Applied Mathematics and Mechanics
(English Edition, Vol. 21, No. 3, Mar 2000)

Published by Shanghai University,
Shanghai, China

Article ID: 0253-4827(2000)03-0265-10

A NOTE ON BIFURCATIONS OF $u'' + \mu(u - u^k) = 0(4 \leqslant k \in \mathbf{Z}^+)$ *

Li Changpin (李常品)

(Department of Mathematics, Shanghai University, Shanghai 200436, P R China)

(Communicated by Liu Zengrong)

Abstract: *Bifurcations of one kind of reaction-diffusion equations*, $u'' + \mu(u - u^k) = 0$(μ *is a parameter*, $4 \leqslant k \in Z^+$), *with boundary value condition* $u(0) = u(\pi) = 0$ *are discussed. By means of singularity theory based on the method of Liapunov-Schmidt reduction, satisfactory results can be acquired.*

Key words: Liapunov-Schmidt reduction; singularity theory; bifurcation

CLC number: O175.1　　**Document code**: A

Introduction

One kind of steady state reaction-diffusion equations is read as ([1, 2], etc),

$$F(u, \mu) = u'' + \mu(u - u^k) = 0 \tag{1}$$

with boundary value condition

$$u(0) = u(\pi) = 0, \tag{2}$$

where μ is a parameter, $1 < k \in \mathbf{Z}^+$.

Let $X = \{u \in C^2[0,\pi] \mid u(0) = u(\pi) = 0\}$, $Y = C^0[0,\pi]$. Then $F(u, \mu)$ is a map from $X \times R$ onto Y. We define inner products on these spaces by $\langle u,v \rangle = \int_0^\pi u(\xi)v(\xi)\mathrm{d}\xi$. For every μ, (1) has a trivial solution $u = 0$.

Consider the linearized equation

$$D_uF(0,\mu) \wedge v = v'' + \mu v = 0 \tag{3}$$

with boundary value condition

$$v(0) = v(\pi) = 0. \tag{4}$$

Obviously, (3) and (4) have nontrivial solutions $v = c\sin nx$ (c is an arbitrary constant) iff $\mu = \mu_n = n^2 (n \in \mathbf{Z}^+)$. For $\mu \neq \mu_n$, (3) has only one solution $u = 0$.

With the increase of the exponent k of u, it is more and more difficult to deal with bifurcations of (1) with (2). This paper will deal with the cases $k \geqslant 4$. In section 1, we apply Liapunov-Schmidt reduction ([3 ~ 6]) to (1) at the bifurcation point $(u, \mu) = (0, n^2)$ to get

* **Received date**: 1998-11-20; **Revised date**: 1999-11-05

Foundation item: the National Natural Science Foundation of China (19971057); Science and Technology Developing Foundation of Shanghai Higher Education (99QA66)

Biography: Li Changpin (1968 ~), Lecturer, Doctor

bifurcation equation. Section 2 is devoted to bifurcation analysis of the bifurcation equation derived in section 1.

1 Liapunov-Schmidt Reduction

Let $L_n = D_u F(0, n^2)$, $\ker L_n = \text{span}\{\sin n\xi\} = \text{span}\{e\}$. $L_n : X \to Y$ is a Fredholm operator of index zero (see [3]). Now we split the spaces into

$$X = \ker L_n \oplus M, \quad Y = N \oplus \text{range} L_n, \tag{5}$$

where $M = (\ker L_n)^{\perp}$, $N = (\text{range} L_n)^{\perp}$. It is evident that L_n is a self-adjoint operator, i.e., $L_n^* = L_n$. According to Fredholm alternative, we have $(\text{range} L_n)^{\perp} = \ker L_n^*$. It immediately follows that $(\text{range} L_n)^{\perp} = \ker L_n$.

Let P_e be the orthogonal projector from Y onto $\text{range} L_n$. According to Liapunov-Schmidt reduction, (1) is equivalent to

$$P_e F(v + w, \mu) = 0, \ v \in \ker L_n, \ w \in M \tag{6}$$

and

$$(I - P_e) F(v + w, \mu) = 0. \tag{7}$$

Due to implicit function theorem (for instance, see [4], (6) is solved for a unique $w(v, \mu)(w(0, n^2) = 0)$. Substituting $w(v, \mu)$ into (7) yields the reduced equation

$$(I - P_e) F(v + w(v, \mu), \mu) = 0, \tag{8}$$

which is called the bifurcation equation. Taking inner product of (8) with e and applying $e \in (\text{range} L_n)^{\perp}$ leads to

$$\langle e, F(v + w(v, \mu), \mu) \rangle = 0. \tag{9}$$

Let $v = xe$, the equation above can be rewritten as

$$g(x, \mu) = \langle e, F(xe + w(x, \mu), \mu) \rangle = 0, \tag{10}$$

which is also called the bifurcation equation of (1) at $(u, \mu) = (0, n^2)$. So the bifurcation phenomena of (1) are reduced to those of (10). However, it is impossible to find the exact expression of the solution to (10), therefore, we need to compute the derivatives of the reduced function $g(x, \mu)$, which is useful for our sequential discussion. Before rendering the derivatives, we define

$$(\mathrm{d}^k G)_{(y, \alpha)}(v_1, \cdots, v_k) = \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_k} G\left(y + \sum_{i=1}^{k} t_i v_i, \alpha\right)\Big|_{t_1 = \cdots = t_k = 0}, \tag{11}$$

$v_i \in R^n (i = 1, 2, \cdots, k)$. It is evident that $(d^k G)_{(y, \alpha)}$ is a symmetric, multilinear function of k arguments.

