Applied Mathematics and Mechanics
(English Edition, Vol. 18, No. 4, Apr. 1997)

Published by SU,
Shanghai, China

A NEW SOLUTION TO STRUCTURAL FUZZY FINITE ELEMENT EQUILIBRIUM EQUATIONS (SFFEEE)*

Lü Enling (吕恩琳)[1]

(Received Jan. 26, 1996; Communicated by Zhang Ruqing)

Abstract

A speedy accurate solution to structural fuzzy finite element equilibrium equations (SFFEEE), by combining the definition of the solution of interval equations with the mechanical meaning of the structural finite element equilibrium equations (SFEEE), was put forward. The fuzzification of the SFFEEE, which is discussed in this paper, originates from that of material property, structural boundary conditions and external loading. The computing quantity of this solution is almost equal to that of the general finite element method (GFEM).

Key words finite element equilibrium equations, fuzzy numbers, interval equations

I. Introduction

In the field of the fuzzy finite element method (FFEM), to study now, to solve the equilibrium equations is of momentous significance. Especially for the application of the FFEM to engineering problems, the key work to do is to explore high-speed and high-effeciency solutions to equilibrium equations. Among available solutions to the SFFEEE in [1, 2], Wang Caihua et al. only took the SFFEEE as a general fuzzy linear algebraic equations, and they also studied its solutions on the basis of operation rules of fuzzy numbers. In this paper, we first transformed the fuzzy equilibrium equations into a interval coefficient equations, and then researched the intrinsic mechanical meaning of the SFFEEE. As a result, a new speedy accurate solution to the SFFEEE was given. The computing quantity of this solution is almost equal to that of the GREM, therefore, a useful tool is provided for deep research on the FFEM and the engineering application of the FFEM.

II. The SFFEEE

In the finite element analysis of engineering structure, since the material property parameters of structure, boundary conditions and external loading are fuzzy, it follows that the stiffness matrix and the loading vector will be fuzzy, so will the unknown nodal displacement vector. And according to the GFEM, we have

$$[\mathbf{K}]\{\mathbf{U}\}=\{\mathbf{P}\}. \tag{2.1}$$

where $[\mathbf{K}]$ is a fuzzy stiffness matrix, $\{\mathbf{U}\}$ is a fuzzy displacement vector, $\{\mathbf{P}\}$ is a fuzzy loading vector. It can easily be seen that if L-R type fuzzy numbers are used to express and deal with the

* Project supported by the National Natural Science Foundation of China

[1] Department of Engineering Mechanics, Chongqing University, Chongqing 630044, P. R, China

fuzzy information, all the elements of $[\mathbf{K}]$, $\{\mathbf{U}\}$ and $\{\mathbf{P}\}$ are L-R type fuzzy numbers. They can be denoted by

$$\left.\begin{aligned} \mathbf{k}_{ij}&=(k^m_{ij},\ k^L_{ij},\ k^R_{ij})_{LR} \\ \mathbf{u}_j&=(u^m_j,\ u^L_j,\ u^R_j)_{LR} \\ \mathbf{p}_i&=(p^m_i,\ p^L_i,\ p^R_i)_{LR} \end{aligned}\right\}$$

where n is the degree of freedom of the structural considered.

$$(i,\ j=1,\ 2,\ \cdots,\ n)$$

By different λ-level cutting Eqs. (2.1), a series of linear equations can be obtained whose coefficients are all interval numbers, especially to a λ, the equations are

$$\begin{bmatrix} [\underline{k}_{11},\ \bar{k}_{11}], & [\underline{k}_{12},\ \bar{k}_{12}], & \cdots, & [\underline{k}_{1n},\ \bar{k}_{1n}] \\ [\underline{k}_{21},\ \bar{k}_{21}], & [\underline{k}_{22},\ \bar{k}_{22}], & \cdots, & [\underline{k}_{2n},\ \bar{k}_{2n}] \\ \vdots & \vdots & \cdots & \vdots \\ [\underline{k}_{n1},\ \bar{k}_{n1}], & [\underline{k}_{n2},\ \bar{k}_{n2}], & \cdots, & [\underline{k}_{nn},\ \bar{k}_{nn}] \end{bmatrix}_\lambda \begin{Bmatrix} [\underline{u}_1,\ \bar{u}_1] \\ [\underline{u}_2,\ \bar{u}_2] \\ \vdots \\ [\underline{u}_n,\ \bar{u}_n] \end{Bmatrix}_\lambda = \begin{Bmatrix} [\underline{p}_1,\ \bar{p}_1] \\ [\underline{p}_2,\ \bar{p}_2] \\ \vdots \\ [\underline{p}_n,\ \bar{p}_n] \end{Bmatrix}_\lambda$$

