THE CONSERVATION AND EXPLOITATION OF VULNERABLE RESOURCES

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The maximum principle of deterministic optimal control, which has proved to be a very useful tool in theoretical bioeconomics, is extended in this article to cover the optimal exploitation of a biological resource vulnerable to catastrophic collapse, the probability of which may depend in general on the state of the resource, the current control and time. A general formulation of the maximum principle for such stochastic problems is presented and a number of applications are outlined. These include: optimal harvesting of a fishery vulnerable to catastrophic collapse; optimal thinning of a forest vulnerable to fire; optimal expenditure and investment in forest fire protection and optimal consumption–pollution tradeoffs in an ecosystem vulnerable to pollution-related collapse. In addition an application of the method to a highly stylized behavioral ecology model is given.

1. Introduction. Colin Clark's two major scientific contributions to date have been in the areas of Bioeconomics and Behavioral Ecology. Although these might at first sight appear as disparate areas, falling respectively within the disciplines of economics and biology, there is in fact a common mathematical thread running through almost all of this work, linking and unifying it. This common thread is the mathematics of dynamic optimization in one form or another.

One of the key insights of Clark's path-breaking 1976 book *Mathematical Bioeconomics* is that the theory of conservation of renewable resources involves resource use over time, and thus that dynamic methods must be used in analysing models of resource management. Prior to the appearance of the book, biologists, and resource managers with a biological training, had tended to think largely in terms of equilibrium concepts such as maximum sustainable yield, thereby ignoring the temporal dimension of resource management. Some economists, more accustomed to dealing with questions of resource allocation

over time, had recognized the essentially dynamic nature of biological resource management and, as in many other areas of economics at the time, had begun to apply the powerful methods of optimal control theory. It was not, however, until the publication of Clark's book that a unified theory of bioeconomics or biological resource management could be said to exist. Running through this work, almost as a central unifying theme, is the analytic application of the methods of continuous-time, deterministic optimal control, i.e. of the *Pontryagin maximum principle*. For example, in the opening chapters Clark showed how the situation of open-access, rent-dissipating bionomic equilibrium, and the situation of the maximization of sustainable biological or economic yield both emerge as special cases of the maximization of discounted present value, corresponding, respectively, to infinite and zero discount rates. He also showed how the preferred policy prescription of many economists to alleviate the distortions caused by open-access, namely the privatization of common-property resources, could lead, in the case of biological resources, to extinction if the discount rate employed by the sole owner was sufficiently high.

For the most part *Mathematical Bioeconomics* dealt with continuous-time deterministic models. Since its publication the field of bioeconomics has grown rapidly, with much of the most interesting work involving stochastic models, mainly in discrete time. Although there is a stochastic version of the maximum principle, it has proved to be not very useful in practice. For stochastic problems *dynamic programming* has proved to be of much greater value. Colin Clark has been heavily involved in this work, and his 1985 book, *Bioeconomic Modelling and Fisheries Management* and the second edition of *Mathematical Bioeconomics* (1990) contain much interesting material utilizing stochastic models. Other books dealing with randomness and uncertainty in resource management are Mangel (1985) and Walters (1986), and a review of some of the earlier work in stochastic bioeconomics is given by Anderson and Sutinen (1984). Dynamic optimization based on stochastic dynamic programming and Bayesian decision theory play a central role in much of this work.

Colin Clark's other great scientific contribution has been in the area of behavioral ecology, in which methods of dynamic optimization are used to model and explain the behavior of biological organisms. Stochastic methods are of the essence here, and Clark's 1988 book (with Marc Mangel) *Dynamic Modelling and Behavioral Ecology* relies heavily on stochastic dynamic programming. In this respect the book can be seen as a natural outgrowth of Clark's earlier pioneering work in bioeconomics. However, whereas much of the earlier work in deterministic bioeconomics was characterized by elegant analytic results, the later work in stochastic bioeconomics and in behavioral ecology has relied much more heavily on the results of numerical optimization from which qualitative insights and conclusions have been drawn. Of course this does not imply any inferiority of dynamic programming with respect to

variational methods such as the maximum principle, but simply the fact that analytic solutions are usually impossible to obtain for stochastic optimization problems.

Nevertheless, while in general terms it is true that stochastic optimization problems are better handled by means of dynamic programming than by variational methods, and that analytic solutions are not usually forthcoming for such problems, there is one class of stochastic problems where this does not hold, and it is this class of problems which is the subject of this paper. This class contains optimization models which are essentially deterministic in all aspects save one, *viz.* the presence of the risk of random catastrophic collapse. It turns out that problems of this sort can be addressed in a deterministic framework, provided that the survival probability is carried along as a state variable. Thus the deterministic maximum principle can be used and frequently analytic results can be obtained. In this article we shall discuss a number of applications of the technique in the resource management field. These include the management of a fishery vulnerable to catastrophic collapse, where the probability of collapse at any time depends on the size of the fish stock; the optimal thinning and rotation of a forest stand vulnerable to destruction by fire where the fire hazard depends on the age of the stand along with the thinning activity; the optimal patterns of expenditure and investment in protecting a forest stand against fire; and finally the optimal tradeoff between consumption and pollution in an economy vulnerable to catastrophic collapse due to the destruction of an ecosystem with the probability of collapse dependent on the current level of pollution. Also included is a highly stylized version of a behavioral ecology model.

All of the applications discussed have appeared or will appear elsewhere and the details are omitted in this article. Its purpose is to illustrate how deterministic formulations of some resource management problems, many of which appeared in Clark's 1976 book, can be extended in a fairly straightforward way to take into account the possibility of random catastrophic collapse.

The method used appears to have been first introduced in the management science literature by Kamien and Schwartz (1971) to deal with problems of machine maintenance and replacement. It has been used by these authors to deal with other problems such as limit pricing and investment in research and development [see Kamien and Schwartz (1981) for reference], and by Sethi (1979) to describe the behavior of a thief. Its use in resource management problems dates from Reed (1987).

