \frac{Ae^{\alpha(\xi+\xi_0)}}{(1 + e^{\alpha(\xi+\xi_0)})^2 + Be^{\alpha(\xi+\xi_0)}} + D, \tag{26}$$

where A, B, α, D are constants to be determined. A direct computation gives

$$u'(\xi) = \frac{A\alpha(\eta - \eta^3)}{[(1 + \eta)^2 + B\eta]^2}, \tag{27}$$

$$u''(\xi) = \frac{A\alpha^2[\eta - (2 + B)\eta^2 - 6\eta^3 - (2 + B)\eta^4 + \eta^5]}{[(1 + \eta)^2 + B\eta]^3}, \tag{28}$$

where $\eta = e^{\alpha(\xi+\xi_0)}$. Substituting the above three equalities into (25) and noticing the linear independent property of $e^{k\alpha(\xi+\xi_0)}$, $0 \leqslant k \leqslant 6$, may lead to

$$\begin{cases} a_3D^3 + a_2D^2 + (a_1 - v^2)D - C = 0, \\ v^2\alpha^2 + 2a_2D + 3a_3D^2 + (a_1 - v^2) = 0, \\ (a_2 + 3a_3D)A - 3v^2\alpha^2(2 + B) = 0, \\ a_3A^2 + (2 + B)(a_2 + 3a_3D)A - v^2\alpha^2(2 + B)^2 - 8v^2\alpha^2 = 0. \end{cases} \tag{29}$$

When D solves $a_3D^3 + a_2D^2 + (a_1 - v^2)D - C = 0$, the above system of algebraic equations has the following two sorts of solutions.

1) If we assume that

$$\begin{cases} a_2 + 3a_3D \neq 0,\ a_1 - v^2 + 3a_3D^2 + 2a_2D < 0, \\ 2a_2^2 - 9a_3(a_1 - v^2) - 9a_3^2D^2 - 6a_2a_3D > 0, \end{cases} \tag{30}$$

the system (29) has two sets of solutions

$$\begin{cases} \alpha = \pm \dfrac{1}{|v|}\sqrt{-(a_1 - v^2 + 3a_3D^2 + 2a_2D)}, \\ A = \pm \dfrac{-6\sqrt{2}(a_1 - v^2 + 3a_3D^2 + 2a_2D)\,|a_2 + 3a_3D|}{\sqrt{2a_2^2 - 9a_3(a_1 - v^2) - 9a_3^2D^2 - 6a_2a_3D(a_2 + 3a_3D)}}, \\ B = -2 \pm \dfrac{2\sqrt{2}\,|a_2 + 3a_3D|}{\sqrt{2a_2^2 - 9a_3(a_1 - v^2) - 9a_3^2D^2 - 6a_2a_3D}}. \end{cases} \tag{31}$$

2) If we assume that

$$a_2 + 3a_3D = 0,\ a_3 > 0,\ a_1 - v^2 + a_2D < 0, \tag{32}$$

the system (29) has two sets of solutions

$$\alpha = \pm \frac{1}{|v|}\sqrt{-(a_1 - v^2 + a_2D)},\ A = \pm\sqrt{\frac{-8}{a_3}(a_1 - v^2 + a_2D)},\ B = -2. \tag{33}$$

Note that (26) may be written as

$$u(\xi) = \frac{A\operatorname{sech}^2\left[\frac{\alpha}{2}(\xi + \xi_0)\right]}{4 + B\operatorname{sech}^2\left[\frac{\alpha}{2}(\xi + \xi_0)\right]} + D. \tag{34}$$

Therefore the above discussion engenders the following result.

Theorem 3 Let v be a constant wave speed and D be a constant. Then

1) If the condition (30) holds, the generalized PC equation (II) has the bell profile solitary-wave solutions

$$u_{PC}^{\pm} = \Bigg\{\Bigg[\mp \frac{3\sqrt{2}(a_1 - v^2 + 3a_3D^2 + 2a_2D)\,|a_2 + 3a_3D|}{\sqrt{2a_2^2 - 9a_3(a_1 - v^2) - 9a_3^2D^2 - 6a_2a_3D(a_2 + 3a_3D)}} \times$$
$$\operatorname{sech}^2\left[\frac{1}{2|v|}\sqrt{-(a_1 - v^2 + 3a_3D^2 + 2a_2D)}(x - vt + \xi_0)\right]\Bigg]\Bigg/$$
$$\Bigg[2 + \left(-1 \pm \frac{\sqrt{2}\,|a_2 + 3a_3D|}{\sqrt{2a_2^2 - 9a_3(a_1 - v^2) - 9a_3^2D^2 - 6a_2a_3D}}\right)\times$$
$$\operatorname{sech}^2\left[\frac{1}{2|v|}\sqrt{-(a_1 - v^2 + 3a_3D^2 + 2a_2D)}(x - vt + \xi_0)\right]\Bigg]\Bigg\} + D; \tag{35}$$

