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A COUPLE STRESS MODEL OF BLOOD FLOW IN THE MICROCIRCULATION

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A simple mathematical model depicting blood flow in the capillary is developed with an emphasis on the permeability property of the blood vessel based on Starling's hypothesis. In this study the effect of inertia has been neglected in comparison with the viscosity on the basis of the smallness of the Reynolds number of the flow in the capillary. The capillary blood vessel is approximated by a circular cylindrical tube with a permeable wall. The blood is represented by a couple stress fluid. With such an ideal model the velocity and pressure fields are determined. It is shown that an increase in the couple stress parameter increases the resistance to the flow and thereby decreases the volume rate flow. A comparison of the results with those of the Newtonian case has also been made.

1. Introduction. Microcirculation is the study of flow in small blood vessels, particularly in the capillary, which range in diameter from 20 μm (micron) to 500 μm in different species. In physiology the most important functions of the circulation of blood through capillaries are to supply nutrient to every living cell of the organism and also to remove various waste products from every cell.

The capillaries are bounded by endothelial cells which have ultramicroscopic pores through which substances of various molecular size can penetrate the surrounding tissue and also the capillary. One of the most important features of the capillary geometry which distinguishes it from arteries is the permeability of the wall. The deposition of cholesterol is believed to increase the permeability of the wall. Such increases in permeability also result from

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dilated, damaged or inflamed capillary walls. Thus it is worthwhile to study the effect of wall permeability of the blood vessel from a fluid-mechanical point of view.

One of the classical papers which include permeability is by Oka and Murata (1970). They considered the flow in the capillaries to be steady and included the wall permeability on the flow characteristics, neglecting the inertial terms in the equation of motion. Though their model is based on a number of simplifying assumptions, it conveys many features which are of physiological importance. They assumed the blood to be a Newtonian fluid. The experimental studies, however, on blood flows by Bugliarello and Sevilla (1970), Cokelet (1972) and Goldsmith and Skalak (1975) indicate that under certain flow conditions, blood flow may exhibit strong deviations from the Newtonian flow behaviour. The deviations occur in the form of non-parabolic velocity profiles for flow through tubes of small diameter (of the order of 50 μm). This is because of the presence of red blood cells in the plasma. Thus, when neutrally buoyant corpuscles are contained in fluid, corpuscles have rotary motion if there is a velocity gradient due to shearing stress. Furthermore, corpuscles have spin angular momentum in addition to orbital angular momentum. Therefore, the symmetry of stress tensor is not held in the fluid which has a spin angular momentum. The fluid containing neutrally buoyant corpuscles, if observed macroscopically, is considered to be a non-Newtonian fluid which has a constitutive equation expressed by the stress tensor. In such fluids the radius of gyration of the corpuscle is different from that of the fluid particle. This difference produces couple stress in the fluid. Thus, the fluid which has couple stress and spin angular momentum is called couple stress fluid.

It has already been demonstrated by Valanis and Sun (1969) and Popel *et al.* (1974) that the couple stress theory presented by Stokes (1966) represents blood flow reasonably well and attempted to explain the rheological abnormalities observed by Bugliarello *et al.* (1965). Since the rheological properties of the blood are mainly due to the suspension of red blood cells, white cells and platelets in the blood plasma, the study of blood flow with couple stresses may play an important role in understanding the rheological anomalies associated with blood flows. The effect of couple stresses on blood flow in the capillary bounded by permeable wall has not been given much attention. Thus the purpose of this paper is to study the combined effect of couple stresses and the exchange of fluids across the capillary walls on the flow of blood in microcirculation. For this purpose we take a simplified model for a capillary and blood and solve the basic equations using Starling's hypothesis of fluid exchange, which states that the difference in hydraulic pressure between the blood and tissue fluid is not only responsible for the process of filtration—the difference in colloidal pressure between the blood and tissue fluid also plays an important role. Using certain physiological experimental data, the computation of the theoretical results is carried out and the results are discussed in detail.

2. Formulation of the Problem.

2.1. Physical configuration and assumptions. As pointed out by Lighthill (1969), the effect of curvature can be neglected in the case of creeping flow. Hence, the capillary between an arteriole and a venule is taken as a tube of uniform circular cross-section with a permeable wall as shown in Fig. 1. We

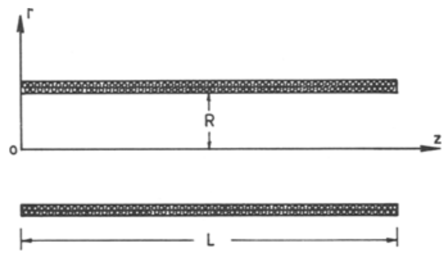

Figure 1. Capillary tube with porous walls.