We rewrite $P_e F(xe + w(x, \mu), \mu) = 0$ for $P_e F(v + w, \mu) = 0$. Repeated application of the chain rule to $g(x, \mu)$ and $P_e F(xe + w(x, \mu), \mu)$ yields the following formulas.

$$g_x = \langle e, \mathrm{d}F(e + w_x) \rangle, \tag{12}$$

$$g_{x^2} = \langle e, \mathrm{d}F(w_{x^2}) + \mathrm{d}^2 F(e + w_x, e + w_x) \rangle, \tag{13}$$

$$g_{x^3} = \langle e, \mathrm{d}F(w_{x^3}) + 3\mathrm{d}^2 F(e + w_x, w_{x^2}) + \mathrm{d}^3 F(e + w_x, e + w_x, e + w_x) \rangle, \tag{14}$$

$$\begin{aligned} g_{x^4} = {} & \langle e, \mathrm{d}F(w_{x^4}) + 3\mathrm{d}^2 F(w_{x^2}, w_{x^2}) + 4\mathrm{d}^2 F(e + w_x, w_{x^3}) + \\ & 6\mathrm{d}^3 F(e + w_x, e + w_x, w_{x^2}) + \mathrm{d}^4 F(\underbrace{e + w_x, \cdots, e + w_x}_{4}) \rangle, \end{aligned} \tag{15}$$

$$\begin{aligned} g_{x^5} = \langle & e, dF(w_{x^5}) + 5d^2F(w_{x^4}, e + w_x) + 10d^2F(w_{x^2}, w_{x^3}) + \\ & 10d^3F(e + w_x, e + w_x, w_{x^3}) + 10d^4F(e + w_x, e + w_x, e + w_x, w_{x^2}) + \\ & 15d^3F(e + w_x, w_{x^2}, w_{x^2}) + d^5F(\underbrace{e + w_x, \cdots, e + w_x}_{5})\rangle, \end{aligned} \tag{16}$$

$$\begin{aligned} g_{x^6} = \langle & e, dF(w_{x^6}) + 6d^2F(e + w_x, w_{x^5}) + 15d^2F(w_{x^2}, w_{x^4}) + \\ & 10d^2F(w_{x^3}, w_{x^3}) + 15d^3F(e + w_x, e + w_x, w_{x^4}) + \\ & 60d^3F(e + w_x, w_{x^2}, w_{x^3}) + 15d^3F(w_{x^2}, w_{x^2}, w_{x^2}) + \\ & 45d^4F(e + w_x, e + w_x, w_{x^2}, w_{x^2}) + 20d^4F(\underbrace{e + w_x, \cdots, e + w_x}_{3}, w_{x^3}) + \\ & 15d^5F(\underbrace{e + w_x, \cdots, e + w_x}_{4}, w_{x^2}) + d^6F(\underbrace{e + w_x \ \cdots, e + w_x}_{6})\rangle, \end{aligned} \tag{17}$$

$$\begin{aligned} g x^7 = \langle & e, dF(w_{x^7}) + 7d^2F(e + w_x, w_{x^6}) + 21d^2F(w_{x^2}, w_{x^5}) + \\ & 35d^2F(w_{x^3}, w_{x^4}) + 21d^3F(e + w_x, e + w_x, w_{x^5}) + \\ & 90d^2F(e + w_x, w_{x^2}, w_{x^4}) + 70d^3F(e + w_x, w_{x^3}, w_{x^3}) + \\ & 105d^3F(w_{x^2}, w_{x^2}, w_{x^3}) + 35d^4F(e + w_x, e + u_x, e + w_x, w_{x^4}) + \\ & 210d^4F(e + w_x, e + w_x, w_{x^2}, w_{x^3}) + 105d^4F(e + w_x, w_{x^2}, w_{x^2}, w_{x^2}) + \\ & 69d^5F(\underbrace{e + w_x, \cdots, e + w_x}_{3}, w_{x^2}, w_{x^2}) + 26d^5F(\underbrace{e + w_x, \cdots, e + w_x}_{4}, w_{x^3}) + \\ & 12d^6F(\underbrace{e + w_x, \cdots, e + w_x}_{5}, w_{x^2}) + d^7F(\underbrace{e + w_x, \cdots, e + w_x}_{7})\rangle, \end{aligned} \tag{18}$$