2. Infinite Time Horizon Model.   Consider a resource under management which is vulnerable at any time to random catastrophic collapse. The probability of collapse at any time can be characterized by a *hazard rate function* (see, for example, Thompson, 1988):

$$h(t) = \lim_{\Delta t \to 0} \{P(\text{resource collapses in } (t, t+\Delta t) \mid \text{alive at } t)/\Delta t\}. \qquad (1)$$

The probability that the resource survives to time t is given by the *survivor function* which is related to the hazard rate by:

$$S(t) = \exp\left\{-\int_0^t h(z)\,\mathrm{d}z\right\}. \qquad (2)$$

Now suppose that up until a collapse (if ever one occurs), the resource is producing a flow of revenues $g(x, u, t)$ where $x(t)$ is the size of the resource stock and $u(t)$ is the control (e.g. harvest) at time t.

We shall assume that the stock dynamics are given by a differential equation:

$$\dot{x} = f(x, u, t). \qquad (3)$$

Let us further suppose that if the stock collapses at time τ a benefit of magnitude $G(x(\tau))$ is earned (usually this will be a negative quantity denoting a cost). If the instantaneous discount rate is δ then the expected net present value of the revenues earned up until the time of collapse is:

$$J = E\left\{\int_0^\tau \mathrm{e}^{-\delta t} g(x, u, t)\,\mathrm{d}t + \mathrm{e}^{-\delta\tau} G(x(\tau))\right\} \qquad (4)$$

where the expectation is taken with respect to the random variable τ. Note that a collapse is not certain. The resource may survive forever if the improper integral $\int_0^\infty h(z)\,\mathrm{d}z$ converges to a finite quantity [see (2)].

To maximize expected present-value we seek to maximize (4) subject to the dynamic constraint (3), and any other constraints on the control u [e.g. in a harvest problem we might have $u(t) \geqslant 0$].

It is shown in the Appendix how the above stochastic optimization problem can be expressed as a problem in deterministic optimal control, by introducing a new state variable:

$$y(t) = -\log S(t). \qquad (5)$$

If the hazard at time t depends on the stock $x(t)$ and control $u(t)$ as well as time, say according to:

$$h(t) = \varphi(x, u, t), \qquad (6)$$

then the dynamics of the variable y are given by:

$$\dot{y} = \varphi(x, u, t): \quad y(0) = 0. \qquad (7)$$

Also the expectation in (4) can be evaluated in terms of $y(t)$ to give

$$J = \int_0^\infty e^{-\delta t - y(t)} [g(x, u, t) - \delta G(x) + G'(x) f(x, u, t)] \, dt + G(x(0)). \tag{8}$$

We seek to maximize (8) subject to the dynamic constraints (3) and (7). This is a standard problem in deterministic optimal control similar to the one that arises if there is no possibility of collapse, except for the inclusion of a new state variable $y(t)$ related to the survival function, and an adjustment to the benefit flow function. The term $G(x(0))$ is constant and can be dropped from the optimization problem. Note that the variable $y(t)$ operates like a premium added to the discount rate δ.

This problem can be solved using the standard maximum principle. However from the point of view of interpreting the results it is perhaps better to introduce a *conditional current value Hamiltonian*:

$$\bar{H} = \tilde{g}(x, u, t) + \rho_1 f(x, u, t) + \rho_2 \varphi(x, u, t) \tag{9}$$

where

$$\tilde{g}(x, u, t) = g(x, u, t) - \delta G(x) + G'(x) f(x, u, t) \tag{10}$$

is the term in square brackets of the integrand in (8).

The conditional current-value Hamiltonian is actually the standard current-value Hamiltonian [see e.g. Clark (1976, p. 105)] divided by the survival probability $e^{-y(t)}$. Likewise the co-state variables ρ_1 and ρ_2 are current shadow prices divided by the survival probability. These variables satisfy the co-state equations:

$$\dot{\rho}_1 = [\delta + \varphi(x, u, t) - f_x(x, u, t)]\rho_1 - \varphi_x(x, u, t)\rho_2 - \tilde{g}_x(x, u, t) \tag{11}$$

(where subscripts denote partial derivatives), and:

$$\dot{\rho}_2 = [\delta + \varphi(x, u, t)]\rho_2 + \tilde{g}(x, u, t). \tag{12}$$

The optimal control $u^*(t)$ maximizes $\bar{H}$ at all t, and in principle can be found by solving the system given by this condition, and equations (3), (7), (11) and (12).

The co-state variable ρ_1 can be interpreted in the usual way as the shadow value of an additional unit of stock at time t, only now it is conditional on the resource still being alive (i.e. not having yet collapsed) at this time. The second co-state variable ρ_2 associated with the new variable $y(t)$ has an interesting interpretation. It can be shown (see Appendix) that:

$$-\rho_2(t) = W(t) - G(x(t)) \tag{13}$$

where $W(t)$ of t is the *value* of the resource at time t conditional on it being alive at that time, i.e. $W(t)$ is the expected present value of the flow of benefits plus

rewards (penalties) earned from any subsequent collapse, conditional on it being alive at time t, and under optimal exploitation.

In particular:

$$-\rho_2(0) = \max_{u(t)} J - G(x_0) = \max_{u(t)} \int_0^\infty e^{-\delta t - y(t)} \tilde{g}(x, u, t) \, dt.$$

In the case when there are no rewards or penalties associated with a collapse $(G(x) \equiv 0)$ the co-state variable ρ_2 represents simply the negative of the value of the resource, $W(t)$, at time t, given that it is alive; or in the behavioral ecology context [see Section 3(c)] ρ_2 represents the negative of the lifetime fitness of an organism at time t. The co-state equation (12) is nothing more than the Bellman equation of dynamic programming for this problem. In the more general case $(G(x) \not\equiv 0)$ the co-state equation (12) can be derived from the Bellman equation for $W(t)$ and *vice versa*.

We now illustrate the technique in two particular applications.

