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GENERAL SOLUTIONS OF COUPLED THERMOELASTIC PROBLEM *

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Abstract: *A new type of general solution of thermoelasticity is derived from the linearized basic equations for coupled thermoelastic problem. In the case of quasi-static problem, the present general solution is simpler since it involves one less potential function than Biot's solution.*

Key words: coupled thermoelasticity; general solution

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Introduction

When the heat conduction equation involves the term of deformation and thermoelastic equations contain temperature, this kind of problem is called coupled thermoelastic problem, which temperature field and elastic field must be solved simultaneously[1,2]. In general, the equation of heat conduction with deformation taken into account is nonlinear. If the temperature in the coupling term is replaced with the reference temperature T_0 (in absolute temperature), the equation of heat conduction will be linearized. When the temperature increment is small as compared with T_0, this simplification will not cause large error. The linearized coupled equations of thermoelasticity for isotropic materials are

$$\mu \nabla^2 \boldsymbol{u} + (\lambda + \mu) \nabla e - \beta \nabla \theta + \boldsymbol{F} = \rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2}, \tag{1}$$

$$k \nabla^2 \theta + h = c \frac{\partial \theta}{\partial t} + T_0 \beta \frac{\partial e}{\partial t}, \tag{2}$$

where $\boldsymbol{u} = [u, v, w]$ is the displacement vector, e the dilation of the body, $\theta = T - T_0$ the temperature increment, T absolute temperature, $\boldsymbol{F}$ the body force vector per unit volume, ρ density, t time, λ and μ are Lamé constants, $\beta = (3\lambda + 2\mu)\alpha$ is the thermal stress coefficient, α thermal expansion coefficient, k thermal conductivity, $c = \rho c_v$, c_v the specific heat at constant

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deformation, and h heat source intensity as well as

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z},\ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

In Ref. [2], in the absence of body force and heat source the displacement vector is expressed in the following form by the Hemholtz decomposition theorem

$$\boldsymbol{u} = \nabla \varphi + \nabla \times \boldsymbol{\varphi}. \tag{3}$$

Substituting Eq.(3) into Eqs.(1) and (2), after some manipulation we find that φ satisfies the wave equation, and ψ satisfies a fourth-order partial differential equation. The sum of order of the equations that φ and ψ satisfy is ten, which is obviously larger than that of Eqs.(1) and (2), i.e. eight. The temperature increment θ can also be expressed by ψ and $\boldsymbol{\varphi}$.

For quasi-static problems, Biot extended Papkovitch-Neuber solution in the absence of the inertia effect, body force and heat source, expressing displacement vector in the following form:

$$\boldsymbol{u} = B\boldsymbol{\psi} - \nabla(\psi_0 + \boldsymbol{r}\cdot\boldsymbol{\psi}), \tag{4}$$

$$B = 2(\lambda + 2\mu + \beta^2 T_0/c)/(\lambda + \mu + \beta^2 T_0/c), \tag{5}$$

where $\boldsymbol{\psi}$ satisfies the harmonic equation, and ψ_0 a fourth-order partial differential equation or $\psi_0 = \psi_1 + \psi_2$, in which ψ_1 and ψ_2 satisfy the harmonic equation and standard heat conduction equation, respectively. Obviously, the sum of equations that $\boldsymbol{\psi}$ and ψ_0 satisfy is also ten. The temperature increment θ can also be expressed in terms of ψ_0 and $\boldsymbol{\psi}$[3]. The completeness of Biot's solution Eq.(4) was first proved by Verruijt[4]. Later, Qing C.B. and Wang M.Z.[5] gave a much simpler proof.

In this paper, displacement functions are first introduced to represent displacement components, then the governing equations are simplified. It is derived that the displacement vector $\boldsymbol{u}$ and temperature increment θ can be expressed in terms of three displacement functions, of which two displacement functions satisfy the same wave equation, another displacement function satisfies a fourth-order partial differential equation. In the case of quasi-static problems, this fourth-order partial differential equation can be decomposed into a harmonic equation and a standard heat condition equation. The sum of order number of all these equations is eight, the same as that of Eqs.(1) and (2).