$\cdots,$

$$g_\mu = \langle e, dF(w_\mu) + F_\mu\rangle, \tag{19}$$

$$g_{x\mu} = \langle e, dF_\mu(e + w_x) + dF(w_{x\mu}) + d^2F(e + w_x, w_\mu)\rangle, \tag{20}$$

$\cdots,$

$$P_e dF(e + w_x) = 0, \tag{21}$$

$$P_e d^2F(e + w_x, e + w_x) + P_e dF(w_{x^2}) = 0, \tag{22}$$

$$P_e d^3F(e + w_x, e + w_x, e + w_x) + 3P_e d^2F(e + w_x, w_{x^2}) + P_e dF(w_{x^3}) = 0, \tag{23}$$

$$\begin{aligned} & P_e d^4F(e + w_x, e + w_x, e + w_x, e + w_x) + 6P_e d^3F(e + w_x, e + w_x, w_{x^2}) + \\ & \quad 3P_e d^2F(w_{x^2}, w_{x^2}) + 4P_e d^2F(e + w_x, w_{x^3}) + P_e dD(w_{x^4}) = 0, \end{aligned} \tag{24}$$

$$\begin{aligned} & P_e d^5F(\underbrace{e + w_x, \cdots, e + w_x}_{5}) + 10P_e d^4F(e + w_x, e + w_x, e + w_x, w_{x^2}) + \\ & \quad 15P_e d^3F(e + w_x, w_{x^2}, w_{x^2}) + 10P_e d^3F(e + w_x, e + w_x, w_{x^3}) + \\ & \quad 10P_e d^2(w_{x^2}, w_{x^3}) + 5P_e d^2F(e + w_x, w_{x^4}) + P_e dF(w_{x^5}) = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} & P_e d^6F(\underbrace{e + w_x, \cdots, e + w_x}_{6}) + 15P_e d^5F(\underbrace{e + w_x, \cdots, e + w_x}_{4}, w_{x^2}) + \\ & \quad 20P_e d^4F(e + w_x, e + w_x, e + w_x, w_{x^3}) + 45P_e d^4F(e + w_x, e + w_x, w_{x^2}, w_{x^2}) + \\ & \quad 60P_e d^3F(e + w_x, w_{x^2}, w_{x^3}) + 15P_e d^3F(w_{x^2}, w_{x^2}, w_{x^2}) + \\ & \quad 10P_e d^2F(w_{x^3}, w_{x^3}) + 15P_e d^3F(e + w_x, e + w_x, w_{x^4}) + \end{aligned}$$

$$P_e dF(w_{x^6}) + 6P_e d^2 F(e + w_x, w_{x^5}) + 15P_e d^2 F(w_{x^2}, w_{x^4}) = 0, \tag{26}$$

$$P_e d^7 F(\underbrace{e + w_x, \cdots, e + w_x}_{7}) + 12P_e d^6 F(\underbrace{e + w_x, \cdots, e + w_x}_{5}, w_{x^2}) +$$
$$26P_e d^5 F(\underbrace{e + w_x, \cdots, e + w_x}_{4}, w_{x^3}) + 69P_e d^5 F(\underbrace{e + w_x, \cdots, e + w_x}_{3}, w_{x^2}, w_{x^2}) +$$
$$210P_e d^4 F(e + w_x, e + w_x, w_{x^2}, w_{x^3}) + 105P_e d^4 F(e + w_x, w_{x^2}, w_{x^2}, w_{x^2}) +$$
$$105P_e d^3 F(w_{x^2}, w_{x^2}, w_{x^3}) + 35P_e d^4 F(e + w_x, e + w_x, e + w_x, w_{x^4}) +$$
$$90P_e d^3 F(e + w_x, w_{x^2}, w_{x^4}) + 70P_e d^3 F(e + w_x, w_{x^3}, w_{x^3}) +$$
$$35P_e d^2 F(w_{x^3}, w_{x^4}) + 21P_e d^3 F(e + w_x, e + w_x, w_{x^5}) +$$
$$P_e dF(w_{x^7}) + 7P_e d^2 F(e + w_x, w_{x^6}) + 21P_e d^2 F(w_{x^2}, w_{x^5}) = 0, \tag{27}$$

$\cdots$,

$$P_e dF(w_\mu) + P_e F_\mu = 0, \tag{28}$$

$$P_e dF_\mu(e + w_x) + P_e dF(w_{x\mu}) + P_e d^2 F(e + w_x, w_\mu) = 0. \tag{29}$$

The main purpose of (21) ~ (29), etc is to determine $w_x, w_{x^2}, \cdots, w_\mu, w_{x\mu}, \cdots$ which are necessary for (12) ~ (20), etc.

2 Bifurcation Analysis

Singularity theory[4~6] based on the method of Liapunov-Schmidt plays an important role in static bifurcation analysis of nonlinear problems. The definitions and results, contained in the recognition problem (one aspect of singularity theory), are introduced as follows.

