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PRECISE INTEGRATION METHOD FOR LQG OPTIMAL MEASUREMENT FEEDBACK CONTROL PROBLEM *

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(Paper from ZHONG Wan-xie, Member of Editorial Committee, AMM)

Abstract: *By using the precise integration method, the numerical solution of linear quadratic Gaussian (LQG) optimal control problem was discussed. Based on the separation principle, the LQG control problem decomposes, or separates, into an optimal state-feedback control problem and an optimal state estimation problem. That is the off-line solution of two sets of Riccati differential equations and the on-line integration solution of the state vector from a set of time-variant differential equations. The present algorithms are not only appropriate to solve the two-point boundary-value problem and the corresponding Riccati differential equation, but also can be used to solve the estimated state from the time-variant differential equations. The high precision of precise integration is of advantage for the control and estimation. Numerical examples demonstrate the high precision and effectiveness of the algorithm.*

Key words: precise integration; LQG measurement feedback control; Riccati differential equation; time-variant differential equation

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Introduction

Consider the linear system of the measurement feedback control

$$\dot{x} = Ax + Bw + B_2 u, \tag{1}$$

$$y = Cx + v, \tag{2}$$

where x is the n-dimensional state vector, y is a q-vector of measurements, u is an m-vector of control inputs, w and v are l-vector, q-vector of white-noise process with known statistical properties respectively, A, B, B_2, C are known time-invariant matrices with the proper sizes and controllable and observable properties, and $w(t)$ and $v(t)$ have the properties as follows

$$\left.\begin{array}{l} \mathrm{E}[v(t_1)v^{\mathrm{T}}(t_2)] = V\delta(t_2 - t_1),\ \mathrm{E}[w(t_1)w^{\mathrm{T}}(t_2)] = W\delta(t_2 - t_1), \\ \mathrm{E}[w] = 0,\ \mathrm{E}[v] = 0,\ \mathrm{E}[v(t_1)w^{\mathrm{T}}(t_2)] = 0, \end{array}\right\} \tag{3}$$

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in which E indicates mean value, intensity matrices W and V are $l \times l$, $q \times q$ positive definite matrices respectively. The initial state can be given as

$$\mathrm{E}[x(0)] = \hat{x}_0, \mathrm{E}[(x(0) - \hat{x}_0)(x(0) - \hat{x}_0)^{\mathrm{T}}] = P_0, \tag{4}$$

where $V(t)$ and $W(t)$ are not correlative with $x(0)$.

In the case of the LQG generalized regulator problem, it is well-known that the synthesis is achieved via a decomposition, or separation, into an optimal state-feedback control problem and an optimal state estimation problem. The optimal state-feedback controller is given in terms of the solution of a Riccati differential equation, which is solved backwards in time from a terminal condition. The optimal state estimator is the Kalman filter and the filter gain is given in terms of the solution of a second Riccati differential equation, which is solved forwards in time from an initial condition. This is known as the separation principle[1~5]. Based on this principle, our solution of the LQG control problem follows standard lines of argument. Therefore, we will solve two kinds of Riccati differential equations

$$\dot{P}(t) = BWB^{\mathrm{T}} + AP + PA^{\mathrm{T}} - PC^{\mathrm{T}}V^{-1}CP, \; P(0) = P_0, \tag{5}$$

$$\dot{S}(t) = -Q - SA - A^{\mathrm{T}}S + SB_2R^{-1}B_2^{\mathrm{T}}S, \; S(t_f) = S_f, \tag{6}$$

and the state estimation needs to solve

$$\dot{\hat{x}}(t) = A\hat{x} + PC^{\mathrm{T}}V^{-1}(y - C\hat{x}) + B_2u, \; \hat{x}(0) = x_0, \tag{7}$$

$$\dot{x}(t) = [A - B_2K(t)]x, \; u = -Kx, \; K = R^{-1}B_2^{\mathrm{T}}S(t). \tag{8}$$

where K is the gain matrix, and $P(t)$ and $S(t)$ are the solutions to Riccati differential equations (5) and (6) respectively.

Although the separation principle is well-known, the corresponding numerical computation still has a number of problems to be solved. In computational mechanics, the precise integration method was presented for the ordinary differential equation appropriate to both initial value problems[6,7] and two-point boundary-value problems of corresponding Riccati differential equations[8]. Even if Eq.(8) is time-variant because of $S(t)$, its precise integration method has been presented[9]. Also the precise integration of Kalman-Bucy filtering problems can be seen Ref.[10]. In Refs.[9] and [10], the analytic characteristics of the Riccati differential equations and the state vector equation of optimal control are applied in deriving the highly precise numerical solution, so that the full computer precision is reached. Since the feedback control vector u is of key importance, the computation of LQG measurement feedback control vectors is followed with great interest. In this paper, the precise integration of LQG control problem is presented.