where $\underline{k}_{ij}$ and $\bar{k}_{ij}$ $(i,\ j=1,\ 2,\ \cdots,\ n)$ are respectively, the lower bound value and upper bound value of the element of the fuzzy stiffness matrix corresponding to the λ-cut, similarly, $\underline{u}_j$ $(j=1,\ 2,\ \cdots,n)$ the lower bound value of the element of fuzzy displacement vector and the $\bar{u}_j$ $(j=1,\ 2,\ \cdots,\ n)$ the upper; $\underline{p}_i$ $(i=1,\ 2,\ \cdots,\ n)$ the lower bound value of the element of fuzzy loading vector and the $\bar{p}_i$ $(i=1,\ 2,\ \cdots,\ n)$ the upper. All those lower bound values and upper bound values form different interval numbers by one-to-one. And now the above formula can simply be denoted by

$$K^I U^I = P^I \tag{2.2}$$

obviously, $U^I=\{[\underline{u}_1,\ \bar{u}_1],\ [\underline{u}_2,\ \bar{u}_2],\ \cdots,\ [\underline{u}_n,\ \bar{u}_n]\}^T$. The lower bound value and the upper bound value of every interval number $[\underline{u}_j,\ \bar{u}_j]$ which is the solution of Eq. (2.2) can be respectively obtained from the following formulae

$$\left.\begin{aligned} \underline{u}_j&=\min\{u_j \mid KU=P,\ K\in K^I,\ P\in P^I\}, \\ \bar{u}_j&=\max\{u_j \mid KU=P,\ K\in K^I,\ P\in P^I\}, \end{aligned}\quad (j=1,\ 2,\ \cdots,\ n)\right\} \tag{2.3}$$

in which K is a general matrix, P and U are all general column arrays.

Denote U^I by U^I_λ, namely $U^I=U^I_\lambda$. In the light of the decomposition theorems of fuzzy set in [4], we can calculate the solution of (2.1) as

$$\mathbf{U}=\bigcup_{\lambda\in[0,1]} \lambda U^I_\lambda \tag{2.4}$$

It is easy to show that for obtaining the solution of (2.1), we only need to solve a series of interval equations as (2.2).

To general interval number linear equations, many scholars have already done a large quantity of research work such as E. Hansen and R. Smith, who, respectively gave how to obtain the interval inverse matrix in [5, 6]. But they attached additional conditions during the course of solving the linear equations, as inevitably placed restrictions on the application of the methods to a more general cases. In [1, 2], Wang Caihua et al. provided one iteractive solution on the basis of interval matrix and the other on the basis of the decomposition of interval numbers. Undoubtfully, all those methods furthered the research on the FFEM and its application. But they had viewed this type of problem only from the pure mathematical standpoint rather than fully utilized the inherent mechanical meaning of (2.2). In fact, there exists laws that disclose the change of the variable U^I with the change of the variables K^I and P^I which we can make full use of to calculate U^I fastly and accurately. In the following sections we will first discuss two specical cases and then synthesize them.

III. The Fuzzification of Material Property and the Solution to the Corresponding FFEM

In this section, we will discuss how to calculate the fuzzy displacement of every nodal when structural material property is fuzzy and the loading is distinct.

We know, the structural stiffness relates to the geometrical parameters, but mainly is hinged on the material property parameters, to isotropic materials, namely, on the modulus of elasticity E and Poisson ratio ν. It is well-known that the change of E affects the mechanical behavior of a structure more greatly than that of ν does. We will maintain that E is fuzzy and ν is non-fuzzy in the following discussion. As for GFEM, the formula of the element stiffness matrix is of the form

$$[k]^e=\int_e [B]^T[D][B]dv \tag{3.1}$$

where $[D]$ is an elastic matrix which embodies the material property of corresponding element. Because of the fuzzification of the modulus of elasticity all the elements of $[D]$ will be fuzzy. To those common problems in elastic mechanics, for example, we have

$$[\mathbf{D}]=\frac{\mathsf{E}(1-\nu)}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1 & \nu/(1-\nu) & \nu/(1-\nu) & 0 & 0 & 0 \\ & 1 & \nu/(1-\nu) & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & (1-2\nu)/2(1-\nu) & 0 & 0 \\ & \text{sys.} & & & (1-2\nu)/2(1-\nu) & 0 \\ & & & & & (1-2\nu)/2(1-\nu) \end{bmatrix}$$

where E is fuzzy modulus of elasticity. E can be described by L-R type fuzzy number as follows

$$\mathsf{E}=(E_m,\ E_L,\ E_R)_{LR} \tag{3.2}$$

where E_m is the main value of E, which equals the value of E under the case of not touching upon fuzzification.