1 General Solutions to Equations of Motion

Assume

$$u = -\frac{\partial \varphi_0}{\partial y} + \frac{\partial G}{\partial x},\ v = \frac{\partial \varphi_0}{\partial x} + \frac{\partial G}{\partial y},\ w = w(x,y,z,t). \tag{6}$$

Substituting (6) into Eqs.(1) and (2), and omitting body force and heat source, from the first and second equations of Eq.(1) we have

$$\left(\mu\nabla^2 - \rho\frac{\partial^2}{\partial t^2}\right)\varphi_0 = 0, \tag{7}$$

$$\mu\nabla^2 G - \rho\frac{\partial^2 G}{\partial t^2} + (\lambda + \mu)\left(\Lambda G + \frac{\partial w}{\partial z}\right) - \beta\theta = 0, \tag{8}$$

where $\Lambda = \partial^2/\partial x^2 + \partial^2/\partial y^2$. The third equation of Eqs.(1) and (2) take the following form:

$$\mu \nabla^2 w + (\lambda + \mu) \frac{\partial}{\partial z}\left(\Delta G + \frac{\partial w}{\partial z}\right) - \beta \frac{\partial \theta}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2}, \tag{9}$$

$$k \nabla^2 \theta = c \frac{\partial \theta}{\partial t} + T_0 \beta \frac{\partial}{\partial t}\left(\Delta G + \frac{\partial w}{\partial z}\right). \tag{10}$$

Differentiating Eq.(8) with respect to z and subtracting it from Eq.(9) yield

$$\left(\mu \nabla^2 - \rho \frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial G}{\partial z} - w\right) = 0. \tag{11}$$

From Eq.(11) we have

$$w = \frac{\partial G}{\partial z} - \varphi_4, \tag{12}$$

$$\left(\mu \nabla^2 - \rho \frac{\partial^2}{\partial t^2}\right) \varphi_4 = 0. \tag{13}$$

Substituting Eq.(12) into Eq.(8) gives

$$\theta = \frac{1}{\beta}\left[(\lambda + 2\mu) \nabla^2 G - \rho \frac{\partial^2 G}{\partial t^2} - (\lambda + \mu) \frac{\partial \varphi_4}{\partial z}\right]. \tag{14}$$

Substituting w and θ in Eqs.(12) and (14) into Eq.(10) leads to

$$k\left[(\lambda + 2\mu) \nabla^2 - \rho \frac{\partial^2}{\partial t^2}\right] \nabla^2 G - [c(\lambda + 2\mu) + T_0\beta^2] \nabla^2 \frac{\partial G}{\partial t} + \rho c \frac{\partial^3 G}{\partial t^3} =$$

$$k(\lambda + \mu) \nabla^2 \frac{\partial \varphi_4}{\partial z} - [c(\lambda + \mu) + T_0\beta^2] \frac{\partial^2 \varphi_4}{\partial z \partial t}. \tag{15}$$

The solution of Eq.(15) can be written as $G = G_0 + G_1$, where G_0 is the general solution of the homogeneous equation, and G_1 is a particular solution of the inhomogeneous equation and a function of φ_4. From Eqs.(12) and (14) it can be seen that w and θ only depend on G and φ_4. Similarly, it is obvious that u and v are related to G and φ_0 from the first two equations of Eq.(6).

2 Two Special Cases

2.1 Quasi-static problem

For quasi-static problem, ρ is set to zero, then those relevant expressions in the last section are simplified

$$\nabla^2 \varphi_0 = 0, \quad \nabla^2 \varphi_4 = 0, \tag{7$'$}$$

$$\theta = \frac{1}{\beta}\left[(\lambda + 2\mu) \nabla^2 G - (\lambda + \mu) \frac{\partial \varphi_4}{\partial z}\right], \tag{14$'$}$$

$$k(\lambda + 2\mu) \nabla^2 \nabla^2 G - [c(\lambda + 2\mu) + T_0\beta^2] \nabla^2 \frac{\partial G}{\partial t} =$$

$$- [c(\lambda + \mu) + T_0\beta^2] \frac{\partial^2 \varphi_4}{\partial z \partial t}. \tag{15$'$}$$