1 Kalman-Bucy Filtering Equation Under Control

For two kinds of Riccati differential equations described by Eqs.(5) and (6), the variational approach with interval mixed energy was developed to derive the whole set of equations and algorithm of precise integration[6~8]. Let the length of a time step be η, time points are as follows

$$t_0 = 0, \cdots, t_k = k\eta, \cdots, t_f = k_f\eta. \tag{9}$$

Suppose that t_k is present time, and S_k and P_k can be computed by the precise integration. For any k, S and P matrices have been the off-line solution of the corresponding Riccati differential equations[9,10].

To obtain the real time integration of the state vector $\hat{x}$, substituting u in Eq.(8) into Eq.(7) gives

$$\dot{\hat{x}}(t) = (A - B_2 R^{-1} B_2^{\mathrm{T}} S)\hat{x} + PC^{\mathrm{T}} V^{-1} \xi(t), \tag{10}$$

$$\xi(t) = y(t) - C\hat{x}(t), \tag{11}$$

where ξ is called as the new information vector[11], and also it is a white-noise process and satisfies

$$\mathrm{E}[\xi(t)\xi^{\mathrm{T}}(t+\tau)] = V\cdot\delta(\tau),\ \mathrm{E}[\xi] = 0.$$

Eq.(10) is Kalman-Bucy filtering equation under control. Solving Eq.(10) is key to LQG measurement feedback control problem.

2 Application of the Precise Integration Method

Eq.(10) is the linear non-homogeneous differential equation of Kalman-Bucy filtering estimation vector $\hat{x}(t)$. If we regard white-noise vectors ξ as external inputs, homogeneous equation of Eq.(10) is the same as Eq.(8). Because of $S(t)$, Eq.(8) is time-variant differential equation. In Ref.[9], the precise integration of the response matrix $F_s^{-1}(t)F_s(0)$ to initial value problem has been presented. In this paper, the sign $\Phi(t,0)$ denotes this response matrix. However, the integration of a non-homogeneous equation need the impulse response function $\Phi(t,t_0)$, and the step-by-step integration for the step size η need $\Phi(t+\eta,t)$($\Phi(t,t) = I$). All these matrices can be computed by the precise integration method (see appendix).

On the basis of ordinary differential equation theory, the solution to a non-homogeneous linear differential equation (10) is

$$\hat{x}(t) = \Phi(t,t_0)\cdot\hat{x}_0 + \int_{t_0}^{t} \Phi(t,t_d)\cdot f(t_d)\mathrm{d}t_d,\ f(t_d) = P(t_d)C^{\mathrm{T}}V^{-1}\xi(t_d).$$

Applying this formula to each time step (t_k, t_{k+1}) and using Newton-Cotes integration method give

$$\begin{aligned}\hat{x}(t_{k+1}) &= \Phi(t_k+\eta,t_k)\cdot\hat{x}_k + \int_{t_k}^{t_k+\eta}\Phi(t_k+\eta,t_d)\cdot P(t_d)C^{\mathrm{T}}V^{-1}\xi(t_d)\mathrm{d}t_d \approx \\ &\Phi(t_k+\eta,t_k)\cdot\hat{x}_k + [\Phi(t_k+\eta,t_k)P_kC^{\mathrm{T}}V^{-1}(y_k - C\hat{x}_k) + \\ &P_{k+1}C^{\mathrm{T}}V^{-1}(y_{k+1} - C\hat{x}_{k+1})]\eta/2.\end{aligned} \tag{12a}$$

Re-write Eq.(12a) as (implicit form)

$$\left.\begin{aligned}&\hat{x}_{k+1} = v_k - N_k\hat{x}_{k+1},\ N_k = (\eta/2)\cdot P_{k+1}C^{\mathrm{T}}V^{-1}C, \\ &v_k = \Phi(t_k+\eta,t_k)\cdot[\hat{x}_k + P_kC^{\mathrm{T}}V^{-1}(y_k - C\hat{x}_k)\cdot(\eta/2)] + \\ &\quad P_{k+1}C^{\mathrm{T}}V^{-1}y_{k+1}\cdot(\eta/2).\end{aligned}\right\} \tag{12}$$

Consequently, we obtain

$$\hat{x}_{k+1} = (I + N_k)^{-1}v_k.$$

This is the precise integration formula of Kalman-Bucy filtering equation under control. Since it is related to measurement value, it needs the real time computation. The quantity of real time should reduce to the lowest limit. Thus, the parts of the off-line computation should be distinguished. Such as S_k, P_k, $\Phi(t_k+\eta,t_k)$ (see Refs.[9], [10] and appendix), $P_{V,k} = P_kC^{\mathrm{T}}V^{-1}\cdot(\eta/2)$, $N_k = P_{V,k+1}\cdot C$, $(I+N_k)^{-1}$ for all k should be computed and stored in advance.