$E_{x,\mu} = \{g \mid g: C^\infty$ map from $\mathbf{R}^2 \times \mathbf{R}$ onto R on some neighborhood of $(0,0)\}$. We call the elements of $E_{x,\mu}$ germs. Obviously, $E_{x,\mu}$ is a linear space.

Consider

$$f(x,\mu) = 0,\ (x,\mu) \in U \times V \subset \mathbf{R}^2, \tag{30}$$

where $f \in E_{x,\mu}$, $(0,0) \in U \times V$ and $f(0,0) = f_x(0,0) = 0$, i.e., $(0,0)$ is a singular point of f.

Suppose that $f, h \in E_{x,\mu}$. We shall say that f and h are equivalents if

$$f(x,\mu) = S(x,\mu)h(\Omega(x,\mu)), \tag{31}$$

where $\Omega(x,\mu) = (X(x,\mu),\ \Lambda(\mu))$ is a C^∞ diffemorphism on the neighborhood of the origin, $S(x,\mu)) \in E_{x,\mu}$ and $X(0,0) = \Lambda(0) = 0$, $\Lambda'(0) > 0$, $X_x(0,0) > 0$, $S(0,0) > 0$. If $\Lambda(\mu) \equiv \mu$, then we say that f and h are strongly equivalents.

The following basic facts are easily verified (see [5]). If f and h are equivalents, then 1) f and h have same singular points, 2) $n_f(\mu) = n_h(\Lambda(\mu))$ ($n_f(\mu)$ denotes the number of solutions of $f(x,\mu) = 0$, so does $n_h(\Lambda(\mu))$, 3) the stability of the equilibrium solution of $\dot{x} = g(x,\mu)$ is the same as that of $\dot{x} = h(x,\mu)$.

Theorem 1 A germ $f \in E_{x,\mu}$ is strongly equivalent to $\varepsilon x^k + \delta\mu x$ iff at $x = \mu = 0$,

$$f = f_x = \cdots = f_{x^{k-1}} = f_\mu = 0 \tag{32}$$

and

$$\varepsilon = \operatorname{sgn} f_{x^k},\ \delta = \operatorname{sgn} f_{x\mu}. \tag{33}$$

The bifurcation diagrams below are known to us.

Now let us discuss the bifurcation Eq.(10).

According to (11), we have

$$(\mathrm{d}F)_{(0,n^2)}\xi = \xi'' + n^2\xi, \tag{34}$$

$$(\mathrm{d}^lF)_{(0,n^2)}(\xi_1,\cdots,\xi_l) = \frac{\partial^l}{\partial t_1\cdots\partial_{t_l}}\Big[\sum_{i=1}^{l}(t_i\xi_i'' + n^2t_i\xi_i) - n^2\Big(\sum_{i=1}^{l}t_i\xi_i\Big)^k\Big]\Big|_{t_1=\cdots=t_l=0} = \begin{cases}0 & (l \neq k \text{ and } l > 1)\\ -n^2k!\prod_{i=1}^{k}\xi_i & (l = k).\end{cases} \tag{35}$$

From (21), we can get $P_eL_n(e + w_x(0,n^2)) = 0$. Since $L_n : M \to \mathrm{range}L_n$ is invertible and $e \in \ker L_n$, $w_x \in M$, then

$$w_x(0,n^2) = 0. \tag{36}$$

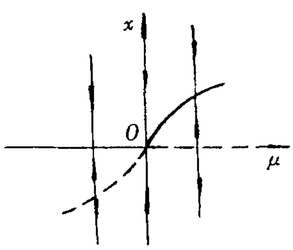

Fig.1 Bifurcation diagram of $\mathrm{d}x/\mathrm{d}t = -x^k + \mu x$ ($4 \leqslant k \in \mathbf{Z}^+$, k is even)

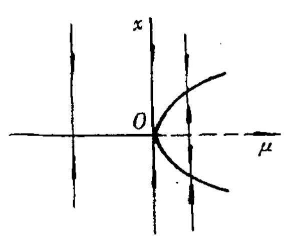

Fig.2 Bifurcation diagram of $\mathrm{d}x/\mathrm{d}t = -x^k + \mu x$ ($4 \leqslant k \in \mathbf{Z}^+$, k is odd)

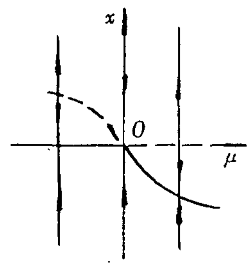

Fig.3 Bifurcation diagram of $\mathrm{d}x/\mathrm{d}t = x^k + \mu x$ ($4 \leqslant k \in \mathbf{Z}^+$, k is even)