Management of a fishery vulnerable to collapse.   Many fisheries (especially of schooling clupeid species) have been known to experience sudden catastrophic collapse. Some examples are the North Sea herring, the Southwest African pilchard, the California sardine, and most spectacular of all the Peruvian anchoveta [see Reed (1988) for references]. It seems reasonable to assume that the probability of collapse depends inversely upon the stock density. Thus, let us suppose that the hazard-rate function φ depends only upon stock density x:

$$h(t) = \varphi(x, u, t) = \psi(x), \tag{14}$$

where ψ is a decreasing function. Following Clark (1976, Chap. 2) assume that the revenue flow function is:

$$g(x, u, t) = [p - c(x)]u \tag{15}$$

where p is the (constant) price of fish, $c(x)$ is the unit cost of harvesting when the population is at size x and u is the rate of harvesting; assume also the stock dynamics are given by:

$$\dot{x} = f(x, u, t) = F(x) - u \tag{16}$$

where $F(x)$ is the familiar population growth function. For the moment we shall assume that there is no penalty associated with a collapse $(G \equiv 0)$.

The conditional current-value Hamiltonian is:

$$\bar{H} = [p - c(x)]u + \rho_1[F(x) - u] + \rho_2\psi(x). \tag{17}$$

This is linear in u and so the optimal control can only involve components of bang-bang or singular control. In fact it can be shown fairly easily [see Reed (1988) for details] that the optimal harvest policy drives the stock as rapidly as possible to an optimum biomass level X^* which is the solution to a modified golden rule (MGR) equation:

$$F'(x) - \frac{c'(x)}{p - c(x)} F(x) = \delta + \psi(x) + \frac{\psi'(x)F(x)}{\delta + \psi(x)}. \tag{18}$$

This is analogous to the golden rule equation derived by Clark (1976, p. 40) save for the inclusion of two additional terms on the right of (18) involving the hazard function ψ. The significance of these terms is discussed in Reed (1988). Here it suffices to note that the second term on the right which is positive, depends on the size of the hazard, while the third which is negative depends on the rate at which the hazard is changing. The effect of the first term is to lower the optimal biomass (optimally one might want to practice less conservation because the population may not survive for later use), while the effect of the second is to increase the optimal biomass (one would want to avoid driving the stock low because it would increase the hazard). Whether the optimal biomass is greater or less than that in a risk-free environment depends on the particular functions, ψ and F.

It is easily verified that the inclusion of a penalty $G(X) \equiv K$ associated with a collapse leads to an additional term:

$$\frac{\psi'(x)}{\delta + \psi(x)} \times \frac{\delta K}{p - c(x)}$$

being included on the right hand side of the MGR equation (18), thereby increasing the optimal stock biomass as one would expect. In Reed (1988) the adjustments required to the model from the inclusion of the possibility of recovery after a collapse, in an uncertain time, are discussed.

Optimal pollution–consumption trade-offs in an environment vulnerable to irreversible collapse. A clean environment is a natural resource, and in many cases, up to a point, it is a renewable one. Natural processes can in time often cleanse pollutants from the environment. Not surprisingly then the methods developed for the optimal management of renewable biological resources have been applied to questions relating to the optimal tradeoffs between consumption and pollution. The seminal papers on this are those of Plourde (1972) and Forster (1973, 1975). [See also Conrad and Clark (1987, Chapt. 4) where the relationship with other renewable resource management problems is made apparent.]

In some cases once threshold levels have been exceeded, the environment

loses its ability to cleanse itself, and an irreversible change takes place. For example, patterns of consumption based on fossil fuel energy may lead to concentrations of greenhouse gases which could cause irreversible climate changes. Similarly, certain toxic substances inhibit the self-purification powers of water by killing the bacteria required to degrade organic wastes. If the pollution level is great enough it may render a waterway biologically dead and unable to cleanse itself.

Threshold levels are largely unknown and a simple way to model this is to assume that the hazard-rate for an irreversible collapse of the environment depends on the level of pollution P. Thus we assume:

$$h(t) = \psi(P(t)) \tag{19}$$

where in this case ψ is non-decreasing and convex.

The standard optimal control model for consumption–pollution tradeoffs [Forster (1973)] assumes that:

$$\dot{P} = Z(C) - F(P) \tag{20}$$

where F is the pollution decay function (positive, concave and increasing), and $Z(C)$ is the net rate of creation of pollution, when consumption is at level C. The objective to be maximized is the present value of the flow of utilities $U(C, P)$ where $U_C > 0$, $U_{CC} < 0$, $U_P < 0$, $U_{PP} < 0$.

If we suppose that after a collapse of the environment the utility flow is reduced to some base level $U_0 \ll 0$, then the expected discounted present value of utility over an infinite time horizon is:

$$J = E\left\{ \int_0^\tau e^{-\delta t} U(C, P)\, dt + \int_\tau^\infty U_0\, e^{-\delta t}\, dt \right\}$$

$$= E\left\{ \int_0^\tau e^{-\delta t} U(C, P)\, dt + e^{-\delta \tau}\, \frac{U_0}{\delta} \right\}. \tag{21}$$

Using the methods described earlier this can be expressed as:

$$J = \int_0^\infty e^{-\delta t - y(t)} [U(C, P) - U_0]\, dt + \frac{U_0}{\delta} \tag{22}$$

where

$$\dot{y} = \psi(P). \tag{23}$$

Consumption $C(t)$ is the control variable and maximization of the expected

present value of utility is obtained by solving the optimization problem of maximizing (22) subject to (20) and (23).

In Reed and Clarke (1990) the optimal solution is discussed, and the optimal levels of pollution and consumption are compared with those that would be prescribed by the model if risk of collapse were ignored. Also the disequilibrium *user costs* which should be charged to polluters in the two cases are compared. It is shown that:

(a) If the hazard-rate does *not* depend on current pollution levels ($\psi(P) \equiv$ constant) then optimal consumption and pollution levels are *higher* when risk of collapse is included in the model, than when not. Furthermore user costs are *lower*.

(b) If the hazard-rate exhibits some dependence on current pollution ($\psi'(P) > 0$) and the disutility ($-U_0$) associated with a collapse is suitably large, then optimal consumption and pollution levels are *lower* and user costs are *higher* when risk of collapse is included in the analysis.