2) If the condition (32) holds, the generalized PC equation (II) has the bell profile solitary-wave solutions

$$u_{PC}^{\pm} = \frac{\pm\sqrt{\frac{-2}{a_3}(a_1 - v^2 + a_2D)}\operatorname{sech}^2\left[\frac{1}{2|v|}\sqrt{-(a_1 - v^2 + a_2D)}(x - vt + \xi_0)\right]}{2 - \operatorname{sech}^2\left[\frac{1}{2|v|}\sqrt{-(a_1 - v^2 + a_2D)}(x - vt + \xi_0)\right]} + D =$$
$$\pm\sqrt{\frac{-2}{a_3}(a_1 - v^2 + a_2D)}\operatorname{sech}\left[\frac{1}{|v|}\sqrt{-(a_1 - v^2 + a_2D)}(x - vt + \xi_0)\right] + D. \tag{36}$$

By now, we have obtained the kink profile solitary-wave solutions and the bell profile solitary-wave solutions of the generalized PC equation (Ⅱ) expected in the first section.

Remark 1) Since sech$x$ is an even function, the solutions with

$$\alpha = \pm (1/|v|)\sqrt{-(a_1 - v^2 + 3a_3D^2 + 2a_2D)}$$

are the same, which are displayed as a solution in the above theorem.

2) The solution $u_{PC}^{+}(\xi)$ defined by (35) is a bounded analytic solution. Moreover the solution $u_{PC}^{-}(\xi)$ defined by (35) is also bounded and analytic provided that $a_3 > 0$.

Let us now provide with some special examples. We first choose $a_2 = 0$, $a_1 = 1$, $a_3 = 1$ and the integration constants $C = 0$, $D = 0$. This moment (36) gives a solitary-wave solution

$$u_{PC}^{\pm}(x,t) = \pm\sqrt{2(v^2-1)}\,\mathrm{sech}\frac{\sqrt{v^2-1}}{v}(x - vt + \xi_0), \qquad v > 0$$

to a particular PC equation

$$u_{tt} - u_{ttxx} - (u + u^3)_{xx} = 0,$$

which is the same as the one defined by (11) in [1]. We second choose $a_2 \neq 0$, $a_1 = 1$, $a_3 = 1$ and the integration constants $C = 0$, $D = 0$. This moment (35) gives a solitary-wave solution

$$u_{PC}^{\pm}(x,t) = \frac{\mp\dfrac{3\sqrt{2}(1-v^2)\,|a_2|}{\sqrt{2a_2^2 - 9(1-v^2)\,a_2}}\mathrm{sech}^2\left[\dfrac{1}{2|v|}\sqrt{v^2-1}(x - vt + \xi_0)\right]}{2 + \left(-1 \pm \dfrac{\sqrt{2}a_2}{\sqrt{2a_2^2 - 9(1-v^2)}}\right)\mathrm{sech}^2\left[\dfrac{1}{2|v|}\sqrt{v^2-1}(x - vt + \xi_0)\right]}$$

to the generalized PC equation (Ⅱ), which possesses the zero asymptotic values.

3 Bell Profile Solitary-Wave Solutions of (Ⅲ)

The solitary-wave solutions $u(x,t) = u(x - vt) = u(\xi)$ to the generalized PC equation (Ⅲ) to possess

$$u(\xi),\ u'(\xi),\ u''(\xi),\ u'''(\xi) \to 0, \qquad |\xi| \to \infty \tag{37}$$

must satisfy

$$u''(\xi) + \frac{a_1 - v^2}{v^2}u(\xi) + \frac{a_3}{v^2}u^3(\xi) + \frac{a_5}{v^2}u^5(\xi) = 0. \tag{38}$$

In order to seek explicit solutions, make a transformation

$$u(\xi) = \sqrt{\phi(\xi)}. \tag{39}$$

Then $\phi(\xi)$ has to satisfy

$$2\phi(\xi)\phi''(\xi) - (\phi'(\xi))^2 + \frac{4(a_1 - v^2)}{v^2}\phi^2(\xi) + \frac{4a_3}{v^2}\phi^3(\xi) + \frac{4a_5}{v^2}\phi^4(\xi) = 0. \tag{40}$$

Similarly we assume that (40) has solutions of the form

$$\phi(\xi) = \frac{Ae^{\alpha(\xi+\xi_0)}}{(1 + e^{\alpha(\xi+\xi_0)})^2 + Be^{\alpha(\xi+\xi_0)}} = \frac{A\,\mathrm{sech}^2\frac{\alpha}{2}(\xi + \xi_0)}{4 + B\,\mathrm{sech}^2\frac{\alpha}{2}(\xi + \xi_0)}, \tag{41}$$

where A, α, B are constants to be determined and ξ_0 is an arbitrary phase shift. Therefore the equality (40) may be reduced to

$$\begin{cases} v^2\alpha^2 + 4(a_1 - v^2) = 0, \\ 2a_3A + 4(a_1 - v^2)(2 + B) - v^2\alpha^2(2 + B) = 0, \\ -5v^2\alpha + 2(a_1 - v^2)[2 + (2 + B)^2] + 2a_3A(2 + B) + 2a_5A^2 = 0. \end{cases} \tag{42}$$