Furthermore, from (22) ~ (27), etc and (35) and exerting the fact that L_n is invertible and $w_{x^j} \in M(j \in Z^+)$, we obtain

$$w_{x^l}(0,n^2) = 0 \qquad (l = 2,\cdots,k-1); \tag{37}$$

and

$$P_e(-n^2k!e^k) + P_eL_nw_{x^k}(0,n^2) = 0, \tag{38}$$

i.e.,

$$w_{x^k}(0,n^2) = n^2k!L_n^{-1}(P_ee^k). \tag{39}$$

Letting (12) ~ (18), etc evaluate on $(0,n^2)$ and applying (35) ~ (37) and (39) to them, we can procure

$$g_{x^l}(0,n^2) = 0\ (l = 1,\cdots,k-1). \tag{40}$$

$$\begin{aligned} g_{x^k}(0,n^2) &= \langle \mathrm{e}, L_nw_{x^k}(0,n^2) + (-n^2k!\mathrm{e}^k)\rangle = \\ &\langle \mathrm{e}, L_n(n^2k!L_n^{-1}P_e\mathrm{e}^k) - n^2k!\mathrm{e}^k\rangle (\text{using } (38)) = \\ &\langle \mathrm{e}, n^2k!P_e\mathrm{e}^k - n^2k!e^k\rangle = \\ &-n^2k!\langle e,e^k\rangle \qquad (P_ee^k \in \mathrm{range}L_n,\ \mathrm{e} \in (\mathrm{range}L_n)^{\perp}). \end{aligned} \tag{41}$$

Lemma 1 Let $I_m = \int_0^{\pi}(\sin n\xi)^m\mathrm{d}\xi\,(m,n \in \mathbf{Z}^+)$. Then

$$I_m = \begin{cases} \frac{(m-1)!!}{m!!} \cdot \frac{1-(-1)^n}{n} & \text{if } m \text{ is odd}, \\ \frac{(m-1)!!}{m!!}\pi & \text{if } m \text{ is even}. \end{cases} \tag{42}$$

Proof $I_m = \int_0^{\pi} (\sin n\xi)^m d\xi = (m-1)(I_{m-2} - I_m)$. So, $I_m = \frac{m-1}{m} I_{m-2}$. By induction,

$$I_m = \begin{cases} \frac{(m-1)!!}{m!!} I_1 & \text{if } m \text{ is odd} \\ \frac{(m-1)!!}{m!!} I_0 & \text{if } m \text{ is even} \end{cases} = \begin{cases} \frac{(m-1)!!}{m!!} \cdot \frac{1-(-1)^n}{n} & \text{if } m \text{ is odd}, \\ \frac{(m-1)!!}{m!!}\pi & \text{if } m \text{ is even}. \end{cases}$$ □

By Lemma 1, (41) is

$$g_{x^k}(0, n^2) = \begin{cases} -\frac{n(k!!)^2}{k+1} \cdot [1-(-1)^n] & \text{if } k \text{ is even} \\ -\frac{n^2(k!!)^2}{k+1}\pi & \text{if } k \text{ is odd}. \end{cases} \tag{43}$$

Let $(u, \mu) = (0, n^2)$, then (28) and (29) can be changed into

$$P_e L_n w_{\mu}(0, n^2) = 0, \tag{44}$$

and

$$P_e e + P_e L_n w_{x\mu}(0, n^2) = 0, \tag{45}$$

i.e.,

$$w_{\mu}(0, n^2) = 0, \tag{46}$$

$$w_{x\mu}(0, n^2) = 0. \tag{47}$$

Substituting (46) and (47) into (19) and (20), respectively, we can get

$$g_{\mu}(0, n^2) = 0, \tag{48}$$

$$g_{x\mu}(0, n^2) = \langle e, e \rangle = \frac{\pi}{2}. \tag{49}$$

Now we list our result except that n is even in (43) (this case is very complicated and is scheduled later).

Case 1 k is odd ($4 \leqslant k \in \mathbf{Z}^+$).

At $(u, \mu) = (0, n^2)$, $g = g_x = \cdots = g_{x^{k-1}} = g_{\mu} = 0$, $g_{x^k} = -\frac{n^2(k!!)^2}{k+1}\pi$, $g_{x\mu} = \frac{\pi}{2}$.

Case 2 k is even and n is odd ($4 \leqslant k \in \mathbf{Z}^+, n \in \mathbf{Z}^+$).

At $(u, \mu) = (0, n^2)$, $g = g_x = \cdots = g_{x^{k-1}} = g_{\mu} = 0, g_{x^k} = -\frac{2n(k!!)^2}{k+1}$, $g_{x\mu} = \frac{\pi}{2}$.

According to theorem 1, we have

Theorem 2 Suppose that k is odd ($4 \leqslant k \in \mathbf{Z}^+$). Then $g(x, \mu)$ is strongly equivalent to $-x^k + (\mu - n^2)x$. Furthermore, the bifurcation diagram of $\frac{\partial u}{\partial t} = F(u, \mu)$ at $(0, n^2)$ is similar to Fig.2.

Theorem 3 Suppose that k is even and n is odd ($4 \leqslant k \in \mathbf{Z}^+, n \in \mathbf{Z}^+$). Then $g(x, \mu)$

is strongly equivalent to $-x^k+(\mu-n^2)x$. Furthermore, the bifurcation diagram of $\frac{\partial u}{\partial t}=F(u,\mu)$ at $(0,\ n^2)$ is similar to Fig.1.