shall consider the steady flow of couple stress fluid in the capillary tube. A cylindrical system of co-ordinates (r, ϕ, z) is chosen, in which the z-axis coincides with the axis of the tube. The flow is assumed to be axisymmetric. The permeability of the wall is governed by Starling's Law, which is a modification of Fick's Law and states that the net filtration pressure is given by the difference between hydrostatic and osmotic pressure between the blood and the tissue fluid. Filtered water which passes into the tissue is either reabsorbed into the capillary blood or returned to the blood via the lymphatic systems. Starling's hypothesis is usually expressed in the form:

$$M = k(p_c - p_i - \pi_c + \pi_i) \tag{1}$$

where M represents the flow rate per unit area of wall surface. The constant k is the measure of the permeability of the capillary wall to water and is called the filtration constant, p_c, p_i, π_c and π_i are hydrostatic capillary blood pressure, interstitial fluid pressure, osmotic pressure of the plasma and proteins in the interstitial fluid, respectively. When M is positive (i.e. when the hydrostatic pressure difference is greater than the osmotic pressure difference) filtration of fluid out of the capillary occurs. When M is negative reabsorption of fluid from the interstitial space into the capillary takes place. If the quantity M is defined as the flow rate per unit area of the total wall surface, then p_c cannot vary along the length of the capillary. In this case p_c should be replaced by an appropriate value.

2.2. Constitutive equations and boundary conditions. The constitutive equations and equations of motion for couple stress fluid flow in the absence of body moment and body couple are

$$\tau_{ji,j}=\rho\,\frac{\mathrm{d}v_i}{\mathrm{d}t} \tag{2}$$

$$e_{ijk}T^{\mathrm{A}}_{jk}+M_{ji,j}=0 \tag{3}$$

$$\tau_{ij}=-p\delta_{ij}+2\mu d_{ij} \tag{4}$$

$$\mu_{ij}=4\eta\omega_{j,i}+4\eta'\omega_{i,j} \tag{5}$$

where ρ is the density, τ_{ij} and T^{A}_{ij} are the symmetrical and antisymmetrical parts of stress tensor T_{ij}, v_i is the velocity vector, M_{ij} is the couple stress tensor, μ_{ij} is the deviatoric part of M_{ij}, ω_i is the vorticity vector, d_{ij} is the symmetric part of the velocity gradient, η and η' are the constants associated with the couple stress, p is the pressure and other terms have their usual meanings in tensor analysis.

As emphasized by Skalak (1972), the rate of inertia is not significant in microcirculation and hence this term can be neglected in the basic equation. Based on the couple stress theory of Stokes (1966) the equations of motion (2) and (3) using (4) and (5) can be written as

$$\nabla p=\nabla^2(\mu q-\eta\nabla^2 q), \tag{6}$$

where

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right).$$

The two-dimensional form of equation (6) is

$$\begin{aligned}p_r={}&\mu\left(v_{rr}+\frac{1}{r}v_r-\frac{v}{r^2}+v_{zz}\right)-l^2\mu\left(v_{rrrr}+v_{zzzz}+2v_{rrzz}+\frac{2}{r}v_{rrr}\right.\\&\left.+\frac{2}{r}v_{rzz}-\frac{3}{r^2}v_{rr}-\frac{2}{r^2}v_{zz}+\frac{3}{r^3}v_r-\frac{3}{r^4}v\right),\end{aligned} \tag{7}$$

$$\begin{aligned}p_z={}&\mu\left(u_{rr}+\frac{1}{r}u_r+u_{zz}\right)-l^2\mu\left(u_{rrrr}+u_{zzzz}+\frac{2}{r}u_{rrr}\right.\\&\left.+2u_{rrzz}+\frac{2}{r}u_{rzz}-\frac{1}{r^2}u_{rr}+\frac{1}{r^3}u_r\right).\end{aligned} \tag{8}$$

The continuity equation is given by

$$\frac{1}{r}(rv)_r + u_z = 0 \tag{9}$$

where μ is the coefficient of viscosity of the fluid having dimension M/LT and η is the material constant characterizing the couple stress property of the fluid and has the dimensions of momentum (ML/T). Thus, $l^2(=\eta/\mu)$ has the dimensions of length squared.

The boundary conditions are

$$u_r = 0,\ v = 0 \qquad \text{at } r = 0 \tag{10a,b}$$

$$u = 0,\ v = k(p - \alpha) \quad \text{at } r = R \tag{11a,b}$$

$$\bar{p} = p_a \qquad \text{at } z = 0 \tag{12}$$

$$\bar{p} = p_v \qquad \text{at } z = L \tag{13}$$

where R is the radius and L is the length of the capillary, $\bar{p}$ is the average of the pressure p over the cross-section of the capillary, p_a and p_v are arterial and venous end pressures, respectively and are taken to be constant. The coefficient k is a measure of the permeability of the wall and α is equal to $\pi_c + p_i - \pi_i$ which is assumed to be a constant. The boundary condition (11a) is the no-slip condition which is assumed to be valid at the porous interface since the permeability of the porous wall is very small and in couple stress fluid both a yield stress and shear-dependent viscosity exist so the fluid element which is in contact with the boundary adheres to it and thus has the same velocity as the boundary. The boundary condition (11b) is Starling's hypothesis which takes care of the smooth transfer of mass across the porous wall.