In the light of dot product rule of L-R type fuzzy numbers, it follows[3]

$$\mathsf{E}=\alpha E_m,\quad \alpha=(1,\ E_L/E_m,\ E_R/E_m)_{LR} \tag{3.3}$$

Replacing $[\mathbf{D}]$ in (3.1) with $[D]$, we can obtain the fuzzy stiffness matrix of an element:

$$[\mathbf{k}]^e=\int_e [B]^T[\mathbf{D}][B]dv$$

$$=\alpha\int_e [B]^T[D][B]dv=\alpha[k_m]^e \tag{3.4}$$

where $[k_m]^e$ is equal to the general element stiffness matrix. By assembling fuzzy element matrices according to the formula (3.4), the structural global fuzzy stiffness matrix is obtained.

$$[\mathbf{K}]=\boldsymbol{\alpha}[K_m] \tag{3.5}$$

where $[K_m]$ is equal to the general structural global stiffness matrix.

Substituting (3.5) into (2.1), and noting that the loading vector is non-fuzzy, we have

$$\boldsymbol{\alpha}[K_m]\{\mathbf{U}\}=\{P\}$$

by λ-level cutting the above formula, we furtherly obtain interval equations as following

$$[\underline{\alpha},\ \bar{\alpha}]_\lambda[K_m]\{\underline{U},\ \bar{U}\}_\lambda=\{P\} \tag{3.6}$$

And its solution can be expressed as

$$\{\underline{U},\ \bar{U}\}_\lambda=[K_m]^{-1}\{P\}/\{\underline{\alpha},\ \bar{\alpha}\}_\lambda$$

Denote

$$\{U_m\}=[K_m]^{-1}\{P\}=\{u_1^m,\ u_2^m,\ \cdots,\ u_n^m\}^T$$

By (2.3), it follows the calculating formula of the jth element of $\{\underline{U},\ \bar{U}\}_\lambda$

$$[\underline{u}_j,\ \bar{u}_j]_\lambda=[u_j^m/a_1,\ u_j^m/a_2] \tag{3.7}$$

in which,

$$\left.\begin{array}{ll} a_1=\bar{\alpha}_\lambda,\ a_2=\underline{\alpha}_\lambda, & \text{when } u_j^m\geqslant 0 \\ a_1=\underline{\alpha}_\lambda,\ a_2=\bar{\alpha}_\lambda, & \text{when } u_j^m<0 \end{array}\quad (j=1,\ 2,\ \cdots,\ n)\right\} \tag{3.8}$$

(3.7) is the accurate solution of (3.6). Choosing a series of $\lambda\in[0,\ 1]$ and following (2.4), we can constitute the fuzzy displacement solution $\{\mathbf{U}\}$. We must point out here that we need not to resolve the equilibrium equations because in (3.7) only a_1 and a_2 have changed when λ is chosen different value which belongs to the closed set [0, 1]. Therefore, the computing quatity of the above method is almost equal to that of the GFEM.

IV. Fuzzification of Loading and the Solution to the Corresponding FFEM

In this section, we will discuss how to compute the fuzzy nodal displacement only when the external loading is fuzzy.

Given that the total number of external loading is l, and let $\mathbf{f}_i$ is the ith fuzzy external loading. Naturally, we can adopt L-R type fuzzy numbers to discribe it as following

$$\mathbf{f}_i=(f_i^m,\ f_i^L,\ f_i^R)_{LR} \qquad (i=1,\ 2,\ \cdots,\ l)$$

If we properly choose the membership function of every loading, its right-extension-form f_i^R and left-extension-form f_i^L, then we can, definitely, introduce a non-dimensional fuzzy number $\boldsymbol{\beta}$:

$$\boldsymbol{\beta}=(1,\ \beta_L,\ \beta_R)_{LR} \tag{4.1}$$

so

$$\mathbf{f}_i=\boldsymbol{\beta}f_i^m=(f_i^m,\ \beta_L f_i^m,\ \beta_R f_i^m)_{LR} \quad (i=1,\ 2,\ \cdots,\ l) \tag{4.2}$$