From (43), $g_{x^k}(0,n^2)=0$ if k and n are even. In order to investigate the case that k and n are even (see(43)), let's elaborate. Let

$$K(k)=\min_l\{l \mid g_{x^l}(0,n^2)\neq 0,\ k\leqslant l\in \mathbf{Z}^+,\ k \text{ and } n \text{ are even}\}. \tag{50}$$

(50) implies that

$$g_{x^l}(0,n^2)=0,\ l=k+1,k+2,\cdots,\ K(k)-1. \tag{51}$$

Several definitions are given as follows. We call $\mathrm{d}^lF(\xi_1,\cdots,\xi_l)$ quasi-item of g_{x^m} if it arises in the right hand of (52)

$$g_{x^m}=\langle e,\sum_{i=1}^{m}\sum_{\xi_1\cdots\xi_i}C_{\xi_1\cdots\xi_i}d^iF(\xi_1\cdots\xi_i)\rangle \tag{52}$$

where $C_{\xi_1\cdots\xi_i}\in\mathbf{Z}^+,\xi_1,\cdots,\xi_i$ denote the derivatives of some order of $xe+w(x,\ \mu)$ with respect to x, $\sum_{\xi_1\cdots\xi_i}$ denotes the sum of all possible items. For example, when $m=5,i=2$, $\sum_{\xi_1\xi_2}C_{\xi_1\xi_2}\mathrm{d}^2F(\xi_1,\xi_2)=5\mathrm{d}^2F(e+w_x,w_{x^4})+10\mathrm{d}^2F(w_{x^2},w_{x^3}),\mathrm{d}^2F(e+w_x,w_{x^4})$ and $\mathrm{d}^2F(w_{x^2},w_{x^3})$ are quasi-items of g_{x^5} (see (16)). We call the quasi-item $\mathrm{d}^lF(\xi_1,\cdots,\xi_l)$ of g_{x^m} dictionary sequencing if $k_1\leqslant k_2\leqslant\cdots\leqslant k_l$, where $\xi_i=\frac{\partial^{k_i}}{\partial x^{k_i}}(xe+w(x,\mu))$. The following lemma is easily verified but important.

Lemma 2 Assume that $\mathrm{d}^lF(\eta_1,\cdots,\eta_l)$ which is dictionary sequencing is an arbitrary quasi-item of g_{x^m}. Then

$$\sum_{i=1}^{l}k_i=m, \tag{53}$$

where $\eta_i=\frac{\partial^{k_i}}{\partial x^{k_i}}(xe+w(x,\mu))$.

Proof 1) $m=1$

Differentiating $F(xe+w(x,\mu),\mu)$ with respect to x one time leads to $\mathrm{d}F(e+w_x)$, which shows that (53) is true for $m=1$.

2) Assume that an arbitrary quasi-item $\mathrm{d}^lF(\eta_1,\cdots,\eta_l)$ satisfies

$$\sum_{i=1}^{l}k_i=m.$$

After differentiating g_{x^m} with respect to x one time again, we can have

$$\frac{\partial}{\partial x}(\mathrm{d}^lF(\eta_1,\cdots,\eta_l))=\mathrm{d}^{l+1}F(e+w_x,\eta_1,\cdots,\eta_l)+\sum_{i=1}^{l}d^lF\left(\eta_1,\cdots,\frac{\partial\eta_i}{\partial x},\cdots,\eta_l\right).$$

Obviously,

$$1+k_1+\cdots+k_l=1+m,$$

which indicates that (53) is true for $m+1$.

By mathematical induction, (53) is true, indeed. □

Remark 1 The condition, dictionary sequencing, is not necessary for our proof.

Now let us find $K(k)$.

By using (35), only when $(d^kF)_{(0,n^2)}(\eta_1,\cdots,\eta_k)$ is a quasi-item of $g_{x^{K'}}(0,n^2)$ (K' is bigger than k.), then $g_{x^{K'}}(0,n^2)$ may be non-zero. Assume that $(d^KF)_{(0,n^2)}(\eta_1,\cdots,\eta_k)$ is dictionary sequencing. According to (37) and (39), only when

$$(d^kF)_{(0,n^2)}(\underbrace{e + w_x,\cdots,e + w_x}_{k-1},w_x{}^k)$$

appears in $g_{x^{K'}}(0,n^2)$, then $g_{x^{K'}}(0,n^2) \neq 0$. By lemma 2, $K' = 2k - 1$ is what we want to seek, i.e., $K(k) = 2k - 1$. Now we can get

$$g_{x^{2k-1}}(0,n^2) = \langle e, L_n w_{x^{2k-1}}(0,n^2) + n^2 N_k(-k!)e^{k-1}w_{x^k}(0,n^2)\rangle =$$
$$- n^4(k!)^2 N_k\langle e, e^{k-1}L_n^{-1}P_e e^k\rangle \qquad (N_k \in \mathbf{Z}^+). \tag{54}$$

(For example, when $k = 2$ and n is even, then at $(u,\mu) = (0, n^2)$, $g = g_x = g_{x^2} = 0$, $g_{x^3} = \dfrac{-5\pi n^2}{2}$, at this moment, $K(2) = 3$. See [3]. When $k = 4$ and n is even, then at $(u,\mu) = (0,n^2)$, $g = g_x = \cdots = g_{x^6} = 0, g_{x^7} = \langle e, n^2 35(-4!)e^3 w_{x^4}(0,n^2)\rangle \neq 0$, at this moment, $K(4) = 7$.)

