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EFFECTS OF PERIPHERAL LAYER VISCOSITY ON BLOOD FLOW THROUGH THE ARTERY WITH MILD STENOSIS

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The effects of peripheral layer viscosity on physiological characteristics of blood flow through the artery with mild stenosis have been studied. It has been shown that the resistance to flow and the wall shear decrease as the peripheral layer viscosity decreases.

1. Introduction. The abnormal and unnatural growth in the lumen of an artery is called stenosis. In recent years considerable attention has been given to the study of blood flow characteristics due to the presence of stenosis in the lumen of the artery (May *et al.*, 1963; Fox and Hugo, 1966; Rodbard, 1966; Fry, 1968; Young, 1968; Forrester and Young, 1970; Lee and Fung, 1970; Caro *et al.*, 1974; Young and Tsai, 1973; Nerem, 1974; Rodkiewicz, 1974; Morgan and Young, 1974; Deshpande *et al.*, 1976). In these studies no attempt has been made to study the effects of peripheral layer viscosity of the blood on the flow characteristics (Whitmore, 1968; Middleman, 1972; Lih, 1969, 1975). Here, an analysis of these effects on the resistance to flow and shear stress at the maximum height of the stenosis, is presented.

2. Analysis. Consider the laminar and steady flow of the fluid whose viscosity varies along the radial direction (Caro *et al.*, 1971, see Fig. 1). The geometry of the stenosis which is assumed to be symmetrical is given by (Young, 1968),

$$\begin{aligned} \frac{R(z)}{R_0} &= 1 - \frac{\delta_s}{2R_0}\left\{1 + \cos\frac{2\pi}{L_0}\left[z - d - \frac{L_0}{2}\right]\right\}; \quad d \leqq z \leqq L_0 + d, \\ &= 1; \end{aligned} \tag{1}$$

otherwise where $R(z)$ is the radius of the tube with stenosis, R_0 is the constant radius, L_0 is the length of the stenosis and δ_s is the maximum height of the stenosis ($\delta_s \ll R_0$).

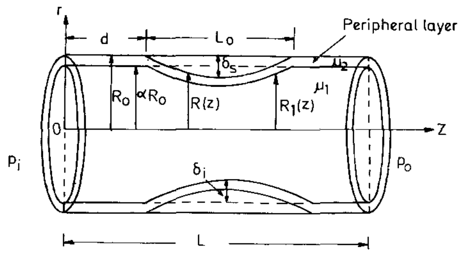

Figure 1. Geometry of arterial stenosis with peripheral layer

The basic equation governing the flow of blood in the arterial system is given by (Young, 1968)

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}\left[\mu(r) r \frac{\partial w}{\partial r}\right], \tag{2}$$

where w is the axial velocity, p is the fluid pressure and $\mu(r)$ is the viscosity function.

Solving equation (2) with boundary conditions $\partial w/\partial r = 0$ at $r = 0$ and $w = 0$ at $r = R(z)$, we get

$$w = \left(-\frac{1}{2}\frac{dp}{dz}\right)\int_r^R \frac{r\,dr}{\mu(r)}. \tag{3}$$

The flow flux, Q, is defined as

$$Q = \int_0^R 2\pi r w\,dr = \pi \int_0^R r^2(-\partial w/\partial r)\,dr, \tag{4}$$

which on using equation (3) gives

$$Q = \left(-\frac{\pi}{2}\frac{dp}{dz}\right)\int_0^R \frac{r^3\,dr}{\mu(r)}. \tag{5}$$

It can be noted from the equation of continuity that Q is a constant.