Obviously, Eq.(15)′ has a solution as follows

$$G = G_0 + \frac{z\varphi_4}{B}. \tag{16}$$

Where B is constant as defined in Eq. (5), and G_0 satisfies the equation below

$$\nabla^2\left(K_0\nabla^2 - \frac{\partial}{\partial t}\right)G_0 = 0, \tag{17}$$

$$K_0 = \frac{k(\lambda + 2\mu)}{c(\lambda + 2\mu) + T_0\beta^2}. \tag{18}$$

The solution of Eq. (17) can be written into the following form:

$$G_0 = \varphi_1 + \varphi_2, \tag{19}$$

where φ_1 and φ_2 satisfy the following two equations, respectively.

$$\nabla^2\varphi_1 = 0, \tag{20}$$

$$\left(K_0\nabla^2 - \frac{\partial}{\partial t}\right)\varphi_2 = 0. \tag{21}$$

Let $\varphi_4 = B\varphi_3$, the following general solution for displacements and temperature increment is obtained

$$\left.\begin{aligned} u &= -\frac{\partial\varphi_0}{\partial y} + \frac{\partial}{\partial x}(\varphi_1 + \varphi_2 + z\varphi_3), \\ v &= \frac{\partial\varphi_0}{\partial x} + \frac{\partial}{\partial y}(\varphi_1 + \varphi_2 + z\varphi_3), \\ w &= -B\varphi_3 + \frac{\partial}{\partial z}(\varphi_1 + \varphi_2 + z\varphi_3), \\ \theta &= \frac{1}{\beta}\left[\frac{(\lambda + 2\mu)}{K_0}\frac{\partial\varphi_2}{\partial t} + [2(\lambda + 2\mu) - (\lambda + \mu)B]\frac{\partial\varphi_3}{\partial z}\right], \end{aligned}\right\} \tag{22}$$

where φ_2 satisfies the standard heat conduction equation Eq. (21); φ_0, φ_1 and φ_3 satisfy the harmonic equation.

2.2 Dynamic problem with temperature increment θ in direct proportion with e^{pt}

Assume

$$\left.\begin{aligned} &\theta = \Theta(x,y,z)e^{pt}, \quad u = U(x,y,z)e^{pt}, \quad v = V(x,y,z)e^{pt}, \\ &w = W(x,y,z)e^{pt}, \quad \varphi_i = \Phi_i(x,y,z)e^{pt}, \quad G = G(x,y,z)e^{pt}. \end{aligned}\right\} \tag{23}$$

Substituting Eq. (23) into relevant formulae in the second section, we have

$$(\mu\nabla^2 - \rho p^2)\Phi_i = 0 \qquad (i = 0,4), \tag{7)'',(13)''}$$

$$\Theta = \frac{1}{\beta}\left[(\lambda + 2\mu)\nabla^2 G - \rho p^2 G - (\lambda + \mu)\frac{\partial\Phi_4}{\partial z}\right], \tag{14)''}$$

$$a\nabla^2\nabla^2 G - b\nabla^2 G + lG = m\frac{\partial\Phi_4}{\partial z}, \tag{15)''}$$

where

$$\left.\begin{aligned} &a = k(\lambda + 2\mu), \; l = \rho c p^3, \; b = k\rho p^2 + [c(\lambda + 2\mu) + T_0\beta^2]p, \\ &m = k(\lambda + \mu)\rho p^2/\mu - [c(\lambda + \mu) + T_0\beta^2]p. \end{aligned}\right\} \tag{24}$$

The solution of Eq. (15)″ can be written as

$$G = \Phi_1 + \Phi_2 + \frac{1}{A}\frac{\partial \Phi_4}{\partial z}, \tag{25}$$

where

$$A = \left[a\left(\frac{\rho p^2}{\mu}\right)^2 - b\frac{\rho p^2}{\mu} + l\right] \Big/ m, \tag{26}$$

and Φ_1 and Φ_2 satisfy the following equation

$$(\nabla^2 - s_i^2)\Phi_i = 0 \qquad (i = 1,2), \tag{27}$$

in which $s_i^2 (i = 1,2)$ are two roots of the equation below

$$as^4 - bs^2 + l = 0. \tag{28}$$

Let $\Phi_4 = A\Phi_3$, then the general solution for U, V, W and Θ expressed in $\Phi_i (i = 0, 1, 2, 3)$ can be obtained, which is similar to Eq. (22) in form with B replaced by A.