The reason for the, at first somewhat surprising, result (a) is the same as in the fishery model. If the risk of collapse is independent of any action users of the resource may take, then its presence acts simply as a premium added to the discount rate (the variable $y(t)$ is simply a constant times t), thereby leading to less conservation of the resource. On the other hand, when the hazard depends on the level of pollution its effect can be reduced by lowering consumption and pollution levels. If the consequences of a collapse are sufficiently horrible it becomes optimal to reduce pollution, overcoming the increased discount rate effect. User costs to be charged to polluters should, in consequence, be higher.

3. Finite Time Horizon Model. We use the same model and notation as in the previous section. However we suppose that if the resource is still surviving at time T, a reward $\Phi(X(T))$ is earned. For instance this could be the revenue earned through clear-cutting a stand of trees at the rotation age. It could also represent the terminal fitness function of an organism in a behavioral ecology model [Mangel and Clark (1988, p. 58)].

It is shown in the Appendix that the expected present value functional is:

$$J = \int_0^T e^{-\delta t - y(t)}[g(x, u, t) - \delta G(x) + G'(x)f(x, u, t)]\, dt$$

$$+ e^{-\delta T - y(T)}[\Phi(x(T)) - G(x(T^-))] + G(x(0)) \tag{24}$$

where $G(x(T^-)) = \lim_{t \nearrow T} G(x(t))$. Since $G(x(0))$ is a constant it can be eliminated from the optimization problem, which is similar to the infinite time horizon one in the benefit flow function, but now includes a terminal reward. To solve this using the maximum principle the conditional current-value Hamiltonian is as

in (9), and the co-state equations as in (11) and (12). However now there are *transversality conditions* which determine the values of the co-state variables at time T. They are:

$$\rho_1(T) = [\Phi'(x(T)) - G'(x(T^-))] \tag{25}$$

and

$$\rho_2(T) = -[\Phi(x(T)) - G(x(T^-))]. \tag{26}$$

If the time T is not specified (but is rather a control variable) then there is an additional *free terminal time condition*:

$$\bar{H}(T) + \frac{\partial}{\partial T}[\Phi(x(T)) - G(x(T^-))] - \delta[\Phi(x(T)) - G(x(T^-))] = 0. \tag{27}$$

We now give some examples from forest management and behavioral ecology.

Optimal protection of a forest against fire.   We consider a forest stand which if clear-cut harvested at age T yields a revenue $V(T)$ but incurs costs c_1. If however a fire occurs before harvest takes place, we assume that costs c_2 are incurred. Suppose that the "natural" hazard (in the absence of any fire protection) depends only on age:

$$h(t) = a(t)$$

say. If on the other hand at time t a flow of expenditure of size $u(t)$ ($ per unit time), is spent on protection, suppose that the hazard is reduced to:

$$\varphi(t, u) = \psi(u(t))a(t) \tag{28}$$

where ψ is a decreasing convex function.

The problem we consider is to choose a harvest age T, and a protection schedule $u(t)$, $0 < t < T$ to maximize the expected present value of harvest revenues net of protection and other costs. In the form of equation (24) this can be expressed as maximize:

$$J = \int_0^T e^{-\delta t - y(t)}[-u(t) + \delta c_2] \, dt + e^{-\delta T - y(T)}[V(T) - c_1 + c_2] - c_2 \tag{29}$$

over T and $u(t)$; $0 < t < T$, subject to:

$$\dot{y} = \psi(u)a(t)$$

and

$$u \geqslant 0. \tag{30}$$

There is only a single state variable $y(t)$ in this optimization problem [$V(T)$ can be assumed to be exogenously determined]. Thus the conditional current-value Hamiltonian will involve only one co-state variable, which we shall denote by ρ_2 for the sake of consistency. It is:

$$\bar{H} = -u(t) + \delta c_2 + \rho_2 \psi(u)a(t). \tag{31}$$

This is maximized over u at:

$$u = 0 \text{ if } \rho_2 \psi'(0)a(t) \leqslant 1$$

otherwise at the solution to:

$$\rho_2 \psi'(u)a(t) = 1. \tag{32}$$

Differentiating this with regard to t and using the co-state equation:

$$\dot{\rho}_2 = [\delta + \psi(u(t))a(t)]\rho_2 \tag{33}$$

it can be shown that when the optimal expenditure is positive it follows the differential equation:

$$\psi''(u)\dot{u} = [\psi'(u)]^2[u - \delta c_2]a(t) - \psi'(u)[\delta + \psi(u)a(t)] - \frac{h'(t)}{h(t)}\psi'(u). \tag{34}$$

Furthermore the transversality condition (26) and the free terminal time condition (27) give that at the optimal cutting time T:

$$\frac{\partial}{\partial T}[V(T) - c_1] - \psi(u(T))a(T)[V(T) - c_1 + c_2] - u(T) = \delta[V(T) - c_1]. \tag{35}$$

This is a version of the Wicksell (1934) condition, which requires that at the optimal cutting time the incremental expected net revenue earned through not cutting the stand [the left hand side of (35)] exactly equal the revenue earned through cutting the stand and investing the proceeds at interest rate δ [the right hand side of (35)].

In the case of an ongoing forest (Faustmann paradigm) it is shown in Reed (1987) that, for a constant hazard [$a(t) \equiv$ constant], the optimal expenditure on protection is increasing over the whole lifetime of a stand, or is initially zero followed by a period of increase. To determine it exactly involves numerical solution. Examples and an interpretation of equation (34) are given in Reed (1987).

We describe now another model for optimal fire protection. Rather than assume that the hazard rate for a protected forest depends solely on the current

expenditure, we assume instead that it depends on the accumulated capital investment in protection, $K(t)$, at that time, i.e. that (28) is replaced by:

$$h(t) = \varphi(t, K) = \theta(K)a(t) \tag{36}$$

where θ is a decreasing convex function.