Of course, non-nodal loading must be replaced equivalently at the nodal. To 2-dimensional plane element, for instance, if the distributed loading over the element is fuzzy then its equivalent fuzzy nodal loading is

$$\{\mathbf{p}\}^e = \int_s [N]^T \{\mathbf{f}\}^e t ds \tag{4.3}$$

where $[N]$ is the corresponding shape function matrix, $\{\mathbf{f}\}^e$ is fuzzy distributed loading, and t is the thick of the element. If the loading over it is fuzzy concentrated loading $\{\mathbf{f}\}^e$, the corresponding equivalent fuzzy nodal loading $\{\mathbf{p}\}^e$, has the following form

$$\{\mathbf{p}\}^e = [N]^T \{\mathbf{f}\}^e \tag{4.4}$$

Comparing (4.2), (4.3) with (4.4), we can describe them with a single formula

$$\{\mathbf{p}\}^e = \boldsymbol{\beta}\{p^m\}^e . \tag{4.5}$$

where $\{p^m\}^e$ is equal to the equivalent nodal loading column array of the element in the GFEM. To other types of elements, similar formulae can also be obtained.

Now, assembling every fuzzy equivalent nodal loading element-by-element the global fuzzy loading column array $\{\mathbf{P}\}$ is obtained. Evidently, we have

$$\{\mathbf{P}\} = \boldsymbol{\beta}\{P^m\} \tag{4.6}$$

where $\{P^m\}$ is composed of all the main value of every element of $\{\mathbf{P}\}$, which is equal to the global loading column array in the GFEM.

By substituting (4.6) into (2.1), it follows

$$[K]\{\mathbf{U}\} = \boldsymbol{\beta}\{P^m\}$$

Similarly, by λ-level cutting the above formula, we can obtain

$$[K]\{\underline{U},\ \overline{U}\}_\lambda = [\underline{\beta},\ \overline{\beta}]_\lambda \{P^m\} \tag{4.7}$$

Of course, the equation is an interval equation.

Denote

$$\{U^m\} = [K]^{-1}\{P^m\} = \{u_1^m,\ u_2^m,\ \cdots,\ u_n^m\}^T$$

on the basis of (2.3), we can obtain the calculating formula of the jth element of the solution $\{\underline{U},\ \overline{U}\}_\lambda$ as follows

$$[\underline{u}_j,\ \overline{u}_j]_\lambda = [\beta_1 u_j^m,\ \beta_2 u_j^m] \qquad (j=1,\ 2,\ \cdots,\ n) \tag{4.8}$$

where

$$\left.\begin{array}{ll} \beta_1 = \underline{\beta}_\lambda,\ \beta_2 = \overline{\beta}_\lambda, & \text{when } u_j^m \geqslant 0 \\ \beta_1 = \overline{\beta}_\lambda,\ \beta_2 = \underline{\beta}_\lambda, & \text{when } u_j^m < 0 \end{array} \quad (j=1,\ 2,\ \cdots,\ n) \right\} \tag{4.9}$$

Here, (4.8) is also the accurate solution of (4.7). Samely, in the light of (2.4), we can constitute the fuzzy displacement solution $\{\mathbf{U}\}$. Clearly, the computing quantity is, also, almost equal to the GFEM.

V. The Solution to the General FFEM

We are now in a position to discuss how to obtain the fuzzy displacement of every nodal when the modulus of elasticity and the loading over the structure are all fuzzy. According to the foregoing methods, at this very case, the equilibrium equations can be written as follows

$$\boldsymbol{\alpha}[K_m]\{\mathbf{U}\} = \boldsymbol{\beta}\{P^m\} \tag{5.1}$$

like the method in Section III and Section IV, we have

$$[\underline{\alpha},\ \bar{\alpha}]_\lambda[K_m]\{\underline{U},\ \bar{U}\}_\lambda=[\underline{\beta},\ \bar{\beta}]_\lambda\{P^m\} \tag{5.2}$$

(5.2) is also an interval equation.