3. Method of Solution for the Flow Field. To solve for the flow field we introduce the dimensionless parameter

$$\varepsilon = \frac{k\mu}{R} \tag{14}$$

where μ is the coefficient of viscosity of fluid. The filtration constant k varies widely for capillaries. The average value of muscle capillaries of dog and cat is 2.5×10^{-8} cm/(sec. cm H_2O). If we take $\eta = 1$ cP and $R = 5$ μm then we get $\varepsilon = 1 \times 10^{-7}$ (Oka, 1981). Hence ε may be regarded as very small. The order of u, v and p may be estimated as 1, ε and 1, respectively. The derivatives of u and v

with respect to z will be of higher order of ε than that of the derivatives with respect to r.

In view of the order analysis, a rough estimate of the orders of magnitude of the various terms in (7)–(9) is given below:

$$
\begin{aligned}
&p_z,\, u_r,\, u_{rr},\, u_{rrr},\, u_{rrrr} && \sim 0(1),\\
&\left.\begin{aligned}&p_r,\, u_z,\, u_{zz},\, v_r,\, v_{rr},\, v_{rrr},\, v_{rrrr},\\ &u_{zzzz},\, u_{rrzz},\, u_{rzzz},\, v_{rzz},\, \frac{1}{r}(rv)_r,\, \frac{v}{r^2}\end{aligned}\right\} && \sim 0(\varepsilon),\\
&v_{zz},\, v_{rrzz},\, v_{zzzz} && \sim 0(\varepsilon^2).
\end{aligned}
\tag{15}
$$

We seek solutions for the basic equations (7)–(9) by splitting the solution into two parts, of the first and second order, in the form,

$$
\begin{aligned}
u(r, z) &= u_1(r, z) + u_2(r, z)\\
v(r, z) &= v_1(r, z) + v_2(r, z)\\
p(r, z) &= p_1(r, z) + p_2(r, z)
\end{aligned}
\tag{16}
$$

where

$$u_1,\, p_1 \sim 0(1);\; u_2,\, p_2,\, v_1 \sim 0(\varepsilon);\; v_2 \sim 0(\varepsilon^2).$$

Substituting (16) in (7)–(9) using the order analysis of the terms, we obtain the following simplified version for the first and second approximation.

(i) First approximation:

$$\frac{\partial p_1}{\partial r} = 0 \tag{17}$$

$$\frac{\partial p_1}{\partial z} = \mu D(1 - l^2 D)u_1 \tag{18}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_1) + \frac{\partial u_1}{\partial z} = 0 \tag{19}$$

where

$$D = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) \tag{20}$$

with boundary conditions

$$\frac{\partial u_1}{\partial r} = 0 \quad \text{and } v_1 = 0, \qquad \text{at } r = 0, \tag{21a,b}$$

$$u_1 = 0 \quad \text{and } v_1 = k(p_1 - \alpha), \quad \text{at } r = R, \tag{22a,b}$$

$$\bar{p}_1 = p_a \qquad \text{at } z = 0 \tag{23}$$

$$\bar{p}_1 = p_v \qquad \text{at } z = L. \tag{24}$$

(ii) Second approximation:

$$\frac{\partial p_2}{\partial r} = \mu\left(D - l^2 D_1^2 - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} - \frac{3}{r^3}\frac{\partial}{\partial r}\right)v_1 \tag{25}$$

$$\frac{\partial p_2}{\partial z} = \mu\left\{\frac{\partial^2 u_2}{\partial r^2} + \frac{1}{r}\frac{\partial u_2}{\partial r} + \frac{\partial^2 u_1}{\partial z^2} - l^2(D_2 u_1 + D_3 u_2)\right\} \tag{26}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_2) + \frac{\partial u_2}{\partial z} = 0 \tag{27}$$

where

$$D_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} \tag{28}$$

$$D_2 = \frac{\partial^4}{\partial z^4} + 2\frac{\partial^4}{\partial r^2 \partial z^2} + \frac{2}{r}\frac{\partial^3}{\partial r \partial z^2} \tag{29}$$

$$D_3 = \frac{\partial^4}{\partial r^4} + \frac{2}{r}\frac{\partial^3}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial r} \tag{30}$$

with boundary conditions:

$$\frac{\partial u_2}{\partial r} = 0 \quad \text{and } v_2 = 0, \quad \text{at } r = 0, \tag{31a,b}$$