Let $L_n^{-1}P_e e^k = \bar{u} \in M$, then $L_n\bar{u} = P_e e^k$. Since $\langle e^k, e\rangle = 0$($k$ and n are even and use lemma 1), therefore, $e^k \in \text{range}L_n$, which implies $P_e e^k = e^k$. So, $L_n\bar{u} = e^k = (\sin nx)^k$, i.e.,

$$\bar{u}'' + n^2\bar{u} = (\sin nx)^k, \tag{55}$$

and the boundary value condition is

$$\bar{u}(0) = \bar{u}(\pi) = 0. \tag{56}$$

Before seeking the solution to (55) and (56), we prove the following lemma which is needed.

Lemma 3 One particular solution to

$$u'' + \alpha u = (\sin\omega x)^l \qquad (\alpha \text{ and } l \text{ are constants, } l \in \mathbf{Z}^+) \tag{57}$$

is

1) if l is even and $\alpha - i^2\omega^2 \neq 0(i = 0,2,\cdots,l-2,l)$, then

$$u = \sum_{i=0}^{l/2} a_{2i}(\sin\omega x)^{2i} \tag{58}$$

where

$$a_l = \frac{1}{\alpha - l^2\omega^2}, \quad a_j = -\frac{(j+2)(j+1)\omega^2}{\alpha - j^2\omega^2}a_{j+2} \qquad (j = l-2, l-4,\cdots,2,0); \tag{59}$$

2) if l is odd and $\alpha - i^2\omega^2 \neq 0(i = 1,3,\cdots,l-2,l)$, then

$$u = \sum_{i=1}^{(l+1)/2} a_{2i-1}(\sin\omega x)^{2i-1} \tag{60}$$

where

$$a_l = \frac{1}{\alpha - l^2\omega^2}, \quad a_j = -\frac{(j+2)(j+1)\omega^2}{\alpha - j^2\omega^2}a_{j+2}(j = l-2, l-4,\cdots,3,1). \tag{61}$$

Proof Suppose that

$$u = \sum_{i=0}^{l} a_i(\sin\omega x)^i \tag{62}$$

satisfies (57), then

$$\sum_{i=2}^{l} i(i-1)\omega^2 a_i(\sin\omega x)^{i-2} + \sum_{i=0}^{l}(\alpha - i^2\omega^2)a_i(\sin\omega x)^i = (\sin\omega x)^l. \tag{63}$$

It immediately follows that

$$\begin{cases}(\alpha - l^2\omega^2)a_l = 1, \\ (i+2)(i+1)\omega^2 a_{i+2} + (\alpha - i^2\omega^2)a_i = 0(i = l-2, l-4, \cdots).\end{cases} \tag{64}$$

If l is even and $\alpha - i^2\omega^2 \neq 0(i = 0,2,\cdots,l-2,l)$, we can get (59). However, $a_1, a_3, \cdots, a_{l-1}$ are arbitrary contants, so we can let them be zero. Hence, we finish the proof of the case 1). Similarly, that case 2) is also true. □

Corollary 1 One particular solution to (55) is

$$u_* = \sum_{i=0}^{k/2} a_i(\sin\omega x)^{2i}, \tag{65}$$

$a_i(i = 0,1,\cdots,k/2)$ is given by

$$a_0 = -C, a_1 = \frac{1}{2}C, a_i = \frac{(2i-3)!!}{(2i)!!}C$$
$$\left(i = 2,\cdots,k/2, C = \frac{k!!}{(k-3)!!n^2(1-k^2)}\right). \tag{66}$$

Proof According to lemma 3, one particular solution of (55) is

$$u_* = \sum_{i=0}^{k/2} a_i(\sin\omega x)^{2i}, \tag{67}$$

where $a_{k/2} = \dfrac{1}{(1-k^2)n^2}, a_i = \dfrac{2i+2}{2i-1}a_{i+1} \quad (i = 0,1,\cdots,k/2-1)$.