Denote

$$\{U^m\}=[K_m]^{-1}\{P^m\}=\{u_1^m,\ u_2^m,\ \cdots,\ u_n^m\}^T$$

indeed $\{U^m\}$ is completely equal to the displacement solution column array in the GFEM, and the value of the jth element of $\{\underline{U},\ \bar{U}\}_\lambda$ is

$$[\underline{u}_j,\ \bar{u}_j]_\lambda=[(\beta_1/\alpha_1)u_j^m,\ (\beta_2/\alpha_2)u_j^m] \qquad (j=1,\ 2,\ \cdots,\ n) \tag{5.3}$$

$\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2$ can respectively be obtained from (3.8) or (4.9).

(5.3) is also the accurate solution of (5.2). We can obtain the fuzzy displacement solution $\{\mathbf{U}\}$ in the same way as discribed in the foregoing sections.

VI. Boundary Conditions and How to Deal with Them

It is known that the finite element equilibrium equations can not be solved if the displacement boundary conditions of the structure are not given. In this section, we will discuss how to deal with the fuzzy support conditions.

The fuzzification of the displacement boundary conditions comes from the support conditions of the boundary. But the support conditions can be simulated by one (or more) elastic bearing, namely, if displacement is generated at the constraind nodal, then the corresponding bearing can exert a force at this nodal to hinder the nodal from generating displacement, the size of which is directly proportional to the size of the displacement, thereby, the effect is equivalent to that of the case that a spring is linked to a stiff bearing. In the GFEM, the method to deal with elastic bearing is that if there exists a spring constrain at rth displacement whose stiffness coefficient is k_s, then, what only need to do is directly add k_s to the rth diagonal element of the global stiffness matrix. Because a stiff bearing can be treated as a elastic bearing whose stiff coeffecient is infinite, k_s can change between zero and infinite. The fuzzification of k_s is just the origin of that of displacement boundary conditions. Therefore we can adopt the fuzzy stiff coefficient $\mathbf{k}_s$ to express the fuzzification of support conditions. To $\mathbf{k}_s$, the membership function and the left-extension-form and the right-extension-form all can be properly chosen, and definitely, there has a constant C_s so

$$\mathbf{k}_s=C_s\mathbf{E}=\boldsymbol{\alpha}(C_sE_m) \tag{6.1}$$

where $\mathbf{E}$ and $\boldsymbol{\alpha}$ is defined by (3.2) and (3.3) respectively. Evidently, after adding $\mathbf{k}_s$ to $\mathbf{k}_{rr}$ the global stiffness matrix still maintains the form as (3.5). It is shown that the method to deal with the fuzzification of support conditions is similar to the method to do that of modulus of elasticity of material. When C_s is a sufficiently big number it is commensurate with the big-number-method to deal with the stiff constraint in the GFEM.

VII. Conclusion

The GFEM is the common method in modern engineering structural analysis, and the FFEM provides more practicable auxiliary information. But the application and dissemination of the FFEM will be seriously restrainted if the computing quantity of the FFEM has increased by tens of times as compared with that of the GFEM only in order to obtain those helpful information. This paper combined fuzzy theorems with interval equation theorems and the mechanical meaning of the finite element equilibrium equations, gave a new speedy accurate solution to the SFFEEE. As a result, the computing quantity of this method is almost equal to that of the GFEM, therefore, this method provides a useful tool for deeply research on the theory of the FFEM and the application and dissemination.

References

[1] Wang Caihua and Zhu Hengshan, An algorithm for structural fuzzy equilibrium equations based on the L-R type fuzzy number compution laws, *Theory and Application of Fuzzy Analysis and Design,* Proceedings of 3rd Conference on Fuzzy Analysis and Design Conference, Architectural and Civil Industry Publition of China (1993), 35~38. (in Chinese)

[2] Yu Zhitao, Lü Enling and Wang Caihua, A method for solving the structural fuzzy finite element equilibrium equations, *Journal of Chongqing University* (*Natural Science Edition*), **19**, 1 (1996), 53~58. (in Chinese)

[3] D. Dubois and H. Prade, *Fuzzy Sets and System: Theory and Applications*, Academic Press, New York (1980).

[4] Wang Caihua and Song Liantian, *Methodology of Fuzzy Method*, Architectural and Civil Industry Publition of China (1988), 70~76. (in Chinese)

[5] E. Hansen, Interval arithmetic in matrix computions, Part I, *SIAM J., Series B, Numerical Analysis*, **2** (1965), 308~320.

[6] E, Hansen, Interval arithmetic in matrix computions, Part II, *SIAM J., Series B, Numerical Analysis*, **2** (1967), 1~9.