$$u_2 = 0 \quad \text{and } v_2 = k p_2 \quad \text{at } r = R, \tag{32a,b}$$

$$\bar{p}_2 = 0 \qquad \text{at } z = 0 \tag{33}$$

$$\bar{p}_2 = 0 \qquad \text{at } z = L. \tag{34}$$

Solutions for small permeability. As per the physiological data, the permeability of the porous capillary is very small, say, of the order of 10^{-6}. Hence, without giving a descriptive method of solution, we directly write the simplified solutions of velocity and pressure fields for small permeability obtained for first and second approximations using the analysis of Oka and Murata (1970) as follows:

$$u_1(\zeta,\xi)=\frac{R^2}{4\mu}\left(\frac{\Delta p}{L}\right)\left[(1-\zeta^2)+\frac{4}{a^2}\{g_0(\zeta)-1\}\right] \times\left[1-\frac{8\varepsilon}{3\beta^2\lambda_0}\left(1-3\frac{\Delta\alpha}{\Delta p}+6\frac{\Delta\alpha}{\Delta p}\xi-3\xi^2\right)\right]. \tag{35}$$

$$v_1(\zeta,\xi)=\frac{\varepsilon R\Delta p}{\mu\lambda_0}\left(\frac{\Delta\alpha}{\Delta p}-\xi\right)\left[2\zeta-\zeta^3+\frac{16}{a^3}\left\{g_1(\zeta)-\frac{1}{2}\zeta a\right\}\right] \tag{36}$$

$$p_1(\zeta.\,\xi)=p_a-\xi\Delta p+\frac{\varepsilon}{\lambda_0}\frac{8\Delta p}{3\beta^2}\left\{\left(1-3\frac{\Delta\alpha}{\Delta p}\right)\xi+3\frac{\Delta\alpha}{\Delta p}\xi^2-\xi^3\right\} \tag{37}$$

$$u_2(\zeta,\xi)=\frac{\varepsilon\Delta pR^2}{4\lambda_0\mu L}\left[\left(1-\frac{8\varepsilon}{a^2\lambda_0}\right)\zeta^4+\frac{8}{a^2}\left(2s_1+\frac{\varepsilon}{\lambda_0}\right)g_0(\zeta) -\left(16\frac{s_1}{a^2}+\frac{1}{3}\frac{m_1}{\beta_1}+\beta_2\right)\zeta^2-\frac{4}{a^2}\left(\frac{1}{3}\frac{m_1}{\beta_1}+\beta_2\right) \times\{1-g_0(\zeta)\}-\frac{64}{a^4}\delta_7(\zeta)+\frac{1}{3}\frac{m_1}{\beta_1}\beta_2-1\right] \tag{38}$$

$$v_2(\zeta,\xi)=0 \tag{39}$$

$$p_2(\zeta,\xi)=-\frac{4\varepsilon\Delta p}{\lambda_0}\left(\frac{\Delta\alpha}{\Delta p}-\xi\right)\left\{\zeta^2-\frac{\beta_2(1+2\beta_1)}{6I_0(a)}h_1(\zeta) +h_2(\zeta)+\beta_3\right\} \tag{40}$$

where, $\xi=z/L$ is the normalized axial distance from the arteriolar end of the capillary, $a=R/l$ is the couple stress parameter,

$$\Delta p=p_a-p_v,\ \Delta\alpha=p_a-\alpha,\ \beta=\frac{R}{L},$$
$$\zeta=\frac{r}{R},\ \lambda_0=1+\frac{16}{a^3}\left[\frac{I_1(a)}{I_0(a)}-\frac{1}{2}a\right],$$
$$g_0(\zeta)=I_0(\zeta a)/I_0(a),\ g_1(\zeta)=I_1(\zeta a)/I_0(a). \tag{41}$$

I is the modified Bessel function and other constants appearing in the equations (38)–(40) are defined in the Appendix.

Finally, combining the first and second approximations for the velocity and pressure fields we obtain the solutions in a neat form as:

$$U(\zeta, \xi) = \frac{u(\zeta, \xi)}{(R^2 \Delta p / 4\mu L)}$$

$$= \left[(1-\zeta^2) + \frac{4}{a^2} \{g_0(\zeta) - 1\} \right] \times [1 + \varepsilon f_1(0, \xi)] + \varepsilon f_2(\zeta, 0) \tag{42}$$

$$v(\zeta, \xi) = \frac{v(\zeta, \xi)}{(R^2 \Delta p / 4\mu L)}$$

$$= \frac{4\varepsilon}{\beta \lambda_0} \left(\frac{\Delta \alpha}{\Delta p} - \xi \right) \left[2\zeta - \zeta^3 + \frac{16}{a^3} \times \left\{ g_0(\zeta) - \frac{1}{2} \zeta a \right\} \right], \tag{43}$$

$$p(\zeta, \xi) = p_a - \xi \Delta p + \varepsilon g_2(\zeta, \xi) \Delta p, \tag{44}$$

where $f_0(\zeta, 0)$, $f_1(0, \xi)$ and $g_2(\zeta, \xi)$ are given in the Appendix.