3 Application

Consider a half-space, which is traction-free and subjected to heating on its surface by $T = f(r)e^{pt}$. This problem is treated as a quasi-static problem. The general solution Eq. (22) can be expressed as follows in cylindrical coordinates

$$\left.\begin{aligned}
u_r &= -\frac{\partial \varphi_0}{r\partial \varphi} + \frac{\partial}{\partial r}(\varphi_1 + \varphi_2 + z\varphi_3),\\
u_\varphi &= \frac{\partial \varphi_0}{\partial r} + \frac{\partial}{r\partial \varphi}(\varphi_1 + \varphi_2 + z\varphi_3),\\
w &= -B\varphi_3 + \frac{\partial}{\partial z}(\varphi_1 + \varphi_2 + z\varphi_3),\\
\theta &= \frac{1}{\beta}\left\{\frac{(\lambda + 2\mu)}{K_0}\frac{\partial \varphi_2}{\partial t} + [2(\lambda + 2\mu) - (\lambda + \mu)B]\frac{\partial \varphi_3}{\partial z}\right\}.
\end{aligned}\right\} \tag{29}$$

Obviously, this is an axisymmetric problem. Assume that $\varphi_0 = 0$, $\varphi_1 = \varphi_1^* e^{pt}$, $\varphi_2 = \varphi_2^* e^{pt}$, $\varphi_3 = \varphi_3^* e^{pt}$ in the general solution, where φ_1^*, φ_3^* are still harmonic function and φ_2^* satisfies $(K_0 \nabla^2 - p)\varphi_2^* = 0$ with $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r} + \frac{\partial^2}{\partial z^2}$. By use of Hankel transform, the following solutions of φ_1^*, φ_2^* and φ_3^* are obtained

$$\varphi_1^* = \int_0^\infty c_1 e^{-\rho z} J_0(\rho r)\rho d\rho, \qquad \varphi_2^* = \int_0^\infty c_2 e^{-\sqrt{\rho^2 + p/K_0}\, z} J_0(\rho r)\rho d\rho,$$

$$\varphi_3^* = \int_0^\infty c_3 e^{-\rho z} J_0(\rho r)\rho d\rho,$$

where $c_1(\rho)$, $c_2(\rho)$ and $c_3(\rho)$ are unknown functions.

Substituting φ_1, φ_2 and φ_3 into the general solution and the following constitutive relation:

$$\left.\begin{aligned}
\sigma_r &= (\lambda + 2\mu)\varepsilon_r + \lambda(\varepsilon_\varphi + \varepsilon_z) - \beta\theta,\\
\sigma_\varphi &= \lambda(\varepsilon_r + \varepsilon_z) + (\lambda + 2\mu)\varepsilon_\varphi - \beta\theta,\\
\sigma_z &= \lambda(\varepsilon_r + \varepsilon_\varphi) + (\lambda + 2\mu)\varepsilon_z - \beta\theta,\\
\tau_{\varphi z} &= \mu\gamma_{\varphi z}, \ \tau_{rz} = \mu\gamma_{rz}, \ \tau_{r\varphi} = \mu\gamma_{r\varphi}.
\end{aligned}\right\} \tag{30}$$

By using boundary conditions $\sigma_z \mid_{z=0} = \tau_{zr} \mid_{z=0} = 0$, $\theta \mid_{z=0} = f(r)e^{pt}$, the following simultaneous equation system in c_1, c_2 and c_3 is obtained with aid of inverse Hankel transform theorem[6]

$$\left.\begin{aligned} &2c_1 + 2c_2 + Bc_3 = 0, \\ &2\rho c_1 + (\rho + \sqrt{\rho^2 + p/K_0})c_2 + (B - 1)c_3 = 0, \\ &\frac{p(\lambda + 2\mu)}{K_0}c_2 - [2(\lambda + 2\mu) - (\lambda + \mu)B]\rho c_3 = \beta\int_0^\infty f(r)J_0(\rho r)r\mathrm{d}r. \end{aligned}\right\} \tag{31}$$

The unknown coefficients $c_1(\rho)$, $c_2(\rho)$ and $c_3(\rho)$ can be determined by solving the above set of equations.

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