The accumulated capital is assumed subject to depreciation at a constant rate γ. In addition new investment at the rate $u(t)$ ($ per unit time) can be made at any time. Thus the state variable $K(t)$ follows the differential equation:

$$\dot{K} = -\gamma K + u(t). \tag{37}$$

As before $u(t)$ is a control variable, subject to the constraint $u(t) \geqslant 0$.

The expected present value net of costs over a single rotation is:

$$J = \int_0^T e^{-\delta t - y(t)} [\delta c_2 - u(t)] \, dt + e^{-\delta T - y(T)} [V(T) - c_1 + c_2] + c_2.$$

This must be maximized over T and $u(t)$, $0 \leqslant t < T$ subject to the dynamic constraints (37) and:

$$\dot{y}(t) = \theta(K)a(t). \tag{38}$$

The conditional current-value Hamiltonian is:

$$\bar{H} = \delta c_2 - u + \rho_1(-\gamma K + u) + \rho_2 \theta(K)a(t) \tag{39}$$

which is linear in u and so the optimal control can only comprise bang-bang or singular components.

In Reed (1989) a differential equation for K on the singular path is derived and is solved numerically in an example. The optimal policy involves an initial pulse of investment followed by a period of increasing investment along the singular path, with a final period of no investment leading up to the optimal harvest age. Also in Reed (1989) the relationship between the equation of the singular path and the equation (34) for the optimal expenditure is established.

Optimal thinning of a forest stand. Clark (1976) describes the Faustmann model for the optimal rotation of a forest stand, and presents an optimal control model for determining optimal thinnings up to the age of clear-cut harvesting. The extension of the rotation model to include fire risk can be posed as a free-terminal time optimal control model but is more easily treated directly [Reed (1984)]. The inclusion of a fire-hazard into the optimal thinning model can be treated using the methods presented in this paper. The important question is specifying how the hazard depends on age, thinning activity and stand density. Following Clark and de Pree (1979) let:

$x(t)$ = volume of timber in stand at age t
$p(t)$ = the unit value (price) of timber at age t
c_0 = the cost per unit volume of thinning
$r(t) = p(t) - c_0$ = the unit net revenue from thinning at age t
c_1 = the cost per unit volume of clear-cut harvesting
$q(t) = p(t) - c_1$ = the unit net revenue from clear-cut harvesting at age t
$u(t)$ = the rate (volume per unit time) of thinning at age t
T = age of clear-cut harvesting

and assume that:

$$\dot{x} = g(t)F(x) - u(t); \quad x(0) = x_0 \tag{40}$$

where g is positive and decreasing, and F is positive and concave.

The value of thinnings plus a clear-cut harvest over a single rotation can be expressed in the form (24) as:

$$J = \int_0^T e^{-\delta t - y(t)} r(t)u(t) \, dt + e^{-\delta T - y(T)} q(T)x(T) \tag{41}$$

which is to be maximized over T and $u(t)$: $0 \leqslant t < T$ subject to (40) and the appropriate dynamic equation for y.

If the hazard depends only on age, then:

$$\dot{y} = a(t), \tag{42a}$$

whereas if it depends on age and volume:

$$\dot{y} = \psi(x, t) \tag{42b}$$

where typically we might expect $\psi_x \leqslant 0$, since dense stands will tend to have more shade and retain more ground moisture than sparse ones.

Another possibility is that the hazard depends on age, and whether thinning is actively taking place or not. In this case an appropriate model might be

$$\dot{y} = a(t) + \lambda H(u) \tag{42c}$$

where $H(u)$ is the step function:

$$H(u) = \begin{cases} 0 & \text{if } u = 0 \\ 1 & \text{if } u > 0. \end{cases} \tag{43}$$

As one might expect when the hazard depends only upon age, the optimal policy is the same as that given in Clark's deterministic analysis, except with the discount rate adjusted upwards at each instant in time by an amount $a(t)$, i.e. it

comprises, in general, a period of no thinning, followed by an interval of thinning along a singular path:

$$g(t)F'(x) + \frac{r'(t)}{r(t)} = \delta + a(t) \tag{44}$$

followed by another period of no thinning leading up to the optimal clear-cut harvest age T which satisfies the Wicksell condition:

$$\frac{g(T)F(x(T))q(T) + x(T)q'(T)}{x(T)q(T)} = \delta + a(T) \tag{45}$$

[see Reed and Apaloo (1991)].

When the hazard depends on age and stand density [equation (42b)] it can be shown that qualitatively the optimal policy is of the same form, although an explicit equation for the singular path cannot apparently be obtained [see Apaloo (1988)]. The optimal rate of thinning when fire risk is present can be either greater or less than that which is optimal when no risk is present. This is similar to the situation in the fishery and pollution models described in Section 2, and the reasons for this are the same—the contrasting effect of discounting the future more heavily at the same time as wishing to avoid driving up the hazard by thinning the forest too much.

In the case when the hazard depends on whether thinning takes place or not [equation (42c)], it can be shown [Reed and Apaloo (1989)] that singular control is no longer optimal. If there is a maximum rate u_{max} at which thinning can take place, then optimal thinning occurs in pulses at this maximum rate, interspersed with periods of no thinning. Switches occur when the switching function:

$$[r(t) - \rho_1]u_{max} + \lambda\rho_2 \tag{46}$$

changes sign. The first term represents the rate at which revenue is generated net of the user cost, through thinning at the maximum rate. Recalling the interpretation of $\rho_2(t)$ given in Section 2 as the negative of the value of the resource at time t, it can be seen that the second term in (46) represents the increase in expected loss in future revenues through fire, caused by thinning. Thinning takes place when the net revenue exceeds the increase in expected future loss.

A behavioral ecology model.   Mangel and Clark (1988) describe a number of models in which an organism's behavior is determined by the maximization of its survival probability or the maximization of its *lifetime fitness* (the expected value of a *terminal fitness function*). They use stochastic dynamic programming

to determine the optimizing behaviour. In this section we describe a highly stylized version of the basic "patch selection" or foraging model used as a paradigm by Mangel and Clark. Rather than using discrete time however we formulate the problem in continuous time and show how the methods described in this paper can be used to solve the problem.