Incidentally, in the limit one can deduce the Poiseuille flow solutions from (42)–(44) by taking $\varepsilon \to 0$ and $a \to \infty$, i.e.

$$u = \frac{R^2 \Delta p}{4\mu L} (1 - \zeta^2),$$

$$v = 0, \quad p = p_a - \xi \Delta p. \tag{45}$$

It is also to be noted that our results coincides well with those of Oka and Murata (1970) as $a \to \infty$ (i.e. Newtonian fluid).

Solution for streamlines. In order to understand the behaviour of the streamline pattern, the streamlines are determined by the equation

$$\frac{dr}{v} = \frac{dz}{u}. \tag{46}$$

Integration of (46) and using (42) and (43), yields

$$\delta_1(\zeta) [1 + \varepsilon f_1(0, \xi)] + \delta_2(\zeta) = C \tag{47}$$

where $\delta_1(\zeta)$ and $\delta_2(\zeta)$ are given in the Appendix and C is an arbitrary constant.

Volume of flow per unit time. It is interesting to calculate the volume of flow per unit time, which is of practical importance. The volume of the fluid per unit time across the cross-section at a point z is given by

$$Q^* = \int_0^R 2\pi ur \, dr. \tag{48}$$

Using (42), Q is obtained from (48) as

$$Q = \frac{Q^*(\xi)}{\left(\dfrac{R^4 \Delta p}{4\mu L}\right)} = \frac{\pi\lambda_0}{2}\left[\{1+\varepsilon f_1(0, \xi)\} + \frac{4\varepsilon A_0}{\lambda_0^2}\right] \tag{49}$$

where λ_0 is given in (41) and A_0 is recorded in the Appendix.

The net outflow M of water into the tissue per unit time across the capillary wall can be calculated by $M = Q^*(0) - Q^*(1)$. By using equation (49), we have

$$M = \frac{2\pi\varepsilon R^4}{\mu\beta^2} \frac{\Delta p}{L}\left(\frac{\Delta\alpha}{\Delta p} - \frac{1}{2}\right) = kS\left(\frac{\Delta\alpha}{\Delta p} - \frac{1}{2}\right) \tag{50}$$

$$\therefore \; m' = \frac{M}{S} = k\left(\frac{\Delta\alpha}{\Delta p} - \frac{1}{2}\right) = k(p_{\mathrm{m}} - \alpha)$$

$$\text{or } m' = k(p_{\mathrm{m}} - p_i - \pi_{\mathrm{c}} + \pi_i) \tag{51}$$

where m' is the net outflow of water per unit time averaged over the whole surface area of the capillary S $(=2\pi RL)$ and p_{m} is the arithmetical mean of p_{a} and p_{v}. It is to be noted that p_{c} in equation (1) is replaced by p_{m}.

From equation (50) it is seen that $M = 0$ when $\Delta\alpha/\Delta p = 1/2$, i.e. outflow and inflow are balanced across the wall and only outflow for $\Delta\alpha/\Delta p > 1/2$ since $M > 0$. For the case when $\Delta\alpha/\Delta p < 1/2$, there is no outflow.

4. Discussion of Results. A mathematical model describing blood flow through a capillary emphasizing the role of permeability of the vessel has been developed and closed form solutions have been obtained. Based on the physiological experimental data (Oka and Murata, 1970), the results have been found out. For example, the average filtration constant k is 60×10^{-8} cm (sec. cm $H_2O)^{-1}$ for normal mesentric capillaries of the frog and 2.5×10^{-8} cm (sec. cm H_2O) for mesentric capillaries of the dog and cat. In general, the parameter ε is very small and in particular, it takes the value 4.9×10^{-8} for mesentric capillaries of the frog.

As seen from equation (50), $\Delta\alpha/\Delta p$ must be equal to 1/2 to have the balance of outflow and inflow across the wall. Then in such a case $\alpha = (p_{\mathrm{a}} + p_{\mathrm{v}})/2$ which is calculated from the relation $\Delta\alpha/\Delta p = (p_{\mathrm{a}} - \alpha)/(p_{\mathrm{a}} - p_{\mathrm{v}})$. Generally, the balance of water is not always kept in capillaries (Oka and Murata, 1970). Here, we are interested in the case when there is only outflow across the wall so the value of $\Delta\alpha/\Delta p$ is chosen to be greater than 1/2, i.e. 0.7 and the value of α is calculated from the relation $\alpha = 0.3\,p_a + 0.7\,p_v$. Since in this case $\Delta\alpha/\Delta p > 1/2$ so $\alpha < (p_a + p_v)/2$. Thus, if we choose $p_a = 35$ mm Hg and $p_v = 15$ mm Hg then we have $\alpha = 21$ mm Hg which is less than half the sum of p_a and p_v i.e. 25 mm Hg.