3 Numerical Examples

Example 1 Suppose that a one-dimensional system is described by

$$n = 1,\ A = -0.8,\ Q = 0.64,\ B_2 R^{-1} B_2 = 25.0;\ t_f = 0.4,\ S_f = 0.01;$$
$$W = 1.0, V = 1.0,\ B = 0.8,\ C = 5.0;\ P_0 = 0.01;$$

$\hat{x}_0 = 1.0$, y is the measurement with inclusion of Gaussian white-noise process. Compute $\hat{x}(t)$ and u.

Example 2 Consider a four-dimensional system

$$n = 4,\ t_f = 16,\ S_f = \mathrm{diag}[10.0,\ 10.0,\ 10.0,\ 10.0],$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 1 & 0 & 0 \\ 0.5 & -0.5 & 0 & 0 \end{bmatrix},\quad Q = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$B_2 R^{-1} B_2^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix},\quad BWB^{\mathrm{T}} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$C = [0,\ 0,\ 0,\ 0.5],\ V = [1],\ P_0 = \mathrm{diag}[0.1,\ 0.1,\ 0.1, 0.1],\ \hat{x}_0 = [1.0,\ 1.0,\ 0,\ 0]^{\mathrm{T}},$$

y is the measurement with inclusion of Gaussian white-noise process. Show $\hat{x}(t)$ and u.

By the precise integration, the numerical results of state estimation vectors $\hat{x}(t)$ and optimal control vectors u see Figs. 1 ~ 6.

We can verify a special case as follows:

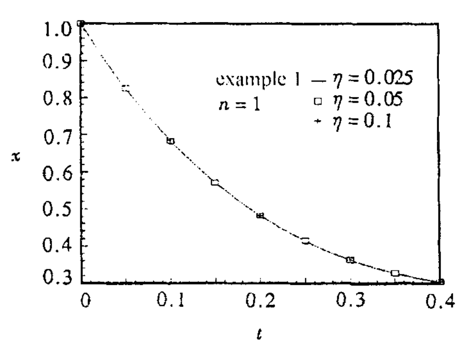
Suppose the y is the determinate measurement $y_k = C\bar{x}_k$, in which $\bar{x}_k$ is the solution of homogeneous equation (8) at t_k. From Eq. (12), that is

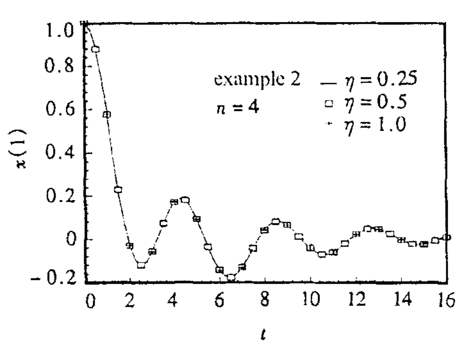
$$y_{k+1} = C\bar{x}_{k+1} = C\Phi(t_k + \eta, t_k)\bar{x}_k,$$

so that

$$\hat{x}_{k+1} = \bar{x}_{k+1}. \tag{13}$$

The numerical results prove this conclusion. See Figs. 1 and 2.