Let $x(t)$ denote the energy reserves of the organism at time t, and let $u(t) \geqslant 0$ be a control variable describing the location at which the organism forages or which it inhabits. For example $u(t)$ could denote the depth selected by an aquatic organism, or the distance from cover selected for feeding by a bird. We shall suppose that the hazard of death through predation or another cause depends on both u and x:

$$h(t) = \psi(x, u) \tag{47}$$

with $\psi_x \leqslant 0$.

Also suppose that energy reserves are used up at a rate $c + \alpha(u)$ with $c > 0$ and $\alpha(u) \geqslant 0$ and that reserves increase at the rate $\beta(u)$ when the organism is foraging at location u. Thus *in toto* reserves change at the rate:

$$\dot{x} = -c + \gamma(u) \tag{48}$$

where $\gamma(u) = \beta(u) - \alpha(u)$. We suppose that $\gamma'(u)$ and ψ_u are of the *same sign*, so that selecting a location where food is more abundant, results in an increase in the risk of death through predation or other cause. For simplicity we shall assume that $\gamma'(u)$ and ψ_u are both negative (for the opposite case consider as control some monotone decreasing function of u), and that $\gamma(u)$ is concave and $\psi(x, u)$ is convex in u.

Let $\Phi(x(T))$, where T is a terminal time, denote the terminal fitness of the organism. We seek the behavior which will maximize the *lifetime fitness*:

$$S(T \mid t) \Phi(x(T)) \tag{49}$$

at all times $t(0 \leqslant t < T)$, where:

$$S(T \mid t) = P \text{ (organism alive at } T \mid \text{alive at } t).$$

This problem can be posed as an optimal control problem in which there are no flow benefits, only terminal rewards. Specifically the problem is:

maximize

$$J = e^{-y(T)} \Phi(x(T))$$

subject to (47) and

$$\dot{y} = \psi(x, u)$$

and $u(t) \geqslant 0$.

The conditional current value Hamiltonian is:

$$\bar{H} = \rho_1 [-c + \gamma(u)] + \rho_2 \psi(x, u) \tag{50}$$

which by assumption is concave in u (ρ_2 is negative). The co-state equations are:

$$\dot{\rho}_1 = \psi(x, u)\rho_1 - \psi_x(x, u)\rho_2 \tag{51}$$

$$\dot{\rho}_2 = \psi(x, u)\rho_2 \tag{52}$$

with transversality conditions:

$$\rho_1(T) = \Phi'(X(T)) \tag{53}$$

$$\rho_2(T) = -\Phi(X(T)). \tag{54}$$

The optimal control is $u^*(t) = 0$ if:

$$\rho_1 \gamma'(0) + \rho_2 \psi_u(x(t), 0) < 0$$

otherwise it is at the solution to:

$$\rho_1 \gamma'(u) + \rho_2 \psi_u(x(t), u) = 0. \tag{55}$$

For simplicity suppose that an interior solution is always optimal [$\psi_u(x, 0) = \infty$ will guarantee this], then from (55), (51), (52) and (48) a pair of differential equations for the behaviour of the optimal $u^*(t)$ and the corresponding $x^*(t)$ can be derived. They are:

$$\dot{u}\left[\psi_{uu} + \frac{\psi_u \gamma''(u)}{\gamma'(u)}\right] = \psi_x \gamma'(u) - \psi_{ux}(-c + \gamma(u)) \tag{56}$$

$$\dot{x} = -c + \gamma(u). \tag{57}$$

Boundary conditions are given by $x(0) = x_0$ and:

$$\Phi'(x(T))\gamma'(u(T)) - \Phi(x(T))\psi_u(u, x(T)) = 0. \tag{58}$$

These equations can be solved numerically as a boundary value problem using, for example, a shooting method. An analytic solution can be obtained in a simpler special case: that in which the hazard depends only on the foraging location $u(t)$ and not on the level of reserves $x(t)$ ($\psi_x = \psi_{xu} = 0$).

In this case since $\dot{\rho}_1 = \psi(u)\rho_1$ and $\dot{\rho}_2 = \psi(u)\rho_2$, we have $\rho_1(t)/\rho_2(t) \equiv$ constant, k say where $k < 0$. The optimal control will be at $u^* \equiv 0$ if $\psi_u(0) + k\gamma'(0) \geqslant 0$. Otherwise it will be at level $u^*(t) \equiv \bar{u}$ where $\bar{u}$ solves:

$$k\gamma'(u) + \psi_u(u) = 0. \tag{59}$$

The value of the constant k can be determined from the transversality conditions (53) and (54):

$$k = -\frac{\Phi'(x(T))}{\Phi(x(T))}. \tag{60}$$

It follows that the optimal control will be at $u^*(t) \equiv 0$ if:

$$\frac{\Phi'(x_0)}{\Phi(x_0)} \geqslant \frac{\psi_u(0)}{\gamma'(0)}. \tag{61}$$

Otherwise it will be at $u^*(t) \equiv \bar{u}$, where $\bar{u}$ solves:

$$\frac{\Phi'(x_0 - (c-u)T)}{\Phi(x_0 - (c-u)T)} = \frac{\psi_u(u)}{\gamma'(u)}. \tag{62}$$

Assuming that Φ is concave, increasing we have the result that the organism will choose a patch where the hazard is minimum ($u^* \equiv 0$) if the initial reserves are sufficiently high. Otherwise it will choose a patch ($u^* \equiv \bar{u} > 0$) at which there is an optimal tradeoff between food intake and the hazard of predation.

The above model is highly stylized, and it is only by making a large simplification that an analytic result can be obtained. This simplified model is perhaps the minimal model that one could contrive and still contain the essence of a behavioral ecology model, *viz.* a tradeoff between benefits and risk. For more realistic models of the type used by Mangel and Clark to describe the behavior of particular organisms, analytic solution is out of the question. Indeed modeling in continuous time when returns are stochastic would lead to optimal control problems with stochastic dynamics for which the methods described in this article would be of no use. The maximum principle appears then to have little prospect of being used successfully in Behavioral Ecology, as Mangel and Clark (1988, p. 239) point out. It has been included here mainly to demonstrate a commonality between problems in resource management and Behavioral Ecology, the two areas in which Colin Clark has made such significant contributions.