Although several assumptions are involved in our theory, the results obtained express fairly well the flow characteristics of blood in a capillary with a permeable wall. Both axial and radial velocity components have been depicted graphically. The axial velocity profile is depicted in Fig. 2 and it is seen that it becomes minimum at the point $\xi = \Delta\alpha/\Delta p$ while it decreases or increases with ξ in the region $\xi < \Delta\alpha/\Delta p$ or $\xi > \Delta\alpha/\Delta p$ respectively. Thus the axial velocity has a decreasing tendency in blood flows as compared to Newtonian fluids. On the other hand, the axial velocity increases as $a(=R/l)$ increases such that it coincides with a Newtonian profile when a is very large (Fig. 3). The streamline

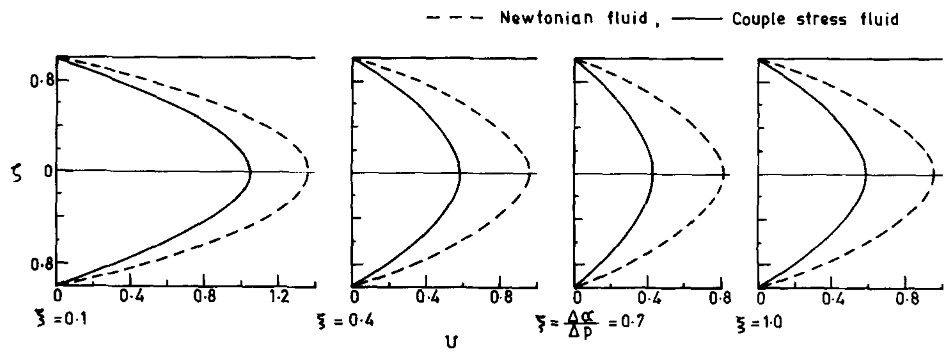

Figure 2. Profiles of longitudinal velocity component U for Newtonian and couple stress fluids for $a = R/l = 3.0$, $\varepsilon = 4.9 \times 10^{-6}$, $\Delta\alpha/\Delta p = 0.7$.

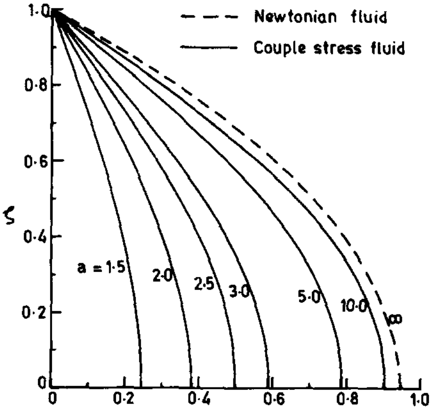

Figure 3. Comparison of profiles of longitudinal velocity u for Newtonian and couple stress fluids for $\Delta\alpha/\Delta p = 0.7$ and $\varepsilon = 4.9 \times 10^{-6}$.

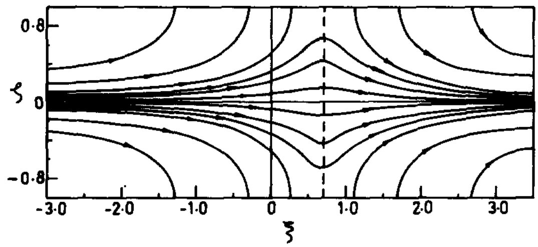

Figure 4. Streamlines for couple stress fluid.

patterns of the flow is shown in Fig. 4. The streamlines direct themselves in such a way that some are clustered along the central part, while the others towards the wall. This might be the reason why the red cells in the blood, in real situations, accumulate near the axis of the capillary and the plasma tends towards the wall.

The radial velocity is shown in Fig. 5, which clearly indicates that it vanishes at $\xi = \Delta\alpha/\Delta p$ while it becomes positive or negative according to $\xi \lessgtr \Delta\alpha/\Delta p$, respectively. Furthermore, there appears outflow and inflow at the wall in the region $\xi < \Delta\alpha/\Delta p$ and $\xi > \Delta\alpha/\Delta p$. On the other hand, the radial velocity