This system has the following solutions

$$\alpha_\pm = \pm 2\sqrt{\frac{v^2 - a_1}{v^2}}, \quad A_\pm = \pm 8\sqrt{\frac{3(a_1 - v^2)^2}{3a_3^2 - 16a_5(a_1 - v^2)}},$$

$$B_\pm = -2 \pm \frac{2\sqrt{3}a_3}{\sqrt{3a_3^2 - 16a_5(a_1 - v^2)}}.$$

Therefore the equation (40) has two solutions

$$\phi_1(\xi) = \frac{4\sqrt{\dfrac{3(a_1 - v^2)^2}{3a_3^2 - 16a_5(a_1 - v^2)}}\operatorname{sech}^2\sqrt{\dfrac{v^2 - a_1}{v^2}}(x - vt + \xi_0)}{2 + \left(-1 + \dfrac{\sqrt{3}a_3}{\sqrt{3a_3^2 - 16a_5(a_1 - v^2)}}\right)\operatorname{sech}^2\sqrt{\dfrac{v^2 - a_1}{v^2}}(x - vt + \xi_0)}, \tag{43}$$

$$\phi_2(\xi) = \frac{-4\sqrt{\dfrac{3(a_1 - v^2)^2}{3a_3^2 - 16a_5(a_1 - v^2)}}\operatorname{sech}^2\sqrt{\dfrac{v^2 - a_1}{v^2}}(x - vt + \xi_0)}{2 - \left(1 + \dfrac{\sqrt{3}a_3}{\sqrt{3a_3^2 - 16a_5(a_1 - v^2)}}\right)\operatorname{sech}^2\sqrt{\dfrac{v^2 - a_1}{v^2}}(x - vt + \xi_0)}. \tag{44}$$

Note that the solutions with $\alpha = -2\sqrt{(v^2 - a_1)/v^2}$ and $\alpha = 2\sqrt{(v^2 - a_1)/v^2}$ are the same and thus they are incorporated into a solution. It is easy to check that if $v^2 - a_1 > 0$, then the condition $a_3 > 0$, $a_5 \geqslant 0$ or $a_3 \leqslant 0$, $a_5 > 0$ guarantees that $\phi_1(\xi) > 0$, but $\phi_2(\xi)$ is generally negative since $\phi_2(\xi, a_3) = -\phi_1(\xi, -a_3)$. Noting that $-u(\xi)$ is also a solution to (38) while $u(\xi)$ is a solution to (38), we can obtain the following theorem.

Theorem 4 Let v denote a constant wave speed and $v^2 - a_1 > 0$. If the condition $a_3 > 0$, $a_5 \geqslant 0$ or the condition $a_3 \leqslant 0$, $a_5 > 0$ holds, then the generalized PC equation (Ⅲ) has the bell profile solitary-wave solutions

$$u(x,t) = \pm\left[\frac{4\sqrt{\dfrac{3(a_1 - v^2)^2}{3a_3^2 - 16a_5(a_1 - v^2)}}\operatorname{sech}^2\sqrt{\dfrac{v^2 - a_1}{v^2}}(x - vt + \xi_0)}{2 + \left(-1 + \dfrac{\sqrt{3}a_3}{\sqrt{3a_3^2 - 16a_5(a_1 - v^2)}}\right)\operatorname{sech}^2\sqrt{\dfrac{v^2 - a_1}{v^2}}(x - vt + \xi_0)}\right]^{\frac{1}{2}}. \tag{45}$$

If we choose $a_1 = 1$, $a_3 = 1/3$, $a_5 = 0$ or $a_1 = 1$, $a_3 = 0$, $a_5 = 1/5$, the above result leads to two solitary-wave solutions

$$u(x,t) = \pm\sqrt{6(v^2 - 1)}\operatorname{sech}\left[\sqrt{\frac{v^2 - 1}{v^2}}(x - vt + \xi_0)\right],$$

$$u(x,t) = \pm\left\{\sqrt{15(v^2 - 1)}\operatorname{sech}\left[2\sqrt{\frac{v^2 - 1}{v^2}}(x - vt + \xi_0)\right]\right\}^{\frac{1}{2}},$$

corresponding to PC equation (1) with $p = 3$ or $p = 5$. They are in agreement with the ones defined by (12) in [2].

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[1] Bogolubsky I L. Some examples of inelastic soliton interaction[J]. *Computer Physics Communications*, 1977, **13**(2):149 ~ 155.

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