Before concluding the discussion of this model it is worth establishing the link between the maximum principle approach and the dynamic programming approach. Let:

$$V(x, t) = \max_{\substack{u(s) \\ 0 \leqslant s \leqslant t}} \{E[\Phi(x(T))] \mid x(t) = x\}$$

[this is the continuous-time analogue of the lifetime fitness function $F(x, t, T)$ used by Mangel and Clark]. From the *principle of optimality* (e.g. Mangel, 1985, p. 42) we have:

$$V(x, t) = \max_{u} \{V[x + (-c + \gamma(u)) \, dt, t + dt] (1 - \psi(x, u) \, dt) + 0 \cdot \psi(x, u) \, dt\}$$

(63)

which on expanding to o (dt) and dividing by dt gives the Bellman partial differential equation:

$$V_t + \max_{u} \{(-c + \gamma(u))V_x - \psi(x, u)V\} = 0.$$ (64)

Making the identification of the co-state variable ρ_1 with the shadow value, V_x of a unit of reserves, given the organism is alive at time t, and of the other co-state variable ρ_2 with the negative of the value function V at t, it can be seen that the expression in braces in (64) is exactly the conditional current value Hamiltonian $\bar{H}$ in (50). The maximum principle and dynamic programming both require that this quantity be maximized.

Furthermore the right hand side of (63) represents the expected value of V at time $t + dt$ conditional on its value at time t. Thus from equation (64) it follows that the *expected growth rate of lifetime fitness is equal to zero*† i.e. that $\{V^*\}$ is a martingale. This is a condition analogous to the familiar condition in resource economics, that the imputed value of a conserved resource should, under optimal management, be growing in expectation at a rate equal to the rate of discount (see e.g. Brock *et al.*, 1988; Reed, 1988). Note that this condition can be derived directly from the co-state equation (52), since $-\rho_2$ represent the lifetime fitness conditional on the organism being alive. The unconditional lifetime fitness is $\mu_2 = e^{-y}\rho_2$.

4. Summary and Conclusions.   One of the major mathematical tools used in analysing problems in resource management and conservation has been the Pontryagin maximum principle of optimal control theory. It was used extensively by Colin Clark in his seminal 1976 book, *Mathematical Bioeconomics*, and its use in problems of intertemporal resource allocation has now become standard. However its use has been almost entirely confined to deterministic models. For stochastic models, dynamic programming (usually

†This result can be verified numerically for the basic patch selection model of Mangel and Clark (1988, pp. 45–57) using the numerical solution obtained therein (p. 55) by discrete dynamic programming.

in discrete time) has proved much more fruitful. The same is true for behavioral ecology models where stochasticity is of the essence.

The purpose of this paper is to demonstrate how the maximum principle can be extended to cover continuous time optimal control models in resource management in which there is a component of stochasticity present *viz.* the possibility of a random catastrophic collapse of the resource. Mathematically this involves the inclusion of an extra state variable corresponding to the survival probability of the resource, and a possible adjustment to the benefit flow function. Corresponding to the additional state variable there is an additional co-state variable which is closely related to the value of the resource, or the lifetime fitness of an organism.

The method has been applied to a number of problems in resource management most of which have been analysed before in a deterministic framework. Application of the method has led to some analytic insight into the consequences of risk. Naively one might expect that the presence of risk would lead to greater conservation, especially if the hazard of collapse increased with lower levels of the resource stock. It is shown that this is not necessarily the case. One consequence of the presence of the risk of collapse is essentially to increase the rate of discount, which leads to less, rather than more, conservative exploitation. In some cases this effect is counterbalanced by other effects which lead to more conservative exploitation. In many cases the overall effect of risk on optimal behavior can lead to either more or less conservation depending on the particular values of the parameters of the model.

A highly stylized behavioral ecology model has also been addressed using the maximum principle method described in the paper. Its purpose is not to suggest that this method is a viable alternative to stochastic dynamic programming in behavioral ecology modelling but rather to demonstrate a link between resource management and behavioral ecology models. Also an analytic insight is given *viz.* that for behavioral ecology models in which fitness depends only on the terminal state, the lifetime fitness is a martingale, i.e. it should grow in expectation at a zero rate.

APPENDIX

The finite-time horizon problem of Section 3 is to maximize the expected present value of rewards. The present-value of rewards is a random variable:

$$\begin{cases} \int_0^Z g(x, u, t)\, \mathrm{e}^{-\delta t}\, \mathrm{d}t + \mathrm{e}^{-\delta Z} G(x(Z)) & \text{if } Z < T \\[2ex] \int_0^Z g(x, u, t)\, \mathrm{e}^{-\delta t}\, \mathrm{d}t + \mathrm{e}^{-\delta T} G(x(T)) & \text{if } Z = T \end{cases} \tag{A1}$$

where Z is a random variable denoting the time of collapse, with $Z = T$ if no collapse occurs. The cumulative distribution function of Z is equal to $1 - S(z)$ for $z < T$ and is one for $z \geqslant T$. Thus the expected present value of rewards can be expressed as:

$$J = \int_0^T \int_0^Z g(x, u, t)\,e^{-\delta t}\,dt\,d(-S(z)) + \int_0^T e^{-\delta z}G(x(z))\,d(-S(z))$$

$$+ S(T)\int_0^T g(x, u, t)\,e^{-\delta t}\,dt + S(T)\,e^{-\delta T}\psi(x(T)). \tag{A2}$$

On reversing the order of the double integral, integrating by parts the second integral and rearranging, one arrives at:

$$J = \int_0^T e^{-\delta t}[g(x, u, t) + G'(x(t))\dot{x}(t) - \delta G(x(t))]S(t)\,dt$$

$$+ G(x(0)) - e^{-\delta T}[G(x(T^-)) - \Phi(x(T))]S(T). \tag{A3}$$

Now letting:

$$y(t) = -\log S(t) \tag{A4}$$

we have

$$\dot{y}(t) = -\frac{\dot{S}(t)}{S(t)} = h(t) = \varphi(x, u, t) \tag{A5}$$

[which is (5)] and the expected present value can be expressed as:

$$J = \int_0^T e^{-\delta t - y(t)}\tilde{g}(x, u, t)\,dt + G(x(0)) + e^{-\delta T - y(T)}[\Phi(x(T)) - G(x(T^-))] \tag{A6}$$

where $\tilde{g}$ is as given in (10).