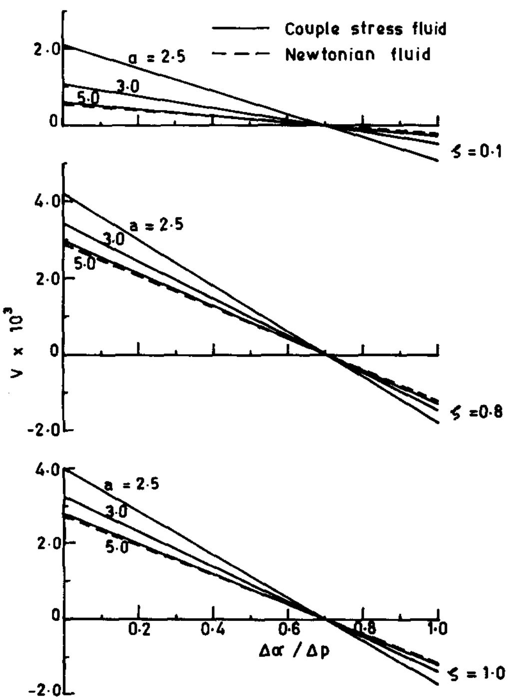

Figure 5. Radial velocity profiles for Newtonian and couple stress fluids for $\Delta\alpha/\Delta p = 0.7$.

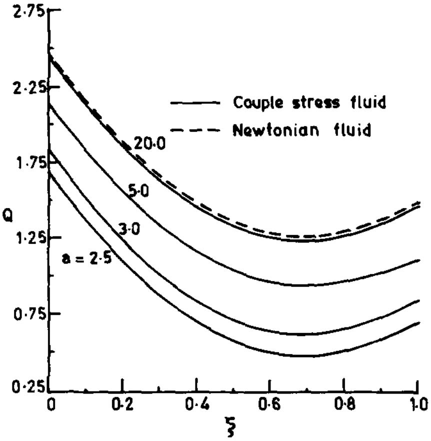

Figure 6. Relationship between Q and ξ for Newtonian and couple stress fluids for $\Delta\alpha/\Delta p = 0.7$ and $\varepsilon = 4.9 \times 10^{-6}$.

decreases as a increases such that it coincides with the Newtonian profile for large a. Just as in the behaviour of axial velocity, the radial velocity also decreases in blood flows.

We can clearly see from Fig. 6 that the volumetric rate is minimum at $\xi = \Delta\alpha/\Delta p$ and the effect of couple stress fluid is to decrease the volumetric rate of fluid flow. It is also interesting to see that as the value of a increases the volumetric rate increases and finally it becomes a constant for a Newtonian fluid. The decrease of the flow rate with increase in ξ in the region $0 < \xi < \Delta\alpha/\Delta p$ is caused by filtration and the increase of the flow rate with ξ in the region $\Delta\alpha/\Delta p < \xi < 1$ is caused by absorption. The net outflow will cause the edema. However, under normal physiological conditions, it may be considered that lymphatics will play a role in protecting the tissues against edema.

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APPENDIX

Constants and functions appearing in (35)–(44), (47) and (49) are as follows:

$$a_1 = R^2\left[\left(1 + \frac{4}{a^2}\right)\left\{1 + \frac{3}{16I_0(a)}\right\} + \frac{1+\lambda^2 l^2}{2a^2 I_0(a)}(a^2+8) - \frac{\lambda^2 R^2}{32a^4}(a^4 + 16a^2 + 64)\right],$$

$$a_2 = \frac{1+\lambda^2 l^2}{2I_0(a)}, \quad a_3 = \frac{3l^2}{4I_0(a)}, \quad b_1 = \frac{a_1 l^2}{I_0(a)},$$

$$b_2 \frac{l^2}{16I_0(a)} \{16I_0(a)(1+a_2)+3\}, \quad b_3 = l^2(4l^2 - 2\lambda^2 l^2 + a_3),$$

$$b_4 = \frac{2}{I_0(a)} \{a_1 + 2l^2(1+\lambda^2 l^2)\},$$

$$b_5 = \frac{l^2}{4I_0(a)} \{11 + 8\lambda^2 l^2 + 8I_0(a)(\lambda^2 l^2 + 1)\}, \quad b_6 = \frac{I_1(a)}{I_0(a)},$$

$$d_1 = 1 - \frac{12b_2}{R^2} + \frac{\lambda^2 l^2}{4a^2}(a^2+24) - \frac{24b_3}{R^4} + 48l^2 a_1 b_6,$$

$$d_2 = 1 - \frac{8}{a^2} + \frac{16}{a^3} b_6,$$

$$d_3 = 1 - \frac{b_1}{2R^2 l^2 a^3} \{2aI_0(a)(a^2+1) - I_1(a)(a^2+4)\} - \frac{2b_2}{R^2 a} - \frac{b_4}{2R^2} I_0(a) + \frac{b_5}{2R^2} - \frac{\lambda^2 l^2}{2},$$

$$d_4 = \frac{2}{a^2}(a^2+1) - \frac{b_6}{a^3}(a^2+4), \quad \beta_1 = \left(\frac{d_4 \lambda_0}{d_2}\right)^{1/2},$$