From equation (5) the pressure gradient is written as follows:

$$dp/dz = -2Q/\pi I(z), \tag{6}$$

where

$$I(z)=\int_0^{R(z)}\frac{r^3\,\mathrm{d}r}{\mu(r)}. \tag{7}$$

Integrating equation (6) using the conditions $p=p_i$ at $z=0$ and $p=p_0$ at $z=L$, we have

$$p_i-p_0=\frac{2Q}{\pi}\int_0^L\frac{\mathrm{d}z}{I(z)}. \tag{8}$$

The resistance to flow, λ, is defined as follows (Young, 1968):

$$\lambda=\frac{p_i-p_0}{Q}. \tag{9}$$

From equations (1), (8) and (9), λ is given by

$$\lambda=\frac{2}{\pi}\left[\frac{L-L_0}{I_0}+\int_d^{d+L_0}\frac{\mathrm{d}z}{I(z)}\right], \tag{10}$$

where

$$I_0=\int_0^{R_0}\frac{r^3\,\mathrm{d}r}{\mu(r)}. \tag{11}$$

The shearing stress at the wall can be defined as

$$\tau_R=\left[-\mu(r)\frac{\partial w}{\partial r}\right]_{r=R(z)}, \tag{12}$$

which on using equations (3) and (6) gives the wall shear, τ_s, at $z=d+(L_0/2)$ (i.e. maximum height of the stenosis) as follows:

$$\tau_s=\left[\frac{R(z)Q}{\pi I(z)}\right]_{z=d+L_0/2}. \tag{13}$$

From equations (10) and (13), the effects of viscosity variation on the resistance to flow and wall shear stress respectively can be studied for any given viscosity function.

3. Effects of Peripheral Layer Viscosity. To see the effects of peripheral layer viscosity on the above-mentioned characteristics, we consider the viscosity function as follows (Lih, 1975, see Fig. 1),

$$\begin{aligned} \mu&=\mu_1, \quad 0\leqq r\leqq R_1(z) \\ \mu&=\mu_2, \quad R_1(z)\leqq r\leqq R(z), \end{aligned} \tag{14}$$

where μ_1 and μ_2 are the viscosities of the central and the peripheral layers respectively. The function $R_1(z)$ represents the shape of the central layer which is assumed to be given by

$$\begin{aligned} \frac{R_1(z)}{R_0}&=\alpha-\frac{\delta_i}{2R_0}\left[1+\cos\frac{2\pi}{L_0}\left(z-d-\frac{L_0}{2}\right)\right], \quad d\leqq z\leqq d+L_0, \\ &=\alpha \end{aligned} \tag{15}$$

otherwise, where α is the ratio of the central core radius to the tube radius in the unobstructed region. The constant δ_i is the maximum bulging of the interface at $z=d+(L_0/2)$ due to the presence of stenosis and is determined as follows.

By using equation (14) in equation (3), velocities w_c and w_p of the central and the peripheral layers respectively can be calculated and then the corresponding fluxes Q_c and Q_p are obtained as

$$Q_c=\int_0^{R_1} 2\pi r w_c \,\mathrm{d}r=\left(\frac{-\pi}{8\mu_2}\frac{\mathrm{d}p}{\mathrm{d}z}\right)2R_1^2\left[R^2-\left(1-\frac{\bar{\mu}_2}{2}\right)R_1^2\right], \tag{16}$$

$$Q_p=\int_{R_1}^{R} 2\pi r w_p \,\mathrm{d}r=\left(-\frac{\pi}{8\mu_2}\frac{\mathrm{d}p}{\mathrm{d}z}\right)(R^2-R_1^2)^2, \tag{17}$$

where

$$\bar{\mu}_2=\mu_2/\mu_1.$$

The total flux, Q, is

$$Q=Q_c+Q_p=\left(-\frac{\pi}{8\mu_2}\frac{\mathrm{d}p}{\mathrm{d}z}\right)[R^4-(1-\bar{\mu}_2)R_1^4]. \tag{18}$$

This expression (18) can also be obtained from equation (5) after using the viscosity function given in equation (14). Further, by using the equation of continuity separately in the two regions ($0\leqq r\leqq R_1$ and $R_1\leqq r\leqq R$), it may be noted that not only the total flux Q but the fluxes Q_c and Q_p are also constants separately.