Letting T pass to infinity the last term vanishes and the infinite time horizon objective is thus:

$$J = \int_0^\infty e^{-\delta t - y(t)}\tilde{g}(x, u, t)\,dt + G(x(0)) \tag{A7}$$

[which is (8)].

We seek to maximize (A6) or (A7) subject to the dynamic equation:

$$\dot{x} = f(x, u, t) \tag{A9}$$

and

$$\dot{y} = \varphi(x, u, t). \tag{A10}$$

The *current-value* Hamiltonian is:

$$H = \tilde{g}(x, u, t)\,e^{-y(t)} + \mu_1 f(x, u, t) + \mu_2\varphi(x, u, t) \tag{A11}$$

with the co-state variables μ_1 and μ_2 satisfying:

$$\dot{\mu}_1 = \delta\mu_1 - e^{-y}\tilde{g}_x - \mu_1 f_x - \mu_2 \varphi_x$$

$$\dot{\mu}_2 = \delta\mu_2 + e^{-y}\tilde{g}. \qquad (A12)$$

Letting $\rho_1 = e^y\mu_1$, $\rho_2 = e^y\mu_2$ and $\bar{H} = e^y H$ gives the *conditional current value Hamiltonian* as:

$$\bar{H} = \tilde{g}(x, u, t) + \rho_1 f(x, u, t) + \rho_2 \varphi(x, u, t) \qquad (A13)$$

and the co-state equations as:

$$\dot{\rho}_1 = \rho_1[\delta + \varphi - f_x] - \rho_2\varphi_x - \tilde{g}_x$$

$$\dot{\rho}_2 = [\delta + \varphi]\rho_2 + \tilde{g}. \qquad (A14)$$

Since $\mu_1(t)$ represents the *shadow value* [i.e. the marginal increment to the optimal value of (A6) or (A7)] corresponding to an extra unit of stock at time t, it follows that $\rho_1(t) = e^y\mu_1(t) = \mu_1(t)/S(t)$, is the shadow value of an extra unit of stock at time t conditional on no collapse having occurred up until that time. It is this conditional shadow value which corresponds more closely to the usual idea of a shadow value, since $\mu_1(t)$, being unconditional, includes the possibility that the resource has collapsed by time t with an additional unit of stock consequently having zero value.

To interpret the second co-state variable $\rho_2(t)$, denote the optimal value of (A6) or (A7) by $J^*(x_0)$. By time $t (0 < t < T$ in the finite time horizon model; $0 < t < \infty$ in the infinite time horizon model) the stock will either have collapsed or not collapsed. It follows by the same derivation as that of (A3) from (A2) that:

$$J^*(x_0) = \int_0^t e^{-\delta z - y(z)}\tilde{g}(x^*, u^*, z)\,\mathrm{d}z + G(x_0) + e^{-\delta t - y(t)}[W^*(x^*(t), t) - G(x^*(t))] \quad (A15)$$

where $u^*(\ )$ and $x^*(\ )$ are the optimal control and corresponding trajectory for x; and $W^*(x^*(t), t)$ represents the expected present value at time t of the flow of benefits plus any rewards (or penalties) associated with a subsequent collapse, conditional on the resource being alive at time t and under optimal management. In short $W^*(x, t)$ is the "value" of a living resource at time t with the stock at level x.

The co-state variable ρ_2 is related to J^* by:

$$\rho_2(t) = e^{\delta t + y(t)}\frac{\partial}{\partial y(t)}J^*$$

$$= -W^*(x^*(t), t) + G(x^*(t)). \qquad (A16)$$

Thus $G(x(t)) - \rho_2(t)$ represents the value $W^*(x, t)$ of the resource at time t.

In particular in the case when there are no penalties or rewards associated with a collapse ($G(x) \equiv 0$), $\rho_2(t)$ is the negative of the value of the resource at time t. Furthermore in this case the co-state equation (A14) for ρ_2 is nothing more than the *Bellman equation* of dynamic programming [e.g. Mangel (1985)]. To verify this, we have from the *Principle of Optimality* (Mangel, *op. cit.*):

$$W^*(x, t) = \max_u \{g(x, u, t)\,\mathrm{d}t + (W^* + \dot{W}^*\,\mathrm{d}t)(1 - \delta\mathrm{d}t)(1 - \varphi\,\mathrm{d}t) + o\,(\mathrm{d}t)\} \qquad (A17)$$

where $\dot{W}^*$ is the *total* derivative of W^*. On dividing by $\mathrm{d}t$ and passing to the limit ($\mathrm{d}t \to 0$) yields the Bellman equation:

$$\dot{W}^* = [\delta + \varphi(x^*, u^*, t)] W^* - g(x^*, u^*, t).$$ (A18)

Identifying W^* with $-\rho_2$, it can be seen that the above is the co-state equation (A14) for ρ_2.

In the case when $G(x) \not\equiv 0$ an extra term of the form $G(x)\varphi \, dt$ has to be included on the right hand side of (A17) yielding a Bellman equation of the form:

$$\dot{W}^* = [\delta + \varphi(x^*, u^*, t)] W^* - g(x^*, u^*, t) - G(x^*(t))\varphi(x^*, u^*, t).$$ (A19)

Identifying W^* with $G(x^*) - \rho_2(t)$ it can be seen that the above equation is:

$$G(x^*(t))\dot{x}^*(t) - \dot{\rho}_2 = -\rho_2[\delta + \varphi(x^*, u^*, t)] + \delta G(x^*(t)) - g(x^*, u^*, t)$$ (A20)

which on rearranging gives the co-state equation (A14) for ρ_2.

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