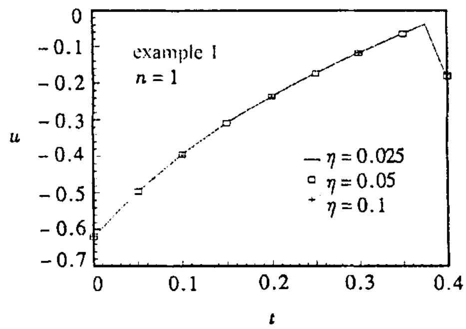


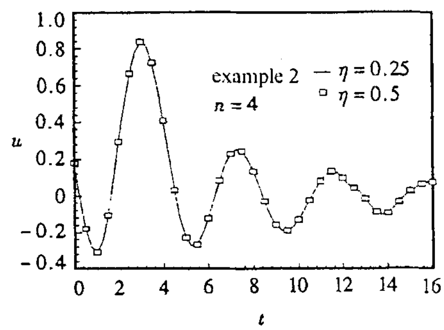


Note: $\xi = 0$

Fig. 1

Fig. 2

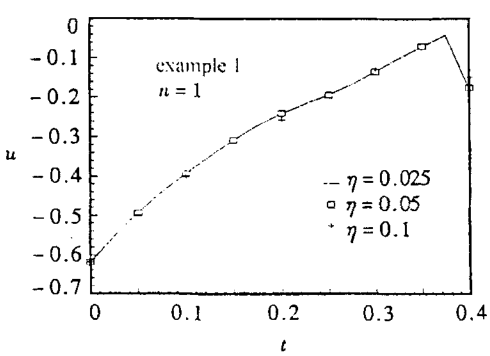


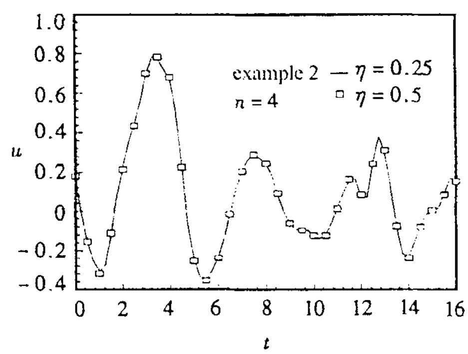


Note: ξ is about 1/10 of $C\hat{x}$

Fig.3

Fig.4





Note: ξ is a almost same as $C\hat{x}$

Fig.5

Fig.6

In Figs.3 ~ 6, some representative results for the optimal control vectors $\boldsymbol{u}$ are presented, in which measurements $\boldsymbol{y}$ are superposed by the different orders of magnitude of the white-noise. When the quantity of the noise is about 1/10 of signal, Figs.3 ~ 4 show that the numerical results at the same time points for the different time step size is very closed. Even if the quantity of the noise is almost the same order of magnitude of signal, Figs.5 ~ 6 show that numerical results at the same time points for the different time step size is still closed. All these show that the precise integration method is not sensitive to the length of a time step. Furthermore, the numerical results show that the corresponding state estimation vectors also have the above characteristics.

4 Concluding Remarks

For LQG measurement feedback control problems, the numerical results in this paper show that the characteristics of high precision of the precise integration method. Even for the time-variant differential equations and two sets of Riccati differential equation, their solutions still are highly precise by combining the precise integration method with the separation principle. Since there are many similarities between the solution of the LQG measurement feedback control problem and the H_∞ generalized regulator problem, further applying the precise integration method to H_∞ control problems has great potentialites.

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Appendix The Precise Computation of Single-step Impulse Response Function $\boldsymbol{\Phi}(t_k+\eta, t_k)$

For the time interval $(t_k,\ t_k+\eta)$, based on the time interval condensation equations in Ref.[9], let the solution matrix S_{k+1} of Riccati differential equation at time $t_{k+1} = t_k + \eta$ be $(\boldsymbol{E}_2, \boldsymbol{G}_2, \boldsymbol{F}_2)$ to constitute virtual time interval $(\boldsymbol{S}_{k+1},\ 0,\ \boldsymbol{I})$; while let single-step mixed energy matrices $[\boldsymbol{E}(\eta),\ \boldsymbol{G}(\eta), \boldsymbol{F}(\eta)]$ be $(\boldsymbol{E}_1,\ \boldsymbol{G}_1,\ \boldsymbol{F}_1)$, so that

$$S_k = \boldsymbol{E}(\eta) + \boldsymbol{F}^{\mathrm{T}}(\eta)[S_{k+1}^{-1} + \boldsymbol{G}(\eta)]^{-1}\boldsymbol{F}(\eta), \tag{A.1}$$

$$\boldsymbol{F}_s(t_k) = [\boldsymbol{I} + \boldsymbol{G}(\eta)\boldsymbol{S}_{k+1}]^{-1}\boldsymbol{F}(\eta),\ \boldsymbol{F}_s(t_{k+1}) = \boldsymbol{I}. \tag{A.2}$$

Applying the corresponding equation in Ref.[9], we have

$$\boldsymbol{x}_{k+1} = \boldsymbol{F}_s^{-1}(t_{k+1})\cdot\boldsymbol{F}_s(t_k)\boldsymbol{x}_k = \boldsymbol{F}_s(t_k)\boldsymbol{x}_k.$$

For any k, this formula is satisfied. Thus, single-step impulse response function is of the form

$$\boldsymbol{\Phi}(t_k+\eta, t_k) = \boldsymbol{F}_s(t_k). \tag{A.3}$$

$\boldsymbol{F}_s(t_k)$ can be obtained from Eq.(A.2).

Note that the matrices $\boldsymbol{E}(\eta), \boldsymbol{G}(\eta), \boldsymbol{F}(\eta)$ and the solution matrix of Riccati differential equation can be computed by the precise integration method (see Ref.[9]). Therefore, for any k, Eq.(A.3) can be calculated precisely.