As before, integrating equations (16), (17) and (18) across the tube length and keeping in mind that the pressure drop is the same in each case, we get

$$Q_c = \frac{(p_i - p_c)\pi R_0^4}{4\mu_2 L} \frac{\alpha^2\{1-[1-(\bar{\mu}_2/2)]\alpha^2\}}{1-(L_0/L)+\alpha^2\{1-[1-(\bar{\mu}_2/2)]\alpha^2\}G_1}, \tag{19}$$

$$Q_p = \frac{(p_i - p_0)\pi R_0^4}{8\mu_2 L} \frac{(1-\alpha^2)^2}{1-(L_0/L)+(1-\alpha^2)^2 G_2}, \tag{20}$$

and

$$Q = \frac{(p_i - p_0)\pi R_0^4}{8\mu_2 L} \frac{1-(1-\bar{\mu}_2)\alpha^4}{1-(L_0/L)+[1-(1-\bar{\mu}_2)\alpha^4]G}, \tag{21}$$

where

$$G = \frac{1}{L}\int_d^{d+L_0} \frac{dz}{(R/R_0)^4 - (1-\bar{\mu}_2)(R_1/R_0)^4}, \tag{22}$$

$$G_1 = \frac{1}{L}\int_d^{d+L_0} \frac{dz}{(R_1/R_0)^2[(R/R_0)^2 - (1-\bar{\mu}/2)(R_1/R_0)^2]}, \tag{23}$$

and

$$G_2 = \frac{1}{L}\int_d^{d+L_0} \frac{dz}{[(R/R_0)^2 - (R_1/R_0)^2]^2}. \tag{24}$$

By using the values of Q_c, Q_p and Q from equations (19), (20) and (21), and the condition $Q = Q_c + Q_p$, the equation determining, δ_i, can be obtained as follows:

$$\frac{1-(1-\bar{\mu}_2)\alpha^4}{1-(L_0/L)+[1-(1-\bar{\mu}_2)\alpha^4]G} = \frac{2\alpha^2\{1-[1-(\bar{\mu}_2/2)]\alpha^2\}}{1-(L_0/L)+\alpha^2\{1-[1-(\bar{\mu}_2/2)]\alpha^2\}G_1} + \frac{(1-\alpha^2)^2}{1-(L_0/L)+(1-\alpha^2)^2 G_2}. \tag{25}$$

It can be seen by direct substitution that $R_1 = \alpha R$ satisfies equation (25). Hence from equations (1) and (1'5), δ_i is given by

$$\delta_i = \alpha\delta_s. \tag{26}$$

Now, for the viscosity function (14), the expressions determining λ and τ_s can be found in the dimensionless form using equations (10) and (13) as follows:

$$\bar{\lambda}=\frac{\bar{\mu}_2}{1-(1-\bar{\mu}_2)\alpha^4}\left\{1-\frac{L_0}{L}+[1-(1-\bar{\mu}_2)\alpha^4]G\right\}, \tag{27}$$

$$\bar{\tau}_s=\frac{\bar{\mu}_2(1-\delta_S/R_0}{\left(1-\frac{\delta_S}{R_0}\right)^4-(1-\bar{\mu}_2)\left(\alpha-\frac{\delta_i}{R_0}\right)^4}, \tag{28}$$

where

$$\bar{\lambda}=\frac{\lambda}{\lambda_0},\ \bar{\tau}_s=\frac{\tau_s}{\tau_0},$$

$$\lambda_0=\frac{8\mu_1 L}{\pi R_0^4},\ \tau_0=\frac{4\mu_1 Q}{\pi R_0^3};$$

and λ_0 and τ_0 are the resistance to flow and wall shear stress for the case of no stenosis with $\bar{\mu}_2=1$.

Using the value of δ_i from equation (26) in equations (27), (28) and evaluating the integral (22), the final expressions for $\bar{\lambda}$ and $\bar{\tau}_s$ can be obtained as

$$\bar{\lambda}=\frac{\bar{\mu}_2}{1-(1-\bar{\mu}_2)\alpha^4}\left[1-\frac{L_0}{L}+\frac{L_0}{L}\frac{a(2a^2+3b^2)}{2(a^2-b^2)^{7/2}}\right], \tag{29}$$

$$\bar{\tau}_s=\frac{\bar{\mu}_2}{1-(1-\bar{\mu}_2)\alpha^4}\cdot\frac{1}{(1-\delta_s/R_0)^3}, \tag{30}$$

where

$$a=1-\frac{\delta_s}{2R_0},\ b=\frac{\delta_s}{2R_0}.$$

When $\bar{\mu}_2=1$, equations (29) and (30) give the same result as obtained by Young (1968).