$$a_i = \frac{2i+2}{2i-1}a_{i+1} = \frac{(2i+2)((2i+4)}{(2i-1)(2i+1)}a_{i+2} = \cdots = \frac{k!!}{(2i)!!}\frac{a_{k/2}}{\prod_{j=i}^{k/2-1}(2j-1)}, \tag{68}$$

$$\prod_{j=i}^{k/2-1}(2j-1) = \begin{cases}\dfrac{(k-3)!!}{(2i-3)!!} & (i = 2,3,\cdots,k/2-1), \\ (k-3)!! & (i = 1), \\ -(k-3)!! & (i = 0),\end{cases} \tag{69}$$

so

$$a_i = \begin{cases}-C & (i = 0), \\ \dfrac{1}{2}C & (i = 1), \\ \dfrac{(2i-3)!!}{(2i)!!}C & (i = 2,3,\cdots,k/2),\end{cases} \tag{70}$$

where $C = \dfrac{k!!}{(k-3)!!}a_{k/2} < 0$. □

Now we can get the solution to (55) and (56)

$$\bar{u} = c_1\sin nx + c_2\cos nx + \sum_{i=0}^{k/2} a_i(\sin nx)^{2i}, \tag{71}$$

$a_i(i = 0,1,\cdots,k/2)$ are given by (70), $\bar{u} \in M, \bar{u}(0) = \bar{u}(\pi) = 0$. Since $\bar{u} \in M$, then $\langle e, \bar{u}\rangle = 0$, which implies that $c_1 = 0$(n is even and apply lemma 1). Due to (56), $c_2 = -a_0$.

Next, we should determine $\langle e, e^{k-1}, L_n^{-1}P_e e^k\rangle$ (see (54)).

$$\langle e, e^{k-1} L_n^{-1} P_e e^k \rangle = \langle e^k, \bar{u} \rangle = \sum_{i=0}^{k/2} a_i \langle (\sin nx)^k, (\sin nx)^{2i} \rangle = - c\pi \left[\frac{(k-1)!!}{k!!} - \frac{(k+1)!!}{2(k+2)!!} - \sum_{i=2}^{k/2} \frac{(2i-3)!!(k+2i-1)!!}{(2i)!!(k+2i)!!} \right] = - \frac{(k-1)!!}{k!!} C_\pi \cdot G_k, \tag{72}$$

where $G_k = \frac{1}{2} + \frac{1}{2(k+2)} - \sum_{i=2}^{k/2} \frac{(2i-3)!!(k+1)(k+3)\cdots(k+2i-1)}{(2i)!!(k+2)(k+4)\cdots(k+2i)}$. So,

$$g_{x^{2k-1}}(0, n^2) = \frac{n^4 (k!)^2 (k-1)!! N_k C\pi}{k!!} G_k. \tag{73}$$

For different k, G_k may have different signs. For example, when $k = 4,6,8,10,12,14,16,18$, $G_k > 0$; when $k = 20$, $G_k < 0$. Since $C < 0$, therefore, the sign of $g_{x^{2k-1}}$ is the same as that of $-G_k$.

Case 3 k and n are even ($4 \leqslant k \in \mathbf{Z}^+$, $n \in \mathbf{Z}^+$). At $(u, \mu) = (0, n^2)$, $g = g_x = \cdots = g_{x^{2k-2}} = g_\mu = 0$, $g_{x^{2k-1}} = \frac{n^4 (k!)^2 (k-1)!! N_k C\pi}{k!!} G_k$, $g_{x\mu} = \frac{\pi}{2}$.

Theorem 4 Suppose that k and n are even ($4 \leqslant k \in \mathbf{Z}^+$, $n \in \mathbf{Z}^+$). If $G_k \neq 0$, then, $g(x, \mu)$ is strongly equivalent to $-\operatorname{sgn} G_k x^{2k-1} + (\mu - n^2) x$. Furthermore, the bifurcation diagram of $\frac{\partial u}{\partial t} = F(u, \mu)$ at $(0, n^2)$ is similar to Fig.1 for the case of $G_k > 0$, or is similar to Fig.3 for the case of $G_k < 0$.

Remark 2 Theorem 4 does not contain the case of $G_k = 0$. If $G_k = 0$, we need the higher order derivative of g with respect to x which is $g_{x^{r(k)}}(0, n^2) \neq 0$.

References:

[1] Fife P C. *Mathematical Aspects of Reaction and Diffusing Systems*[M]. *Lecture Notes in Biomathematics*, Vol 28 Berlin/Heidelberg/New York: Springer-Verlag, 1979.

[2] Ye Qixao, Li Zhengyuan. *Introduction to Reaction-Diffusion Equations*[M]. Beijing: Science Press, 1994. (in Chinese)

[3] Chow S N, Hale J K. *Methods of Bifurcation Theory*[M]. New York: Springer-Verlag, 1982.

[4] Golubitsky M, Schaeffer D G. *Singularities and Groups in Bifurcation Theory*[M]. Vol 1, New York: Springer-Verlag, 1985.

[5] Lu Qishao. *Bifurcation and Singularity*[M]. Shanghai: Shanghai Scientific and Technological Education Publishing House, 1995. (in Chinese)

[6] Tang Yun. *Foundation of Symmetry Bifurcation Theory*[M]. Beijing: Science Press, 1998. (in Chinese)