$$\beta_2 = \frac{3\lambda_0 d_3 - d_1}{\beta_1^2 d_2}, \quad \beta_3 = 1 - \frac{\beta_2}{6} - \frac{\beta_1 \beta_2}{3},$$

$$\beta_4 = \frac{l^2}{a} \{I_1(a)(a^2+2) - aI_0(a)\},$$

$$\beta_5 = \frac{R^2}{2}\left[b_5 - R^2 \lambda^2 l^2 - \frac{2b_4}{a} I_1(a)\right] + \frac{2b_1 \beta_4}{l^2},$$

$$\beta_6 = 2\beta_3 - \frac{2\beta_2 \beta_4}{3R^2 I_0(a)}(1+2\beta_1) + \frac{4}{R^4}\beta_5,$$

$$\beta_7 = \frac{2\beta_4}{R^2 I_0(a)}, \quad \beta_8 = \beta_1 \beta_2 \beta_7, \quad g_1(\zeta) = \frac{I_1(\zeta a)}{I_0(a)},$$

$$m_1 = \frac{\beta_1(1+\beta_6)}{1+\beta_7}, \quad m_2 = \beta_1^2 \beta_2 \frac{(1+\beta_8)}{(1+\beta_7)},$$

$$m_3 = 1 + \frac{11}{16I_0(a)} - \frac{8\varepsilon}{a^2 \lambda_0}\left\{1 - \frac{1}{I_0(a)}\right\}, \quad \lambda_1 = b_6 - \frac{a}{2},$$

$$A_0 = \frac{16\varepsilon}{a^3 \lambda_0} + \frac{32}{a^3}\left\{m_3\left(b_6 - \frac{a}{4}\right) + \frac{8\lambda_1}{a^2}\left(1 - \frac{8\varepsilon}{a^2 \lambda_0} + \frac{3}{16I_0(a)}\right)\right\} + \frac{1}{2a^3}\left(\frac{m_1}{\beta_1} - \frac{\beta_2}{3}\right)(16\lambda_1 + a^3)$$
$$-\left(b_6 - \frac{a}{5} + \frac{2}{3}\right)$$

$$h_1(\zeta)=I_0(\zeta a)\left(1+\frac{2}{\zeta^2 a^2}\right)-\frac{I_1(\zeta a)}{\zeta^3 a^3}(\zeta^2 a^2-4)$$

$$h_2(\zeta)=\frac{2b_1}{l^2}h_1(\zeta)-\frac{4b_2}{\zeta^2 a^2}-\frac{I_0(\zeta a)}{l^2}(b_4 l^2-2b_1)-\lambda^2 l^2 R^2+b_5,$$

$$\delta_1(\zeta)=\frac{1}{\zeta a}g_1(\zeta)-\frac{1}{2},\quad \delta_2(\zeta)=\frac{1}{\zeta a}g_1(\zeta)-\frac{\zeta^3}{5},$$

$$\delta_3(\zeta)=\left\{1-\frac{8\varepsilon}{a^2\lambda_0}+\frac{3}{16I_0(a)}\right\}\{1-g_0(\zeta)\},$$

$$f_0(\zeta,0)=\frac{4}{a^2}\{g_0(\zeta)-1\},$$

$$f_1(0,\xi)=\frac{8}{\lambda_0\beta^2}\left(\frac{\Delta\alpha}{\Delta p}-2\frac{\Delta\alpha}{\Delta p}\xi+\xi^2-\frac{1}{3}\right),$$

$$f_2(\zeta,0)=\frac{1}{\lambda_0}\left[\frac{16\varepsilon}{a^2\lambda_0}\{g_0(\zeta)-\zeta^4\}-2(1-\zeta^4)+\frac{32}{a^2}\{g_0(\zeta)-\zeta^2\}m_3-\frac{256}{a^4}\delta_3(\zeta)\right.$$
$$\left.+\frac{2\beta_2}{3a^2}\{4(1-g_0(\zeta))-a^2\times(1-\zeta^2)\}+\frac{8m_1}{\beta_1 a^2}\left\{1-g_0(\zeta)-\frac{a^2}{4}(1-\zeta^2)\right\}\right],$$

$$g_2(\zeta,\xi)=\frac{8}{\lambda_0\beta^2}\left[\left(\frac{1}{3}-\frac{\Delta\alpha}{\Delta p}\right)\xi+\frac{\Delta\alpha}{\Delta p}\xi^2-\frac{\xi^3}{3}-\frac{\beta^2}{2}\left\{\left(\frac{\Delta\alpha}{\Delta p}-\xi\right)\right.\right.$$
$$\left.\left.\times\left(\zeta^2+\beta_3+h_2(\zeta)-\frac{\beta_2}{6}(1+2\beta_1)\frac{h_1(\zeta)}{I_0(a)}\right)\right\}\right].$$

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