4. Discussion. The expression for $\bar{\lambda}$ in equation (29) is plotted for different values of $\bar{\mu}_2$, δ_s/R_0 and L_0/L in Figure 2. It is seen from this graph that the resistance to flow decreases as $\bar{\mu}_2$ (i.e. the peripheral layer viscosity) decreases for fixed stenosis size, but increases as the size (height and length) of the stenosis increases for fixed $\bar{\mu}_2$.

The behaviour of $\bar{\tau}_s$ is shown in Figure 3. It is noted from the graph that the wall shear stress decreases as the peripheral layer viscosity decreases; however, it increases as the height of the stenosis increases.

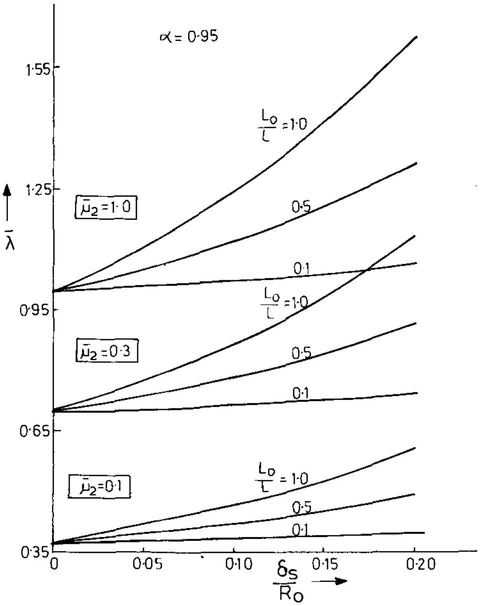

Figure 2. Variation of $\bar{\lambda}$ with δ_s/R_0 for different values of $\bar{\mu}_2$ and L_0/L

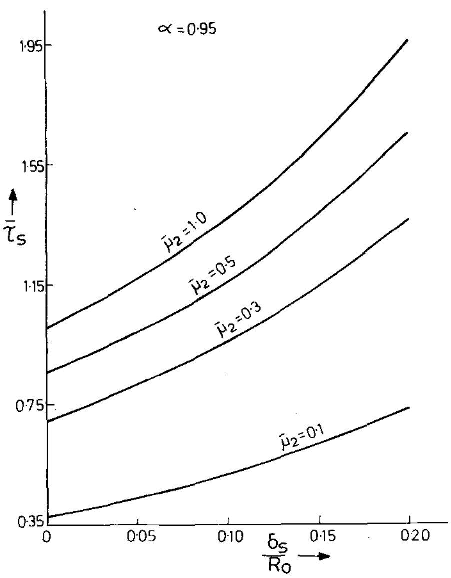

Figure 3. Variation of $\bar{\tau}_s$ with δ_s/R_0 for different $\bar{\mu}_2$

Further, using the data $\bar{\mu}_2=0.3$, $L_0/L=1.0$, $\alpha=0.95$ and $\delta_s/R_0=0.1$ (Middleman, 1972) in equations (29) and (30), it may be noted that the resistance to flow and wall shear stress are decreased by 13% and 4% respectively when compared with the case of no stenosis with $\bar{\mu}_2=1$. However, in the absence of peripheral layer these characteristics are increased by 25% and 37% respectively for the same stenosis size and $\bar{\mu}_2=1$, as pointed out by Young (1968).

5. *Conclusion.* The effects of peripheral layer viscosity on the blood flow in the presence of mild stenosis in the lumen of the artery has been investigated. It is concluded that the resistance to flow and wall shear stress decreases as the viscosity of the peripheral layer decreases but these characteristics increase as the size of the stenosis increases (height and length).

Thus, it may be remarked that the presence of the peripheral layer helps in the functioning of the diseased arterial system.

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